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COINCIDENCE THEOREMS VIA CONTRACTIVE MAPPINGS IN ORDERED NON-ARCHIMEDEAN FUZZY METRIC SPACES

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Abstract. In this article, we prove coincidence point theorems for comparable ψ-contractive mappings in ordered non-Archimedean fuzzy metric spaces utilizing the recently established concept of $\mathcal{T}$-comparability and relatively weaker order theoretic variants. With a view to show the usefulness and applicability of this work, we solve the system of ordered Fredholm integral equations as an application. In the process, this presentation generalize and improve some prominent recent results obtained in Mihet [Fuzzy Sets Syst., 159 (6), 739-744, (2008)], Altun and Mihet [Fixed Point Theory Appl. 2010, 782680, (2010)], Alam and Imdad [Fixed Point Theory, 18(2), 415-432, (2017)] and several others in the settings of partially ordered non-Archimedean fuzzy metric spaces.

1. Introduction

In 1988, Grabiec [8] revisited the Banach contraction principle to study the survival of fixed point in the setting of fuzzy metric spaces. Thereafter, George and Veeramani [6] presented slight modification of fuzzy metric initiated by Kramosil and Michalek [14] by obtaining Hausdroff topology on fuzzy metric spaces. Subsequently, numerous authors (see for instance, [3-4], [9-10], [16-17], [22] and [24]) obtained many useful results in this direction. On another point of note, several mathematicians followed fixed point results of Nieto-López [18] and Ran-Reurings [21] in the last fifteen years. The respective authors investigated the fixed point by considering the monotonicity of mappings and some monotone iterative techniques in the setting of ordered metric spaces and ordered fuzzy metric spaces, see for instance ([1-2], [5], [26]). However, the first try to investigate the analogous version of the Banach contraction principle in this setting was performed by Turinici [25].

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Meanwhile, Altun [4] presented a fixed point results under Lukasiewicz t-norm by considering the partially ordered set with non-Archimedean fuzzy metric settings and Altun and Mihet [3] presented fuzzy Ψ-contractive mappings in the same setting.

One of the objectives of this presentation is to generalize, improve and extend the results for a fuzzy Ψ-contractive mapping [16] to a pair of mappings in an ordered non-Archimedean fuzzy metric settings exploiting the notion of $\mathcal{T}$-comparable mapping presented by Alam et al.[1]. To prove the coincidence of a pair of mapping satisfying comparable fuzzy ψ-contraction, we use the comparable iterative technique together with the traditional technique exploiting relatively weaker order theoretic variants. Further, we present two corollaries which are novel and sharpened versions of celebrated and contemporary results existing in the literature (see, [3], [9-10], [16] and references therein) as underlying ψ-contraction is presumed to hold only on the comparable elements of the ordered set. In order to vindicate the usefulness and applicability of such theorems, we furnish a non-trivial example and solve the system of Fredholm integral equations.

2. PRELIMINARIES

Firstly, we present basic definitions which will work as a relevant necessary background for further presentations. Throughout this paper, we use the notation $\mathbb{N}$, to denote the set of natural numbers and $\mathbb{N}_0$, to denote the set of whole numbers (i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Definition 2.1 ([23]). Let $* : [0,1] \times [0,1] \to [0,1]$ be a binary operation. Then '$*$' is a continuous t-norm if $([0,1], *)$ is a commutative topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

A continuous t-norm $*$ is a Hadžić-type norm if we have a strictly increasing sequence $\{b_n\} \subset (0,1)$ and the condition $b_n * b_n = b_n$, $n \in \mathbb{N}$.

Definition 2.2 ([14]). Let $M : \mathcal{X} \times \mathcal{X} \times [0,\infty) \to [0,1]$ be a fuzzy set defined on a non-empty set $\mathcal{X}$ and '$*$' be a continuous t-norm. Then a triplet $(\mathcal{X}, \mathcal{M}, *)$ is a fuzzy metric space if for $x, y, z \in \mathcal{X}$ and for all $t, s > 0$, the subsequent assumptions hold :
(KM_1) $\mathcal{M}(x, y, 0) = 0$,
(KM_2) $\mathcal{M}(x, y, t) = 1$ iff $x = y$,

(KM_3) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$,
(KM_4) $\mathcal{M}(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,
(KM_5) $\mathcal{M}(x, z, t + s) \geq \mathcal{M}(x, y, t) * \mathcal{M}(y, z, s)$.
Moreover, if the triangular inequality (KM_5) is restored by

$$\mathcal{M}(x, z, \max\ \{t, s\}) \geq \mathcal{M}(x, y, t) * \mathcal{M}(y, z, s),\ (NA)$$

then the triplet $(\mathcal{X}, \mathcal{M}, *)$ is a non-Archimedean fuzzy metric space [11]. It can be easily observed that (NA) implies (KM_5).

Example 2.3 ([3])**.** Let $(\mathcal{X}, d)$ be a metric space and $\theta : (0, \infty) \to (0, 1)$ be a continuous non-decreasing function so that $\lim_{t\to\infty} \theta(t) = 1$. Let $a * b \leq ab,\ a, b \in [0, 1]$. Define

$$\mathcal{M}(x, y, t) = [\theta(t)]^{d(x,y)},$$

$t > 0$, and $x, y \in \mathcal{X}$. Then observe that $(\mathcal{X}, \mathcal{M}, *)$ is a non-Archimedean fuzzy metric space.

Definition 2.4 ([6, 8])**.** Let $\{x_n\}$ be a sequence in a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ and $t > 0$. Then
(a) $\{x_n\}$ is an $\mathcal{M}$-Cauchy sequence, if for $\epsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ so that $\mathcal{M}(x_n, x_m, t) > 1 - \epsilon$, $m, n \geq n_0$.
(b) $\{x_n\}$ is convergent, if $\lim\limits_{n\to\infty} \mathcal{M}(x_n, x, t) = 1,\ x \in \mathcal{X}$.
(c) $(\mathcal{X}, \mathcal{M}, *)$ is $\mathcal{M}$-complete if every $\mathcal{M}$-Cauchy sequence is convergent to a point in $\mathcal{X}$.
(d) $\{x_n\}$ is a $\mathcal{G}$-Cauchy sequence if $\lim\limits_{n\to\infty} \mathcal{M}(x_n, x_{n+1}, t) = 1$.
(e) $(\mathcal{X}, \mathcal{M}, *)$ is $\mathcal{G}$-complete if every $\mathcal{G}$-Cauchy sequence is convergent to a point in $\mathcal{X}$.

Lemma 2.5 ([3])**.** *An $\mathcal{M}$-complete non-Archimedean fuzzy metric space together with Hadžić norm is $\mathcal{G}$-complete.*

Now we present some ordered theoretic definitions.

