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EMPLOYING GENERALIZED (ψ, θ, φ)-CONTRACTION ON PARTIALLY ORDERED FUZZY METRIC SPACES WITH APPLICATIONS

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Abstract. We establish fixed point and multidimensional fixed point results satisfying generalized (ψ, θ, φ)-contraction on partially ordered non-Archimedean fuzzy metric spaces. By using this result we obtain the solution for periodic boundary value problems and give an example to show the degree of validity of our hypothesis. Our results generalize, extend and modify several well-known results in the literature.

1. Introduction

In [21], Shaddad et al. study the existence and uniqueness of fixed points for complete partially ordered metric spaces, which extends the main results of Harjani and Sadarangani [13], Nieto and Rodríguez-López [17] and Ran and Reurings [18]. They also establish coupled fixed point theorems, which extend and generalize the results of Harjani et al. [14], Bhaskar and Lakshmikantham [3] and Luong and Thuan [16]. Some of our basic references are [4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 20].

In this paper, we prove a fixed point theorem for $G-$non-decreasing mappings satisfying generalized (ψ, θ, φ)-contraction on partially ordered non-Archimedean fuzzy metric spaces. By using this result, we obtain the solution for periodic boundary value problems and give an example to show the degree of validity of our hypothesis. In the process, some multidimensional fixed point results are derived from our main results. We improve and generalize the results of Alotaibi and Alsulami [1], Alsulami [2], Harjani and Sadarangani [13], Harjani et al. [14], Luong and Thuan [16], Nieto and Rodriguez-Lopez [17], Razani and Parvaneh [19] and many other results in the literature.

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2. Fixed Point Results

In the sequel, X is a non-empty set and $G : X \to X$ is a mapping. For simplicity, we denote $G(x)$ by Gx where $x \in X$.

Definition 2.1 ([21])**.** An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfied the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose $T, G : X \to X$ are two mappings such that the following properties are fulfilled:*

(i) $T(X) \subseteq G(X)$,

(ii) T is $(G, \preceq)-$non-decreasing,

(iii) there exists $x_0 \in X$ such that $Gx_0 \preceq Tx_0$,

(iv) there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \leq \theta\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right),$$

for all $x, y \in X$ with $Gx \preceq Gy$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Also assume that, at least, one of the following conditions holds.

(a) (X, M) is complete, T and G are continuous and the pair (T, G) is compatible,

(b) (X, M) is complete, T and G are continuous and commuting,

(c) $(G(X), M)$ is complete and $(X, M, \preceq)$ is non-decreasing-regular,

(d) (X, M) is complete, $G(X)$ is closed and $(X, M, \preceq)$ is non-decreasing-regular,

(e) (X, M) is complete, G is continuous and monotone non-decreasing, the pair (T, G) is compatible and $(X, M, \preceq)$ is non-decreasing-regular.

Then T and G have a coincidence point. Moreover, if for every $x, y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty, and also the pair (T, G) is weakly compatible. Then T and G have a unique common fixed point.

Proof. We divide the proof into five steps.

Step 1. We claim that there exists a sequence $\{x_n\}_{n\geq 0} \subseteq X$ such that $\{Gx_n\}$ is $\preceq$-non-decreasing and $Gx_{n+1} = Tx_n$, for all $n \geq 0$. Let $x_0 \in X$ be arbitrary and since

$Tx_0 \in T(X) \subseteq G(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = Gx_1$. Then $Gx_0 \preceq Tx_0 = Gx_1$, as T is $(G, \preceq)$-non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq G(X)$, so there exists $x_2 \in X$ such that $Tx_1 = Gx_2$. Then $Gx_1 = Tx_0 \preceq Tx_1 = Gx_2$. Since T is $(G, \preceq)$-non-decreasing, $Tx_1 \preceq Tx_2$. Continuing this process, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{Gx_n\}$ is $\preceq$-non-decreasing, $Gx_{n+1} = Tx_n \preceq Tx_{n+1} = Gx_{n+2}$ and

$$Gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{2.1}$$

Step 2. We claim that $\{M(Gx_n, Gx_{n+1}, t)\} \to 1$. By contractive condition (iv) and by the monotonicity of ψ, we have