Definition 2.6 ([15, 26])**.** (a) Let $\mathcal{X}$ be a non-empty set endowed with a partial order relation (anti-symmetric, reflexive and transitive) denoted by '$\preceq$'. Then the pair $(\mathcal{X}, \preceq)$ is partially ordered set (or an ordered set).
(b) The element x is comparable to y if either $x \preceq y$ or $x \succeq y$ and is denoted by the symbol '$\prec\succ$'.
(c) $\mathcal{X}$ is linearly ordered or totally ordered if any two elements of $\mathcal{X}$ are comparable.

Definition 2.7 ([2])**.** Let $(\mathcal{X}, \preceq)$ be an ordered set and $\{x_n\} \subset \mathcal{X}$.
(a) A sequence $\{x_n\}$ is term-wise bounded if for $z \in \mathcal{X}$, each term of $\{x_n\}$ is comparable with z, i.e.,

$$x_n \prec\succ z,\ n \in \mathbb{N}_0$$

and z is a c-bound of $\{x_n\}$.
(b) A sequence $\{x_n\}$ is term-wise monotonic sequence if consecutive terms of $\{x_n\}$ are comparable, i.e.,

$$x_n \prec\succ x_{n+1},\ n \in \mathbb{N}_0.$$

In the light of above definitions, it is easy to observe that all bounded below and bounded above sequences are term-wise bounded and monotonic sequences are term-wise monotonic.

Definition 2.8 ([2])**.** The space $(\mathcal{X}, \mathcal{M}, *)$ endowed with partial order '$\preceq$' has a term-wise monotone convergence-c-bound property or in short, TCC-property if each term-wise monotonic convergent sequence $\{x_n\} \subset \mathcal{X}$ has a subsequence, which is term-wise bounded by the limit of $\{x_n\}$ as a c-bound, i.e. $x_n \updownarrow x \Rightarrow$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_{n_k} \prec\succ x,\ k \in \mathbb{N}_0$.

Definition 2.9 ([2])**.** A self-mapping $\mathcal{T}$ on $(\mathcal{X}, \mathcal{M}, *)$ equipped with a partial order '$\preceq$' has a term-wise monotone convergence-c-bound property or in short, $\mathcal{T}$-TCC-property if each term-wise monotonic convergent sequence $\{x_n\}$ in $\mathcal{X}$ has a subsequence, whose $\mathcal{T}$-image is term-wise bounded by the $\mathcal{T}$-image of the limit of $\{x_n\}$ as a c-bound, i.e. $x_n \updownarrow x \Rightarrow$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $\mathcal{T}(x_{n_k}) \prec\succ \mathcal{T}(x),\ k \in \mathbb{N}_0$.

Definition 2.10 ([2])**.** Let $\mathcal{S}$ and $\mathcal{T}$ be self-mappings of an ordered set $(\mathcal{X}, \preceq)$. Then for $x, y \in \mathcal{X}$, (a) $\mathcal{S}$ is $\mathcal{T}$-increasing if

$$\mathcal{T}(x) \preceq \mathcal{T}(y) \Rightarrow \mathcal{S}(x) \preceq \mathcal{S}(y),$$

(b) $\mathcal{S}$ is $\mathcal{T}$-decreasing if

$$\mathcal{T}(x) \preceq \mathcal{T}(y) \Rightarrow \mathcal{S}(x) \succeq \mathcal{S}(y),$$

(c) $\mathcal{S}$ is $\mathcal{T}$-monotone if $\mathcal{S}$ is either $\mathcal{T}$-increasing or $\mathcal{T}$-decreasing.

If $\mathcal{T} = \mathcal{I}$, the identity mapping on $\mathcal{X}$ in above definitions then, these reduces to the notions of increasing, decreasing and monotone mapping respectively (see, Alam and Imdad [2]). Turinici [26] presented the notion of weakly monotonic or comparable mappings.

Definition 2.11. Let $\mathcal{S}$ be a self-mapping on an ordered set $(\mathcal{X}, \preceq)$. Then
(1) ([26]). $\mathcal{S}$ is comparable ($\prec\succ$-preserving or weakly monotonic) mapping if,

$$x \prec\succ y \Rightarrow \mathcal{S}(x) \prec\succ \mathcal{S}(y),\ x, y \in \mathcal{X}.$$

(2) ([1]). $\mathcal{S}$ is $\mathcal{T}$-comparable ($\prec\succ$-preserving or weakly monotonic) mapping if,

$$\mathcal{T}(x) \prec\succ \mathcal{T}(y) \Rightarrow \mathcal{S}(x) \prec\succ \mathcal{S}(y),\ x, y \in \mathcal{X}.$$

Definition 2.12 ([1, 26])**.** Let $\mathcal{E}$ be a subset of an ordered set $(\mathcal{X}, \preceq)$ and $p, q \in \mathcal{E}$. A subset $\{e_1, e_2, ..., e_k\}$ of $\mathcal{E}$ is said to be $\prec\succ$-chain between p and q in $\mathcal{E}$ if
(i) $k \geq 2$,
(ii) $e_1 = p$ and $e_k = q$,
(iii) $e_1 \prec\succ e_2 \prec\succ ... \prec\succ e_{k-1} \prec\succ e_k$.

Let $\mathbb{C}(p, q, \prec\succ, \mathcal{E})$ denotes the family of $\prec\succ$-chains between p and q in E. Particularly, if $\mathcal{E} = \mathcal{X}$, we abbreviate $\mathbb{C}(p, q, \prec\succ)$ for $\mathbb{C}(p, q, \prec\succ, \mathcal{X})$.

Definition 2.13 ([7, 13])**.** Let $\mathcal{S}$ and $\mathcal{T}$ be self-mappings defined on a non-empty set $\mathcal{X}$. Then
(a) $x \in \mathcal{X}$ is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$ if $\mathcal{T}(x) = \mathcal{S}(x)$,
(b) if $z \in \mathcal{X}$ is any point so that $z = \mathcal{S}(x) = \mathcal{T}(x)$, then z is a point of coincidence of the mappings $\mathcal{S}$ and $\mathcal{T}$.
(c) the pair ($\mathcal{S}$, $\mathcal{T}$) is weakly compatible if $\mathcal{S}$ and $\mathcal{T}$ commute at their coincidence points, i.e., $\mathcal{T}(\mathcal{S}(x)) = \mathcal{S}(\mathcal{T}(x))$, whenever $\mathcal{S}(x) = \mathcal{T}(x)$.

Definition 2.14 ([17])**.** Let $\mathcal{S}$ and $\mathcal{T}$ be self-mappings on a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$. Then $\mathcal{S}$ and $\mathcal{T}$ is compatible if for $t > 0$

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n, t) = 1, \tag{2.1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ so that $\underset{n\to\infty}{lim} \mathcal{S}x_n = \underset{n\to\infty}{lim} \mathcal{T}x_n = z,\ z \in \mathcal{X}$.

Definition 2.15 ([19])**.** Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings on a fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$. Then $\mathcal{S}$ and $\mathcal{T}$ are reciprocally continuous, if $\lim_{n\to\infty} \mathcal{T}(\mathcal{S}x_n) = Tz$ and $\lim_{n\to\infty} \mathcal{S}(\mathcal{T}x_n) = Sz$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ so that $\underset{n\to\infty}{lim} \mathcal{S}x_n = \underset{n\to\infty}{lim} \mathcal{T}x_n = z,\ z \in \mathcal{X}$.

Lemma 2.16 ([12]). *Let $\mathcal{S}$ be a self-mapping defined on a non-empty set $\mathcal{X}$. Then there exists a subset $\mathcal{E} \subseteq \mathcal{X}$ such that $\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{X})$ and $\mathcal{S} : \mathcal{E} \to \mathcal{X}$ is one-one.*

Let $\mathbf{\Psi}$ be the class of nondecreasing and left continuous functions $\psi : [0,1] \to [0,1]$, so that $\psi(t) > t,\ t \in (0,1)$. Then we refer the following lemma :

Lemma 2.17 ([10]). *If $\psi \in \mathbf{\Psi}$ then $\lim_{n\to\infty} \psi^n(t) = 1$ for each $t \in (0,1]$.*

3. Main Results

Firstly, we present three coincidence point results for comparable pair of ψ-contractive mappings and then applicability of these results by solving the ordered integral equations.

Theorem 3.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings on an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ together with Hadžić-type norm and equipped with a partial order '$\preceq$'. Let the subsequent hypotheses hold:*
(a) $\mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$,
(b) *$\mathcal{S}$ is $\mathcal{T}$-comparable,*
(c) $\mathcal{T}(x_0) \prec\succ \mathcal{S}(x_0),\ \lim_{n\to\infty} \psi^n(\mathcal{M}(\mathcal{T}x_0, \mathcal{S}x_0, t)) = 1,\ x_0 \in \mathcal{X}$,
(d) *there exists $\psi \in \mathbf{\Psi}$ so that*

$$\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(\mathcal{T}x, \mathcal{T}y, t)),\ \forall\ x, y \in \mathcal{X} \text{ with } \mathcal{T}(x) \prec\succ \mathcal{T}(y),\ t > 0,$$

(e) *a pair $(\mathcal{S}, \mathcal{T})$ is compatible,*
(f) *$\mathcal{S}$ and $\mathcal{T}$ are continuous mappings,*
or alternately
(f′) *$(\mathcal{X}, \mathcal{M}, *)$ has $\mathcal{T}$-TCC property and $\mathcal{S}$ is continuous.*
Then $\mathcal{S}(x) = \mathcal{T}(x),\ x \in \mathcal{X}$, that is, the pair $(\mathcal{S}, \mathcal{T})$ has a coincidence point.

Proof. In pursuance of condition (d), the assumption $\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(\mathcal{T}x, \mathcal{T}y, t))$ is satisfied under two cases : either $\mathcal{T}(x) \preceq \mathcal{T}(y)$ or $\mathcal{T}(x) \succeq \mathcal{T}(y),\ x, y \in \mathcal{X}$. If it holds for the first case, then by the symmetry condition (KM_3) of fuzzy metric space it must holds for second case too and the same is true for converse consideration. So, on applying the given contractive condition these two cases are the same. Therefore, we consider only the first to explore our further investigations.
In the light of condition (c), if $\mathcal{T}(x_0) = \mathcal{S}(x_0)$, then x_0 is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$. So the proof is accomplished.
If $\mathcal{T}(x_0) \neq \mathcal{S}(x_0)$, then we have $\mathcal{T}(x_0) \prec \mathcal{S}(x_0)$. Exploiting the condition (a) (i.e.

$\mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$), we are able to select $x_1 \in \mathcal{X}$ so that $\mathcal{T}(x_1) = \mathcal{S}(x_0)$. Also in the light of the same condition we may select $x_2 \in \mathcal{X}$ so that $\mathcal{T}(x_2) = \mathcal{S}(x_1)$. Following this pattern, we are able to construct a sequence of Picard's iterates $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{T}(x_{n+1}) = \mathcal{S}(x_n),\ n \in \mathbb{N}_0. \tag{3.1}$$

Now, we assert that $\{\mathcal{T}x_n\}$ is a term-wise monotone sequence, that is,

$$\mathcal{T}(x_n) \prec\succ \mathcal{T}(x_{n+1}),\ n \in \mathbb{N}_0. \tag{3.2}$$

We establish this assertion by using mathematical induction.

Consider assumption (c) and equation (3.1) with $n = 0$, we obtain $\mathcal{T}(x_0) \prec\succ \mathcal{S}(x_0) = \mathcal{T}(x_1)$. Therefore, (3.2) is true for $n = 0$. If equation (3.2) is true for $n = r > 0$, i.e.,

$$\mathcal{T}(x_r) \prec\succ \mathcal{T}(x_{r+1}) \tag{3.3}$$

then we show that (3.2) is true for $n = r + 1$ too. To assay this, we consider (3.1), (3.3) and assumption (b) so that

$$\mathcal{T}(x_{r+1}) = \mathcal{S}(x_r) \prec\succ \mathcal{S}(x_{r+1}) = \mathcal{T}(x_{r+2}).$$

Hence, in the light of mathematical induction (3.2) is valid for all $n \in \mathbb{N}_0$.

If $n_0 \in \mathbb{N}$, so that $\mathcal{M}(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0+1}, t) = 1$, this implies that $\mathcal{T}(x_{n_0}) = \mathcal{T}(x_{n_0+1})$, that is, $\mathcal{T}(x_{n_0}) = \mathcal{S}(x_{n_0})$, then x_{n_0} is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$. So the proof is accomplished.

On the other hand, if $\mathcal{T}(x_n) \neq \mathcal{T}(x_{n+1}), i.e., \mathcal{M}(\mathcal{T}x_n, \mathcal{T}x_{n+1}, t) \neq 1,\ n \in \mathbb{N}_0$. On utilizing (3.1) and (3.2) in the light of contractive condition (d), we obtain

$$\begin{aligned} \mathcal{M}(\mathcal{T}x_n, \mathcal{T}x_{n+1}, t) &= \mathcal{M}(\mathcal{S}x_{n-1}, \mathcal{S}x_n, t) \\ &\geq \psi(\mathcal{M}(\mathcal{T}x_{n-1}, \mathcal{T}x_n, t)),\ n \in \mathbb{N},\ t > 0, \end{aligned} \tag{3.4}$$

so that

$$\mathcal{M}(\mathcal{T}x_n, \mathcal{T}x_{n+1}, t) \geq \psi^n(\mathcal{M}(\mathcal{T}x_o, \mathcal{S}x_0, t)),\ n \in \mathbb{N},\ t > 0. \tag{3.5}$$