$$\begin{aligned} &\psi\left(\frac{1}{M(Gx_{n+1},\ Gx_{n+2},\ t)} - 1\right) \\ =\ &\psi\left(\frac{1}{M(Tx_n,\ Tx_{n+1},\ t)} - 1\right) \\ \leq\ &\theta\left(\frac{1}{M(Gx_n,\ Gx_{n+1},\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx_n,\ Gx_{n+1},\ t)} - 1\right), \end{aligned} \tag{2.2}$$

but we have $\psi(\delta_n) - \theta(\delta_n) + \varphi(\delta_n) > 0$, where $\delta_n = \frac{1}{M(Gx_n,\ Gx_{n+1},\ t)} - 1$. Then

$$\frac{\psi(\delta_{n+1})}{\psi(\delta_n)} \leq \frac{\theta(\delta_n) - \varphi(\delta_n)}{\psi(\delta_n)} < 1.$$

Therefore we take

$$\psi(\delta_{n+1}) < \psi(\delta_n). \tag{2.3}$$

Since ψ is non-decreasing, we obtain

$$\delta_{n+1} < \delta_n. \tag{2.4}$$

Thus the sequence $\{\delta_n\}_{n\geq 0}$ is a decreasing sequence of positive numbers. Hence, there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}\left(\frac{1}{M(Gx_n,\ Gx_{n+1},\ t)} - 1\right) = \delta. \tag{2.5}$$

We claim that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Taking $n \to \infty$ in (2.2), by using the property of ψ, θ, φ and (2.5), we obtain

$$\psi(\delta) \leq \theta(\delta) - \varphi(\delta) \Rightarrow \psi(\delta) - \theta(\delta) + \varphi(\delta) \leq 0,$$

which is a contradiction. Thus, by (2.5), we get

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}\left(\frac{1}{M(Gx_n,\ Gx_{n+1},\ t)} - 1\right) = 0, \tag{2.6}$$

or

$$\lim_{n\to\infty} M(Gx_n,\ Gx_{n+1},\ t) = 1. \tag{2.7}$$

STEP 3. We now claim that $\{Gx_n\}_{n\geq 0}$ is a Cauchy sequence in X. If possible, suppose that $\{Gx_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and

$$M(Gx_{n(k)},\ Gx_{m(k)},\ t) \leq 1-\varepsilon \text{ for } n(k) > m(k) > k. \tag{2.8}$$

Let $n(k)$ be the smallest such positive integer, we get

$$M(Gx_{n(k)-1},\ Gx_{m(k)},\ t) > 1-\varepsilon. \tag{2.9}$$

Now, by (2.8) and (2.9), we have

$$\begin{aligned} 1-\varepsilon &\geq r_k = M(Gx_{n(k)},\ Gx_{m(k)},\ t) \\ &\geq M(Gx_{n(k)},\ Gx_{n(k)-1},\ t) * M(Gx_{n(k)-1},\ Gx_{m(k)},\ t) \\ &> M(Gx_{n(k)},\ Gx_{n(k)-1},\ t) * (1-\varepsilon). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (2.7), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} M(Gx_{n(k)},\ Gx_{m(k)},\ t) = 1-\varepsilon. \tag{2.10}$$

By (NAFM-4), we have

$$\begin{aligned} & M(Gx_{n(k)+1},\ Gx_{m(k)+1},\ t) \\ \geq\ & M(Gx_{n(k)+1},\ Gx_{n(k)},\ t) * M(Gx_{n(k)},\ Gx_{m(k)},\ t) * M(Gx_{m(k)},\ Gx_{m(k)+1},\ t). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (2.7) and (2.10), we have

$$\lim_{k\to\infty} M(Gx_{n(k)+1},\ Gx_{m(k)+1},\ t) = 1-\varepsilon. \tag{2.11}$$

Since $n(k) > m(k)$, $x_{n(k)} \succeq x_{m(k)}$, therefore by using contractive condition (iv), we have

$$\begin{aligned} & \psi\left(\frac{1}{M(Gx_{n(k)+1},\ Gx_{m(k)+1},\ t)} - 1\right) \\ =\ & \psi\left(\frac{1}{M(Tx_{n(k)},\ Tx_{m(k)},\ t)} - 1\right) \\ \leq\ & \theta\left(\frac{1}{M(Gx_{n(k)},\ Gx_{m(k)},\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx_{n(k)},\ Gx_{m(k)},\ t)} - 1\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (2.10), (2.11), we have

$$\psi\left(\frac{\varepsilon}{1-\varepsilon}\right) \leq \theta\left(\frac{\varepsilon}{1-\varepsilon}\right) - \varphi\left(\frac{\varepsilon}{1-\varepsilon}\right),$$

which is a contradiction due to $\varepsilon > 0$. Thus $\{Gx_n\}_{n\geq 0}$ is a Cauchy sequence in X.