Now,

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{T}x_n, \mathcal{T}x_{n+1}, t) = 1, \tag{3.6}$$

$t > 0$, i.e., $\{\mathcal{T}x_n\}$ is a $\mathcal{G}$-Cauchy sequence. As $\mathcal{X}$ is $\mathcal{M}$-complete, so in the light of Lemma 2.5, $\mathcal{X}$ must be $\mathcal{G}$-complete. So,

$$\lim_{n\to\infty} \mathcal{T}(x_n) = z,\ z \in \mathcal{X}. \tag{3.7}$$

By using (3.1) and (3.7), we attain

$$\lim_{n\to\infty} \mathcal{S}(x_n) = \lim_{n\to\infty} \mathcal{T}(x_{n+1}) = z. \tag{3.8}$$

In the light of assumption (f) (that is, if $\mathcal{T}$ is continuous), we attain

$$\lim_{n\to\infty} \mathcal{T}(\mathcal{T}x_n) = \mathcal{T}(\lim_{n\to\infty} \mathcal{T}x_n) = \mathcal{T}(z), \tag{3.9}$$

$$\lim_{n\to\infty} \mathcal{T}(\mathcal{S}x_n) = \mathcal{T}(\lim_{n\to\infty} \mathcal{S}x_n) = \mathcal{T}(z). \tag{3.10}$$

As $\lim_{n\to\infty} \mathcal{S}(x_n) = \lim_{n\to\infty} \mathcal{T}(x_n) = z$ (due to (3.7) and (3.8)) on using assumption (e), we have

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n, t) = 1. \tag{3.11}$$

Now, consider condition (f). Then by utilizing (3.7) and continuity of $\mathcal{S}$, we obtain

$$\lim_{n\to\infty} \mathcal{S}(\mathcal{T}x_n) = \mathcal{S}(\lim_{n\to\infty} \mathcal{T}x_n) = \mathcal{S}(z). \tag{3.12}$$

By using (3.10), (3.11), (3.12) and continuity of $\mathcal{M}$, we obtain

$$\begin{aligned} \mathcal{M}(\mathcal{T}z, \mathcal{S}z, t) &= \mathcal{M}(\lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n, \lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n, t) \\ &= \lim_{n\to\infty} \mathcal{M}(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n, t) \\ &= 1. \end{aligned}$$

Therefore $\mathcal{T}z = \mathcal{S}z$, i.e., z is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$. Thus, the proof is done. Alternately, let $\mathcal{S}$ be continuous and $(\mathcal{X}, \mathcal{M}, *)$ has $\mathcal{T}$-TCC property, on account of (3.2) and (3.7), we have $\mathcal{T}x_n \updownarrow z$, then there exists a subsequence $\{x_{n_k}\}$ of $\{\mathcal{T}x_n\}$, with the result that

$$\mathcal{T}(x_{n_k}) \prec\succ \mathcal{T}(z),\ k \in \mathbb{N}_0.$$

Now $\{\mathcal{T}x_n\} \subset \mathcal{T}(\mathcal{X})$ and $\{x_{n_k}\} \subset \{\mathcal{T}x_n\}$, then there exists $\{x_{n_k}\} \subset \mathcal{X}$ so that $\{x_{n_k}\} = \mathcal{T}(x_{n_k})$. Thus, we obtain

$$\mathcal{T}(\mathcal{T}x_{n_k}) \prec\succ \mathcal{T}(z),\ \text{for all } k \in \mathbb{N}_0. \tag{3.13}$$

Since $\mathcal{T}(x_{n_k}) \to z$, equations (3.7) to (3.12) remain true for $\{x_{n_k}\}$, instead of $\{x_n\}$. Using (3.13) and assumption (d), we obtain

$$\mathcal{M}(\mathcal{S}(\mathcal{T}x_{n_k}), \mathcal{S}(z), t) \geq \psi(\mathcal{M}(\mathcal{T}(\mathcal{T}x_{n_k}), \mathcal{T}(z), t)),\ t > 0,\ k \in \mathbb{N}_0. \tag{3.14}$$

By using triangular inequality, equations (3.9), (3.10), (3.11) and (3.14), we have

$$\begin{aligned}
&\mathcal{M}(\mathcal{T}z, \mathcal{S}z, t)\\
&\geq \mathcal{M}(\mathcal{T}z, \mathcal{T}(\mathcal{S}x_{n_k}), t) * \mathcal{M}(\mathcal{T}(\mathcal{S}x_{n_k}), \mathcal{S}(\mathcal{T}x_{n_k}), t) * \mathcal{M}(\mathcal{S}(\mathcal{T}x_{n_k}), \mathcal{S}z, t)\\
&\geq \mathcal{M}(\mathcal{T}z, \mathcal{T}(\mathcal{S}x_{n_k}), t) * \mathcal{M}(\mathcal{T}(\mathcal{S}x_{n_k}), \mathcal{S}(\mathcal{T}x_{n_k}), t) * \psi(\mathcal{M}(\mathcal{T}(\mathcal{T}x_{n_k}), \mathcal{T}z, t))\\
&\to 1 \text{ (as } k \to \infty),
\end{aligned}$$

that is, $\mathcal{S}z = \mathcal{T}z$. Hence $z \in X$ is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$. □

Theorem 3.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings of an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space$(\mathcal{X}, \mathcal{M}, *)$ equipped with a partial order '$\preceq$'. Let the subsequent assumptions hold:*
(a) $\mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$,
(b) *$\mathcal{S}$ is $\mathcal{T}$-comparable,*
(c) *there exists $x_0 \in X$ so that $\mathcal{T}(x_0) \prec\succ \mathcal{S}(x_0),\ \mathcal{M}(\mathcal{T}x_0, \mathcal{S}x_0, t)) > 0$,*
(d) *there exists $\psi \in \mathbf{\Psi}$ so that*

$$\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(\mathcal{T}x, \mathcal{T}y, t)),\ \forall\ x, y \in \mathcal{X} \text{ with } \mathcal{T}(x) \prec\succ \mathcal{T}(y),\ t > 0,$$

(e) *a pair $(\mathcal{S}, \mathcal{T})$ is compatible,*
(f) *$\mathcal{S}$ and $\mathcal{T}$ are continuous mappings,*
or alternately
(f′) *$(\mathcal{X}, \mathcal{M}, *)$ has $\mathcal{T}$-TCC property and $\mathcal{S}$ is continuous.*
Then $\mathcal{S}(x) = \mathcal{T}(x),\ x \in \mathcal{X}$, i.e., the pair $(\mathcal{S}, \mathcal{T})$ has a coincidence point.