STEP 4. We claim that T and G have a coincidence point distinguishing between cases $(a) - (e)$.

Suppose now that (a) holds, that is, $(X,\ M)$ is complete, T and G are continuous and the pair $(T,\ G)$ is compatible. Since $(X,\ M)$ is complete, therefore there exists $z \in X$ such that $\{Gx_n\} \to z$. Since $Tx_n = Gx_{n+1}$ for all $n \geq 0$, therefore $\{Tx_n\} \to z$. As T and G are continuous, so $\{TGx_n\} \to Tz$ and $\{GGx_n\} \to Gz$. Since the pair $(T,\ G)$ is compatible, therefore we conclude that

$$M(Gz,\ Tz,\ t) = \lim_{n\to\infty} M(GGx_{n+1},\ TGx_n,\ t) = \lim_{n\to\infty} M(GTx_n,\ TGx_n,\ t) = 1,$$

that is, z is a coincidence point of T and G.

Suppose now that (b) holds, that is, $(X,\ M)$ is complete, T and G are continuous and commuting. Thus (a) is applicable.

Suppose now that (c) holds, that is, $(G(X),\ M)$ is complete and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Now, since $\{Gx_n\}$ is a Cauchy sequence in the complete space $(G(X),\ M)$. Therefore there exist $y \in G(X)$ such that $\{Gx_n\} \to y$. Let $z \in X$ be any point such that $y = Gz$, then $\{Gx_n\} \to Gz$. Since $(X,\ M,\ \preceq)$ is non-decreasing-regular and $\{Gx_n\}$ is $\preceq$ $-$non-decreasing and converging to Gz, we obtain that $Gx_n \preceq Gz$ for all $n \geq 0$. Using the contractive condition (iv), we have

$$\begin{aligned}\psi\left(\frac{1}{M(Gx_{n+1},\ Tz,\ t)} - 1\right) &= \psi\left(\frac{1}{M(Tx_n,\ Tz,\ t)} - 1\right)\\ &\leq \theta\left(\frac{1}{M(Gx_n,\ Gz,\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx_n,\ Gz,\ t)} - 1\right).\end{aligned}$$

Taking $n \to \infty$ in the above inequality, we get $M(Gz,\ Tz,\ t) = 1$, that is, z is a coincidence point of T and G.

Suppose now that (d) holds, that is, $(X,\ M)$ is complete, $G(X)$ is closed and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since a closed subset of a complete metric space is also complete. Therefore, $(G(X),\ M)$ is complete and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Thus (c) is applicable.

Suppose now that (e) holds, that is, $(X,\ M)$ is complete, G is continuous and monotone non-decreasing, the pair $(T,\ G)$ is compatible and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since $(X,\ M)$ is complete, therefore there exists $z \in X$ such that $\{Gx_n\} \to z$. As $Tx_n = Gx_{n+1}$ for all $n \geq 0$ and so $\{Tx_n\} \to z$. Also G is continuous, then $\{GGx_n\} \to Gz$. Furthermore, since the pair (T, G) is compatible and $\{GGx_n\} \to Gz$, it follows that $\{TGx_n\} \to Gz$.

Again, since $(X, M, \preceq)$ is non-decreasing-regular and $\{Gx_n\}$ is $\preceq$-non-decreasing and converging to z, we obtain that $Gx_n \preceq z$ for all $n \geq 0$, which, by the monotonicity of G, implies $GGx_n \preceq Gz$. Applying the contractive condition (iv), we get

$$\begin{aligned} &\psi\left(\frac{1}{M(TGx_n,\ Tz,\ t)} - 1\right) \\ \leq\ &\theta\left(\frac{1}{M(GGx_n,\ Gz,\ t)} - 1\right) - \varphi\left(\frac{1}{M(GGx_n,\ Gz,\ t)} - 1\right). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, we get $M(Gz, Tz, t) = 1$, that is, z is a coincidence point of T and G.