Proof. Firstly, we recall Theorem 3.1 above and follow similar pattern till the following contractive condition (3.4) holds :

$$\begin{aligned}
\mathcal{M}(\mathcal{T}x_n, \mathcal{T}x_{n+1}, t) &\geq \psi(\mathcal{M}(\mathcal{T}x_{n-1}, \mathcal{T}x_n, t))\\
&\geq \mathcal{M}(\mathcal{T}x_{n-1}, \mathcal{T}x_n, t),\ n \in \mathbb{N},\ t > 0.
\end{aligned} \tag{3.15}$$

So, $\{\mathcal{M}(\mathcal{T}x_{n-1}, \mathcal{T}x_n, t)\}$ is a nondecreasing sequence of real numbers in $(0, 1]$. In the light of Lemma 2.17, we obtain

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{T}x_{n-1}, \mathcal{T}x_n, t) = 1. \tag{3.16}$$

Next, we assert that $\{\mathcal{T}x_n\}$ is an $\mathcal{M}$-Cauchy sequence. Let $\{\mathcal{T}x_n\}$ be not an $\mathcal{M}$-Cauchy. Then there exists $\epsilon \in (0, 1)$ so that for each $k \in \mathbb{N}$, $m_k > n_k \geq k$, $m_k, n_k \in \mathbb{N}$ and

$$\mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}, t) \leq 1 - \epsilon. \tag{3.17}$$

Let m_k be the least integer exceeding n_k so that

$$\mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_{k-1}}, t) > 1 - \epsilon. \tag{3.18}$$

Then for each k

$$\begin{aligned} 1 - \epsilon &\geq \mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}, t) \\ &\geq \mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k-1}, t) * \mathcal{M}(\mathcal{T}x_{m_k}, \mathcal{T}x_{m_k-1}, t) \\ &\geq (1-\epsilon) * \mathcal{M}(\mathcal{T}x_{m_k}, \mathcal{T}x_{m_k-1}, t). \end{aligned}$$

Letting $k \to \infty$ in the light of (3.16), we obtain

$$\lim_{n\to\infty} \mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}, t) = 1 - \epsilon. \tag{3.19}$$

However, $\mathcal{T}x_{n_k} \prec\succ \mathcal{T}x_{m_k},\ k \in \mathbb{N}$ and $t > 0$, we obtain

$$\begin{aligned} &\mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}, t) \\ &\quad \geq \mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{n_k+1}, t) * \mathcal{M}(\mathcal{T}x_{n_k+1}, \mathcal{T}x_{m_k+1}, t) * \mathcal{M}(\mathcal{T}x_{m_k+1}, \mathcal{T}x_{m_k}, t), \\ &\quad \geq \mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{n_k+1}, t) * \psi(\mathcal{M}(\mathcal{T}x_{n_k}, \mathcal{T}x_{m_k}, t)) * \mathcal{M}(\mathcal{T}x_{m_k+1}, \mathcal{T}x_{m_k}, t). \end{aligned}$$

Letting $k \to \infty$ in the light of (3.16) and (3.19), we have

$$1 - \epsilon \geq 1 * \psi(1-\epsilon) * 1 = \psi(1-\epsilon) > 1 - \epsilon,$$

this is a contradiction. Therefore, $\{\mathcal{T}x_n\}$ is an $\mathcal{M}$-Cauchy sequence. $\mathcal{M}$-completeness of $\mathcal{X}$ implies that

$$\lim_{n\to\infty} \mathcal{T}x_n = z,\ z \in \mathcal{X},$$

which is equivalent to (3.7) of Theorem 3.1. Again tracing back the proof of Theorem 3.1 along with the assumptions (a)- (f') this theorem may be proved. □

Theorem 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings of an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ equipped with a partial order '$\preceq$'. Let the subsequent assumptions hold:*
(a) $\mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$,
(b) *$\mathcal{S}$ is $\mathcal{T}$-comparable,*
(c) *there exists $x_0 \in X$ so that $\mathcal{T}(x_0) \prec\succ \mathcal{S}(x_0),\ \mathcal{M}(\mathcal{T}x_0, \mathcal{S}x_0, t)) > 0$,*
(d) *there exists $\psi \in \mathbf{\Psi}$ so that*

$$\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(\mathcal{T}x, \mathcal{T}y, t)),\ \forall\ x, y \in \mathcal{X} \text{ with } \mathcal{T}(x) \prec\succ \mathcal{T}(y),\ t > 0,$$

(e) *a pair* $(\mathcal{S},\mathcal{T})$ *is compatible,*
(f) $\mathcal{S}$ *and* $\mathcal{T}$ *are reciprocally continuous mappings.*
Then $\mathcal{S}(x)=\mathcal{T}(x),\ x\in\mathcal{X}$*, i.e., the pair* $(\mathcal{S},\mathcal{T})$ *has a coincidence point.*

Proof. On the pattern of Theorem 3.1 and 3.2, we are able to define a sequence $\{x_n\}\subset\mathcal{X}$ so that, we obtain equations (3.7) and (3.8). Therefore,

$$\lim_{n\to\infty}\mathcal{S}(x_n)=\lim_{n\to\infty}\mathcal{T}(x_n)=z.$$

In the light of assumption (f) (that is, if $\mathcal{S}$ and $\mathcal{T}$ are reciprocally continuous), we obtain

$$\lim_{n\to\infty}\mathcal{T}(\mathcal{S}x_n)=Tz,\ \lim_{n\to\infty}\mathcal{S}(\mathcal{T}x_n)=Sz, \tag{3.20}$$

and using assumption (e), we have

$$\lim_{n\to\infty}\mathcal{M}(\mathcal{T}\mathcal{S}x_n,\mathcal{S}\mathcal{T}x_n,t)=1. \tag{3.21}$$

Now, on utilizing (3.20), (3.21), we obtain

$$\begin{aligned}\mathcal{M}(\mathcal{T}z,\mathcal{S}z,t) &= \mathcal{M}(\lim_{n\to\infty}\mathcal{T}\mathcal{S}x_n,\lim_{n\to\infty}\mathcal{S}\mathcal{T}x_n,t)\\ &= \lim_{n\to\infty}\mathcal{M}(\mathcal{T}\mathcal{S}x_n,\mathcal{S}\mathcal{T}x_n,t)\\ &= 1.\end{aligned}$$

Therefore $\mathcal{T}z=\mathcal{S}z$, i.e., z is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$. □

Remark 3.4. We also highlight the fact that Theorems 3.1, 3.2 and 3.3 are valid if we restore the assumption (a) $\mathcal{S}(\mathcal{X})\subseteq\mathcal{T}(\mathcal{X})$ by (a$'$) $\mathcal{S}(\mathcal{X})\subseteq\mathcal{T}(\mathcal{X})\cap\mathcal{Y},\ \mathcal{Y}\subseteq\mathcal{X}$, where the necessity of $\mathcal{X}$ to be an $\mathcal{M}$-complete is relaxed. Alternatively, we need at least, one of the subspace of $\mathcal{X}$ is to be $\mathcal{M}$-complete,i.e., any one from these three subspaces $(\mathcal{S}(\mathcal{X}),\mathcal{M},*)$, $(\mathcal{Y},\mathcal{M},*)$, $(\mathcal{T}(\mathcal{X}),\mathcal{M},*)$ must be $\mathcal{M}$-complete and other assumptions of the theorems remain the same. Then the proof of these newly identified results may be obtained by utilizing the Lemma 2.16 of Haghi et al.[12], in the light of the proof of the Theorem 3.5 presented in [1].