STEP 5. As the set of coincidence points of G and T is non-empty, so suppose that x and y are coincidence points of T and G, that is, $Tx = Gx$ and $Ty = Gy$. Now, we claim that $Gx = Gy$. Since, there exists $u \in X$ such that Tu is comparable with Tx and Ty. Put $u_0 = u$ and choose $u_1 \in X$ so that $Gu_0 = Tu_1$. Then, we can inductively define sequences $\{Gu_n\}$ where $Gu_{n+1} = Tu_n$ for all $n \geq 0$. Hence $Tx = Gx$ and $Tu = Tu_0 = Gu_1$ are comparable. Suppose that $Gu_1 \preceq Gx$ (the proof is similar to that in the other case). We claim that $Gu_n \preceq Gx$ for each $n \in \mathbb{N}$. In fact, we will use mathematical induction. Since $Gu_1 \preceq Gx$, our claim is true for $n = 1$.

We assume that $Gu_n \preceq Gx$ holds for some $n > 1$. Since T is G-nondecreasing with respect to $\preceq$, we get $Gu_{n+1} = Tu_n \preceq Tx = Gx$ and this proves our claim. Since $Gu_n \preceq Gx$, by (2.1) and (iv), we have

$$\begin{aligned} (2.12)\qquad &\psi\left(\frac{1}{M(Gx,\ Gu_{n+1},\ t)} - 1\right) \\ =\ &\psi\left(\frac{1}{M(Tx,\ Tu_n,\ t)} - 1\right) \\ \leq\ &\theta\left(\frac{1}{M(Gx,\ Gu_n,\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx,\ Gu_n,\ t)} - 1\right), \end{aligned}$$

but we have $\psi(\Delta_n) - \theta(\Delta_n) + \varphi(\Delta_n) > 0$ where $\Delta_n = \frac{1}{M(Gx,\, Gu_n,\, t)} - 1$. Then

$$\frac{\psi(\Delta_{n+1})}{\psi(\Delta_n)} \leq \frac{\theta(\Delta_n) - \varphi(\Delta_n)}{\psi(\Delta_n)} < 1.$$

Thus

$$\psi\left(\Delta_{n+1}\right) \leq \left(\Delta_n\right).$$

Since ψ is non-decreasing, therefore

$$\Delta_{n+1} < \Delta_n.$$

This shows that the sequence $\{\Delta_n\}_{n\geq 0}$ defined by

$$\Delta_n = \frac{1}{M(Gx,\ Gu_n,\ t)} - 1,$$

is a decreasing sequence of positive numbers. Then there exists $\Delta \geq 0$ such that

$$\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty}\left(\frac{1}{M(Gx,\ Gu_n,\ t)} - 1\right) = \Delta. \tag{2.13}$$

We claim that $\Delta = 0$. Suppose to the contrary that $\Delta > 0$. Taking $n \to \infty$ in (2.12), by using the property of ψ, θ, φ and (2.13), we obtain

$$\psi(\Delta) \leq \theta(\Delta) - \varphi(\Delta) \Rightarrow \psi(\Delta) - \theta(\Delta) + \varphi(\Delta) \leq 0,$$

which is a contradiction. Thus, by (2.13), we get

$$\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty}\left(\frac{1}{M(Gx,\ Gu_n,\ t)} - 1\right) = 0.$$

It follows that $\lim_{n\to\infty} M(Gx,\ Gu_n,\ t) = 1$. Similarly, one can prove that

$$\lim_{n\to\infty} M(Gy, Gu_n, t) = 1.$$

Hence, we get $Gx = Gy$. Since $Tx = Gx$, therefore by weak compatibility of T and G, we have $TGx = GTx = GGx$. Let $z = Gx$, then $Tz = Gz$. Thus z is a coincidence point of T and G. Then $y = z$, it follows that $Gx = Gz$, that is, $Tz = Gz = z$. Therefore, z is a common fixed point of T and G. To prove the uniqueness, assume that w is another common fixed point of T and G. Then, we have $w = Gw = Gz = z$. Hence the common fixed point of T and G is unique. □

Take $\psi(t) = t$ and $\varphi(t) = 0$ for all $t \geq 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Suppose $T,\ G : X \to X$ are two mappings satisfying $(i) - (iii)$ of Theorem 2.1 and there exists an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq \theta\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right),$$

for all $x,\ y \in X$ such that $Gx \preceq Gy$, where $\theta(0) = 0$ and $t - \theta(t) > 0$ for all $t > 0$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 2.1 holds. Then T and G have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty and also the pair $(T,\ G)$ is weakly compatible. Then T and G have a unique common fixed point.

Take $\varphi(t) = 0$ and $\theta(t) = k\psi(t)$ with $0 \leq k < 1$, for all $t \geq 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Suppose $T,\ G : X \to X$ are two mappings satisfying $(i) - (iii)$ of Theorem 2.1 and there exists an altering distance function ψ such that*

$$\psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \leq k\psi\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right),$$

for all $x,\ y \in X$ such that $Gx \preceq Gy$, where $0 \leq k < 1$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 2.1 holds. Then T and G have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty and also the pair $(T,\ G)$ is weakly compatible. Then T and G have a unique common fixed point.

If we take $\psi(t) = \theta(t)$ for all $t \geq 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Suppose $T,\ G : X \to X$ are two mappings satisfying $(i) - (iii)$ of Theorem 2.1 and there exist an altering distance function ψ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} &\psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \\ \leq\ &\psi\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right) - \varphi\left(\frac{1}{M(Gx,\ Gy,\ t)} - 1\right), \end{aligned}$$

for all $x,\ y \in X$ such that $Gx \preceq Gy$, where $\varphi(0) = 0$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 2.1 holds. Then T and G have a coincidence point. Moreover, if for every $x,\ y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty and also the pair $(T,\ G)$ is weakly compatible. Then T and G have a unique common fixed point.

If we take $\psi(t) = \theta(t) = t$ for all $t \geq 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Suppose $T, G : X \to X$ are two mappings satisfying $(i) - (iii)$ of Theorem 2.1 and there exists a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that*

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \left(\frac{1}{M(Gx, Gy, t)} - 1\right) - \varphi\left(\frac{1}{M(Gx, Gy, t)} - 1\right),$$

for all $x, y \in X$ such that $Gx \preceq Gy$, where $\varphi(0) = 0$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 2.1 holds. Then T and G have a coincidence point. Moreover, if for every $x, y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty and also the pair (T, G) is weakly compatible. Then T and G have a unique common fixed point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ for all $t \geq 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Suppose $T, G : X \to X$ are two mappings satisfying $(i) - (iii)$ of Theorem 2.1 such that*

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(Gx, Gy, t)} - 1\right),$$

for all $x, y \in X$ such that $Gx \preceq Gy$, where $k < 1$. Also assume that, at least, one of the conditions $(a) - (e)$ of Theorem 2.1 holds. Then T and G have a coincidence point. Moreover, if for every $x, y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty and also the pair (T, G) is weakly compatible. Then T and G have a unique common fixed point.

Example 2.1. Suppose that $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$ and $*$ is defined by $a * b = ab$, for all $a, b \in [0, 1]$. Define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for all } x, y \in X \text{ and } t > 0.$$

Clearly $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Let $T,\ G: X \to X$ be defined as

$$Tx = \frac{x^2}{3} \text{ and } Gx = x^2 \text{ for all } x \in X.$$

Let $\psi(t) = \theta(t) = t$ and $\varphi(t) = \frac{2t}{3}$ for $t \geq 0$. Clearly, T and G satisfied the contractive condition of Theorem 2.1. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = 0$ is a unique common fixed point of T and G.