Now, we present uniqueness results related to a coincidence point corresponding to earlier above mentioned results.

Theorem 3.5. *Besides the assumptions* (a)-(f) *together with* (f') *of Theorem 3.1, 3.2 and 3.3, if the subsequent assumption holds :*
(g) $\mathbb{C}(\mathcal{S}x,\mathcal{S}y\prec\succ,\mathcal{T}(\mathcal{X}))$ *is non-empty,* $x,y\in\mathcal{X}$*.*
Then the pair $(\mathcal{S},\mathcal{T})$ *has a unique point of coincidence.*

Proof. In Theorems 3.1, 3.2 and 3.3, suppose that $\bar{x}$ and $\bar{y}$ be two points of coincidences of the mappings $\mathcal{S}$ and $\mathcal{T}$, then there exist $x, y \in \mathcal{X}$, so that

$$\bar{x} = \mathcal{T}(x) = \mathcal{S}(x) \text{ and } \bar{y} = \mathcal{T}(y) = \mathcal{S}(y). \tag{3.22}$$

As $\mathcal{S}(x), \mathcal{S}(y) \in \mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, in the light of assumption (g), we obtain a $\prec\succ$-chain between $\mathcal{S}(x)$ and $\mathcal{S}(y)$ in $\mathcal{T}(\mathcal{X})$ as $\{\mathcal{T}(z_1), \mathcal{T}(z_2), ..., \mathcal{T}(z_{k-1}), \mathcal{T}(z_k)\}$ for $z_1,\ z_2, ..., z_k \in \mathcal{X}$. By using (3.22), we are able to choose $z_1 = x$ and $z_k = y$. Thus, we obtain

$$\mathcal{T}(z_1) \prec\succ \mathcal{T}(z_2) \prec\succ ..... \prec\succ \mathcal{T}(z_{k-1}) \prec\succ \mathcal{T}(z_k). \tag{3.23}$$

Now define $z_n^1 = x$ and $z_n^k = y$ as a constant sequences and utilizing (3.22), we obtain $\mathcal{T}(z_{n+1}^1) = \mathcal{S}(z_n^1) = \bar{x}$ and $\mathcal{T}(z_{n+1}^k) = \mathcal{S}(z_n^k) = \bar{y}$, $\mathbb{N}_0$. Put $z_2 = z_0^2,\ z_3 = z_0^3, ..., z_{k-1} = z_0^{k-1}$. Again as $\mathcal{S}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, recalling Theorem 3.1 above, we are able to construct a sequence $\{z_n^2\},\ \{z_n^3\}, ..., \{z_n^{k-1}\}$ in $\mathcal{X}$, so that $\mathcal{T}(z_{n+1}^2) = \mathcal{S}(z_n^2),\ \mathcal{T}(z_{n+1}^3) = \mathcal{S}(z_n^3), ..., \mathcal{T}(z_{n+1}^{k-1}) = \mathcal{S}(z_n^{k-1}),\ n \in \mathbb{N}_0$. Thus, we obtain

$$\mathcal{T}(z_{n+1}^i) = \mathcal{S}(z_n^i),\ n \in \mathbb{N}_0, \text{and } i\ (1 \leq i \leq k-1). \tag{3.24}$$

We assert that

$$\mathcal{T}(z_n^1) \prec\succ \mathcal{T}(z_n^2) \prec\succ ..... \prec\succ \mathcal{T}(z_n^{k-1}) \prec\succ \mathcal{T}(z_n^k),\ n \in \mathbb{N}_0. \tag{3.25}$$

We determine this assertion by using mathematical induction. In the light of (3.23), (3.25) is valid for $n = 0$. Let (3.25) be true for $n = r > 0$, i.e.,

$$\mathcal{T}(z_r^1) \prec\succ \mathcal{T}(z_r^2) \prec\succ \mathcal{T}(z_r^3) \prec\succ ..... \prec\succ \mathcal{T}(z_r^{k-1}) \prec\succ \mathcal{T}(z_r^k).$$

On utilizing the $\mathcal{T}$-comparability of $\mathcal{S}$, we obtain

$$\mathcal{S}(z_r^1) \prec\succ \mathcal{S}(z_r^2) \prec\succ \mathcal{S}(z_r^3) \prec\succ ..... \prec\succ \mathcal{S}(z_r^{k-1}) \prec\succ \mathcal{S}(z_r^k),$$

by using (3.24), produces

$$\mathcal{T}(z_{r+1}^1) \prec\succ \mathcal{T}(z_{r+1}^2) \prec\succ \mathcal{T}(z_{r+1}^3) \prec\succ ..... \prec\succ \mathcal{T}(z_{r+1}^{k-1}) \prec\succ \mathcal{T}(z_{r+1}^k).$$

It results that (3.25) is valid for $n = r + 1$. Consequently, using induction, (3.25) is true for all $n \in \mathbb{N}_0$.

Now we define

$$\begin{cases} m_n^1 &= \mathcal{M}(\mathcal{T}z_n^1, \mathcal{T}z_n^2, t) \\ m_n^2 &= \mathcal{M}(\mathcal{T}z_n^2, \mathcal{T}z_n^3, t) \\ . \\ . \\ . \\ m_n^{k-2} &= \mathcal{M}(\mathcal{T}z_n^{k-2}, \mathcal{T}z_n^{k-1}, t) \\ m_n^{k-1} &= \mathcal{M}(\mathcal{T}z_n^{k-1}, \mathcal{T}z_n^{k}, t), \ n \in \mathbb{N}_0, \ t > 0. \end{cases}$$

By using (3.25), in the light of contractive assumption (d), we obtain

$$m_{n+1}^i \geq \psi(m_n^i) \ \forall \ n \in \mathbb{N}_0 \ , \text{ for each } t > 0 \text{ and}, i \ (1 \leq i \leq k-1).$$

By applying mathematical induction, we have

$$m_{n+1}^i \geq \psi(m_n^i) \geq \psi^2(m_{n-1}^i) ... \geq \psi^{n+1}(m_0^i)$$

so that

$$m_{n+1}^i \geq \psi^{n+1}(m_0^i).$$

Letting $n \to \infty$ and in the light of the Lemma 2.17, we obtain

$$\lim_{n\to\infty} m_n^i = 1. \tag{3.26}$$

Finally, by using triangular inequality and (3.26), we have

$$\mathcal{M}(\bar{x}, \bar{y}, t) \geq m_n^1 * m_n^2 * ... m_n^{k-1} \to 1.$$

As $n \to \infty$, we have

$$\bar{x} = \bar{y}.$$

Consequently, the pair $(\mathcal{S}, \mathcal{T})$ has a unique point of coincidence. □

Remark 3.6. In Theorems 3.1, 3.2 and 3.3, $\mathcal{S}$ and $\mathcal{T}$ have unique common fixed point if we assume these self-mappings to be weakly compatible. These results are extensions and improvements of Altun and Mehit [3], Mehit [16] and references therein to ordered non-Archimedean fuzzy metric space as ψ-contraction for a pair of mappings holds only on comparable elements of the underlying partial ordering set.