3. Coupled Fixed Point Results

Next, we deduce the two dimensional version of Theorem 2.1. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). For the ordered fuzzy metric space $(X,\ M,\ \preceq)$, let us consider the ordered fuzzy metric space $(X^2,\ M_\delta,\ \sqsubseteq)$, where $M_\delta : X^2 \times X^2 \times [0,\ \infty) \to [0,\ 1]$ defined by

$$M_\delta(Y,\ V,\ t) = \min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\},\ \forall Y = (x,\ y),\ V = (u,\ v) \in X^2.$$

It is easy to check that M_δ is a non-Archimedean fuzzy metric on X^2. Moreover, $(X,\ M,\ *)$ is complete if and only if $(X^2,\ M_\delta,\ *)$ is complete and $\sqsubseteq$ was introduced in

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v,\ \text{ for all } (u,\ v),\ (x,\ y) \in X^2.$$

We define the mapping $T_F,\ T_G : X^2 \to X^2$, for all $(x,\ y) \in X^2$, by

$$T_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } T_G(x,\ y) = (Gx,\ Gy).$$

Under these conditions, the following properties hold.

Lemma 3.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Let $F : X^2 \to X$ and $G : X \to X$ be two mappings. Then*

(1) $(X,\ M)$ is complete if and only if $(X^2,\ M_\delta)$ is complete.

(2) If $(X,\ M,\ \preceq)$ is regular, then $(X^2,\ M_\delta,\ \sqsubseteq)$ is also regular.

(3) If F is $M-$continuous, then T_F is M_δ-continuous.

(4) F has the mixed monotone property with respect to $\preceq$ if and only if T_F is $\sqsubseteq$-non-decreasing.

(5) F has the mixed $G-$monotone property with respect to $\preceq$ if and only if then T_F is $(T_G,\ \sqsubseteq)$-non-decreasing.

(6) If there exist two elements $x_0, y_0 \in X$ *with* $Gx_0 \preceq F(x_0, y_0)$ *and* $Gy_0 \succeq F(y_0, x_0)$, *then there exists a point* $(x_0, y_0) \in X^2$ *such that* $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(7) If $F(X^2) \subseteq G(X)$, *then* $T_F(X^2) \subseteq T_G(X^2)$.

(8) If F *and* G *are commuting in* $(X, M, \preceq)$, *then* T_F *and* T_G *are also commuting in* $(X^2, M_\delta, \sqsubseteq)$.

(9) If F *and* G *are compatible in* $(X, M, \preceq)$, *then* T_F *and* T_G *are also compatible in* $(X^2, M_\delta, \sqsubseteq)$.

(10) If F *and* G *are weak compatible in* $(X, M, \preceq)$, *then* T_F *and* T_G *are also weak compatible in* $(X^2, M_\delta, \sqsubseteq)$.

(11) A point $(x, y) \in X^2$ *is a coupled coincidence point of* F *and* G *if and only if it is a coincidence point of* T_F *and* T_G.

(12) $(x, y) \in X^2$ *is a coupled fixed point of* F *if and only if it is a fixed point of* T_F.

Proof. Items (1), (2), (3), (4), (5), (6), (7), (11) and (12) are obvious.

(8) Let $(x, y) \in X^2$. Since G and F are commutative, by the definition of T_G and T_F, we have $T_G T_F(x, y) = T_G(F(x, y), F(y, x)) = (GF(x, y), GF(y, x)) = (F(Gx, Gy), F(Gy, Gx)) = T_F(Gx, Gy) = T_F T_G(x, y)$, which shows that T_G and T_F are commutative.

(9) Let $\{(x_n, y_n)\} \subseteq X^2$ be any sequence such that $T_F(x_n, y_n) \overset{M_\delta}{\to} (x, y)$ and $T_G(x_n, y_n) \overset{M_\delta}{\to} (x, y)$. Therefore,

$$(F(x_n,\ y_n),\ F(y_n,\ x_n)) \overset{M_\delta}{\to} (x,\ y) \Rightarrow F(x_n,\ y_n) \overset{M}{\to} x \text{ and } F(y_n,\ x_n) \overset{M}{\to} y,$$

and

$$(Gx_n,\ Gy_n) \overset{M_\delta}{\to} (x,\ y) \Rightarrow Gx_n \overset{M}{\to} x \text{ and } Gy_n \overset{M}{\to} y.$$

Thus

$$\begin{aligned}\lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} Gx_n = x \in X,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} Gy_n = y \in X.\end{aligned}$$