Remark 3.7. Theorems 3.1(together with Hadžić-type norm) and 3.2 are extensions and generalizations of Alam and Imdad [1] to ordered non-Archimedean fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ wherein continuity of both the self mappings is not essential. Further, in Theorem 3.3 continuity of self mappings is completely relaxed by replacing it by its weaker variant, reciprocal continuity.

Also, if we set the identity mapping $\mathcal{I}$, in place of the mapping $\mathcal{T}$, that is, $\mathcal{T} = \mathcal{I}$ in Theorem 3.1 and Theorem 3.2 respectively (along with the Theorem 3.5), then the consequences are as follows :

Corollary 3.8. *Let $\mathcal{S}$ be a self-mapping of an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space with Hadžić-type norm and a partial order '$\preceq$'. Let the subsequent assumptions hold:*
(a) *$\mathcal{S}$ is a comparable mapping,*
(b) *there exists $x_0 \in \mathcal{X}$ so that $x_0 \prec\succ \mathcal{S}(x_0)$, $\lim_{n\to\infty} \psi^n(\mathcal{M}(x_0, \mathcal{S}x_0, t)) = 1$,*
(c) *there exists $\psi \in \mathbf{\Psi}$ so that*

$$\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(x, y, t)),\ \forall\ x, y \in \mathcal{X} \text{ with } x \prec\succ y,\ t > 0,$$

(d) *$\mathcal{S}$ is a continuous mapping or $(\mathcal{X}, \mathcal{M}, *)$ has TCC-property,*
(e) *$\mathbb{C}(\mathcal{S}x, \mathcal{S}y \prec\succ)$ is non-empty.*
Then $\mathcal{S}$ has a unique fixed point.

Corollary 3.9. *Let $\mathcal{S}$ be a self-mapping of an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space with a partial order '$\preceq$'. Let the subsequent assumptions hold:*
(a) *$\mathcal{S}$ is a comparable mapping,*
(b) *there exists $x_0 \in \mathcal{X}$ so that $x_0 \prec\succ \mathcal{S}(x_0)$, $\mathcal{M}(x_0, \mathcal{S}x_0, t) > 0$,*
(c) *there exists $\psi \in \mathbf{\Psi}$ so that*

$$\mathcal{M}(\mathcal{S}x, \mathcal{S}y, t) \geq \psi(\mathcal{M}(x, y, t)),\ \forall\ x, y \in X \text{ with } x \prec\succ y,\ t > 0,$$

(d) *$\mathcal{S}$ is continuous mapping or $(\mathcal{X}, \mathcal{M}, *)$ has TCC-property,*
(e) *$\mathbb{C}(\mathcal{S}x, \mathcal{S}y \prec\succ)$ is non-empty.*
Then the mapping $\mathcal{S}$ has a unique fixed point.

Now, we provide an example to appreciate the hypotheses of Corollaries 3.8 and 3.9, which assure the survival of a unique fixed point.

Example 3.10. Let $\mathcal{X} = \mathbb{R}$ and the order relation be defined by $x \preceq y \iff |x| \leq |y|$ and $xy \geq 0$. Let $a * b = ab$ and

$$\mathcal{M}(x, y, t) = \begin{cases} \frac{min\{x,y\}}{max\{x,y\}}, & \text{if } x > 0,\ t > 0 \\ \frac{max\{x,y\}}{min\{x,y\}}, & \text{if } x \leq 0,\ t > 0. \end{cases} \tag{3.27}$$

Then $(\mathcal{X}, \mathcal{M}, *)$ is an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space, where $\mathcal{M}(x, y, t) > 0$, $t > 0$ holds. Define a self-mapping $\mathcal{S}$ on $\mathcal{X}$ as

$$\mathcal{S}(x) = \begin{cases} 3x+3 & \text{if } x < -1, \\ 1+x & \text{if } -1 \leq x \leq 0, \\ 1-\frac{1}{2}x & \text{if } x > 0. \end{cases} \tag{3.28}$$

Then $\mathcal{S}$ is a comparable (i.e, weakly monotone) continuous mapping but not monotone. For $x_0 = 0$, we have $\mathcal{S}x_0 = 1$, i.e. $0 = x_0 \preceq \mathcal{S}x_0 = 1$, however if we take $x_0 = 1$ we obtain $1 = x_0 \succeq \mathcal{S}x_0 = \frac{1}{2}$. Now we observe that the mapping $\mathcal{S}$ satisfies an ordered fuzzy ψ-contraction condition with $\psi(t) = \sqrt{t}$. If $x, y \in \mathbb{R}$ with $x \preceq y$, then

$$\begin{aligned} \mathcal{M}(x,y,t) &= \frac{max\{\mathcal{S}x, \mathcal{S}y\}}{min\{\mathcal{S}x\mathcal{S}y\}} \\ &= \frac{max\{3x+3, 3y+3\}}{min\{3x+3, 3y+3\}} \\ &= \frac{3x+3}{3y+3} \\ &\geq \sqrt{\frac{x}{y}} \\ &= \psi(\mathcal{M}(x,y,t)). \end{aligned}$$

Hence $\mathcal{S}$ is an ordered fuzzy ψ-contractive mapping. Thus all the assumptions of Corollary 3.8 and 3.9 are verified and $\mathcal{S}$ has a unique fixed point at $x = -\frac{3}{2}$.

In consideration of the above non-trivial example, we ensure that Corollary 3.8 and 3.9 are generalized and sharpened versions of Theorem 2.3 and 2.4 respectively, presented in the paper [3]. We annotate the following considerations :

- In respect to Theorem 2.3 and Theorem 2.4 [3], the nondecreasing (i.e. monotonic) self-mapping $\mathcal{S}$ is replaced by comparable (weakly monotonic) self-mapping which is relatively a weaker assumption.
- The condition (2.3) of the mentioned theorems is replaced by TCC-property introduced by Alam and Imdad [1] which is again relatively weak property.
- In consideration of an additional hypothesis (e) in Theorem 3.5, (that is, $\prec\succ$-chain), we provide the uniqueness of the corresponding results.

An Application

As an application, we present a unique solution for the system of ordered Fredholm integral equations wherein our main results are applicable. Consider the system of integral equations with partial order as follows:

$$\begin{cases} x(\rho) = \phi(\rho) + \int_0^1 \mathcal{K}_1(\rho, \omega, x(\omega))d\omega, \\ y(\rho) = \phi(\rho) + \int_0^1 \mathcal{K}_2(\rho, \omega, y(\omega))d\omega, \end{cases} \tag{3.29}$$

where the function $\phi \in \mathcal{X} = (C[0,1], \mathbb{R})$ and the kernel $\mathcal{K}_1, \mathcal{K}_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$. For $x(\rho), y(\rho) \in \mathcal{X}$, we define the partial order relation as : $x(\rho) \preceq y(\rho) \iff |x(\rho)| \leq |y(\rho)|$, for all $\rho \in [0,1]$. Let $a * b = ab$ and

$$\mathcal{M}(x(\rho), y(\rho), t) = e^{-\frac{d(x,y)}{t}}$$

where, $d(x,y) = ||x(\rho) - y(\rho)||_\infty = \max_{0 \leq \rho \leq 1} |x - y|$ with $t > 0$. Then $(\mathcal{X}, \mathcal{M}, *)$ is an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space, with $\mathcal{M}(x(\rho), y(\rho), t) > 0,\ t > 0$.

Theorem 3.11. *Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings of an $\mathcal{M}$-complete ordered non-Archimedean fuzzy metric space $(\mathcal{X}, \mathcal{M}, *)$ equipped with a partial order '$\preceq$'. Consider the problem (3.29), and let for $x(\rho), y(\rho) \in \mathcal{X}$ with $x(\rho) \preceq y(\rho)$, the subsequent assumptions hold :*

(1) $$\left| \int_0^1 [\{\mathcal{K}_1^2(\rho, \omega, x(\omega)) - \mathcal{K}_1^2(\rho, \omega, y(\omega))\} + 2\phi(\rho)\{\mathcal{K}_1(\rho, \omega, x(\omega)) - \mathcal{K}_1(\rho, \omega, y(\omega))\}] d\omega \right| \leq \left| \int_0^1 \{\mathcal{K}_2(\rho, \omega, x(\omega)) - \mathcal{K}_2(\rho, \omega, y(\omega))\} d\omega \right|,$$

(2) *there exists a sequence $\{x_n(\rho)\}$ in $\mathcal{X}$ satisfying*

$$\lim_{n\to\infty} \mathcal{S}x_n(\rho) = \lim_{n\to\infty} \mathcal{T}x_n(\rho) = z(\rho),\ z(\rho) \in \mathcal{X},$$

so that $\lim_{n\to\infty} \mathcal{T}(\mathcal{S}x_n(\rho)) = Tz(\rho)$, $\lim_{n\to\infty} \mathcal{S}(\mathcal{T}x_n(\rho)) = Sz(\rho)$, *and* $\lim_{n\to\infty} \mathcal{S}\mathcal{T}x_n(\rho) = \lim_{n\to\infty} \mathcal{T}\mathcal{S}x_n(\rho)$,

(3) $\mathcal{S}x(\rho) = \mathcal{T}x(\rho)$ *implies* $\mathcal{T}(\mathcal{S}x(\rho)) = \mathcal{S}(\mathcal{T}x(\rho))$, $x(\rho) \in \mathcal{X}$.

Then then the problem (3.29) has a solution.

Proof. Define $\mathcal{S}$ and $\mathcal{T}$ as

$$\mathcal{S}x(\rho) = \phi(\rho) + \int_0^1 \mathcal{K}_1(\rho, \omega, x(\omega))d\omega$$

$$\text{and } \mathcal{T}y(\rho) = \phi(\rho) + \int_0^1 \mathcal{K}_2(\rho, \omega, y(\omega))d\omega.$$

For $x(\rho), y(\rho) \in \mathcal{X}$, $\mathcal{T}x(\rho) \prec\succ \mathcal{T}y(\rho)$, that is, either $|\mathcal{T}x(\rho)| \leq |\mathcal{T}y(\rho)|$ or $|\mathcal{T}x(\rho)| \geq |\mathcal{T}y(\rho)|$. We consider the first $|\mathcal{T}x(\rho)| \leq |\mathcal{T}y(\rho)|$. Then in the light of the assumption

(a) of the Theorem 3.1, we have $\mathcal{S}x(\rho) \prec\succ \mathcal{S}y(\rho)$, that is, $\mathcal{S}$ is $\mathcal{T}$ comparable. For the contractive condition, using assumption (1), we have

$$\begin{aligned}|\mathcal{S}^2x(\rho) - \mathcal{S}^2y(\rho)| &= \bigg| \int_0^1 [\{\mathcal{K}_1^2(\rho,\omega,x(\omega)) - \mathcal{K}_1^2(\rho,\omega,y(\omega))\} + 2\phi(\rho)\{\mathcal{K}_1(\rho,\omega,x(\omega)) \\ &\qquad - \mathcal{K}_1(\rho,\omega,y(\omega)\}]d\omega\bigg| \\ &\leq \bigg| \int_0^1 \{\mathcal{K}_2(\rho,\omega,x(\omega)) - \mathcal{K}_2(\rho,\omega,y(\omega))\}d\omega\bigg| \\ &= |\mathcal{T}x(\rho) - \mathcal{T}y(\rho)|.\end{aligned}$$

Therefore, for $\rho \in [0,1]$, we obtain

$$e^{-\frac{d(\mathcal{S}^2x,\mathcal{S}^2y)}{t}} \geq e^{-\frac{d(\mathcal{T}x,\mathcal{T}y)}{t}}.$$

Now consider a non-decreasing function $\psi \in \mathbf{\Psi}$ such that $\psi(s) = \sqrt{s}$, we obtain

$$\begin{aligned}\mathcal{M}(\mathcal{S}x(\rho),\mathcal{S}y(\rho),t) = \psi(e^{-\frac{d(\mathcal{S}^2x,\mathcal{S}^2y)}{t}}) &\geq \psi(e^{-\frac{d(\mathcal{T}x,\mathcal{T}y)}{t}}) \\ &= \psi(\mathcal{M}(\mathcal{T}x(\rho),\mathcal{T}y(\rho),t)).\end{aligned}$$

So, for $x(\rho), y(\rho) \in \mathcal{X}$ and $\mathcal{T}x(\rho) \prec\succ \mathcal{T}y(\rho)$, we obtain

$$\mathcal{M}(\mathcal{S}x(\rho),\mathcal{S}y(\rho),t) \geq \psi(\mathcal{M}(\mathcal{T}x(\rho),\mathcal{T}x(\rho),t)).$$

Using assumption (2), the other assumptions of the Theorem 3.1, 3.2, and 3.3 may be easily verified. Moreover, if the assumption (3) holds, i.e, if the pair $(\mathcal{S},\mathcal{T})$ is weakly compatible, then the system of ordered Fredholm integral equations (3.29) have a solution. □

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