Since the pair $\{F, G\}$ is compatible, we have

$$\begin{aligned}\lim_{n\to\infty} M(F(Gx_n,\ Gy_n),\ GF(x_n,\ y_n),\ t) &= 1,\\ \lim_{n\to\infty} M(F(Gy_n,\ Gx_n),\ GF(y_n,\ x_n),\ t) &= 1.\end{aligned}$$

In particular, we have

$$\begin{aligned}
& \lim_{n\to\infty} M_\delta(T_G T_F(x_n,\ y_n),\ T_F T_G(x_n,\ y_n),\ t) \\
= & \lim_{n\to\infty} M_\delta(T_G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ T_F(Gx_n,\ Gy_n),\ t) \\
= & \lim_{n\to\infty} M_\delta\left((GF(x_n,\ y_n),\ GF(y_n,\ x_n)),\ (F(Gx_n,\ Gy_n),\ F(Gy_n,\ Gx_n)),\ t\right) \\
= & \lim_{n\to\infty} \min\left\{ \begin{array}{c} M(GF(x_n,\ y_n),\ F(Gx_n,\ Gy_n),\ t), \\ M(GF(y_n,\ x_n),\ F(Gy_n,\ Gx_n),\ t) \end{array} \right\} \\
= & \ 1.
\end{aligned}$$

Hence, the mappings T_F and T_G are compatible in $(X^2,\ M_\delta,\ \sqsubseteq)$.

(10) Let $(x,\ y) \in X^2$ be a coincidence point T_G and T_F. Then $T_G(x,\ y) = T_F(x,\ y)$, that is, $(Gx,\ Gy) = (F(x,\ y),\ F(y,\ x))$, that is, $Gx = F(x,\ y)$ and $Gy = F(y,\ x)$. Since G and F are weak compatible, by the definition of T_G and T_F, we have $T_G T_F(x,\ y) = T_G(F(x,\ y),\ F(y,\ x)) = (GF(x,\ y),\ GF(y,\ x)) = (F(Gx,\ Gy),\ F(Gy,\ Gx)) = T_F(Gx,\ Gy) = T_F T_G(x, y)$, which shows that T_G and T_F commute at their coincidence point, that is, T_G and T_F are weak compatible. □

Theorem 3.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $G : X \to X$ are two mappings such that F has the mixed $G-$monotone property with respect to $\preceq$ on X for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying*

$$\begin{aligned}
& \psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\
\leq & \ \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right) \\
& -\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right),
\end{aligned} \tag{3.1}$$

for all $x,\ y,\ u,\ v \in X$, with $Gx \preceq Gu$ and $Gy \succeq Gv$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Suppose that $F(X^2) \subseteq G(X)$, G is continuous and monotone non-decreasing and the pair $\{F,\ G\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X,\ M,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ *with*

$$Gx_0 \preceq F(x_0,\ y_0) \text{ and } Gy_0 \succeq F(y_0,\ x_0).$$

Then F *and* G *have a coupled coincidence point. In addition, suppose that for every* $(x,\ y),\ (x^*, y^*) \in X^2$*, there exists a* $(u,\ v) \in X^2$ *such that* $(F(u,\ v),\ F(v,\ u))$ *is comparable to* $(F(x,\ y),\ F(y,\ x))$ *and* $(F(x^*,\ y^*),\ F(y^*,\ x^*))$*, and also the pair* $(F,\ G)$ *is weakly compatible. Then* F *and* G *have a unique coupled common fixed point, that is, there exists a unique* $(x,\ y) \in X^2$ *such that* $x = Gx = F(x,\ y)$ *and* $y = Gy = F(y,\ x)$*.*

Proof. Let $x,\ y,\ u,\ v \in X$, with $Gx \preceq Gu$ and $Gy \succeq Gv$. Then, by using (3.1), we have

$$\begin{aligned} & \psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\ \leq\ & \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right) \\ & -\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right). \end{aligned}$$

Furthermore taking into account that $Gy \succeq Gv$ and $Gx \preceq Gu$, (3.1) also guarantees that

$$\begin{aligned} & \psi\left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)} - 1\right) \\ \leq\ & \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right) \\ & -\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right),\ \psi\left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)} - 1\right)\right\} \\ \leq\ & \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right) \\ & -\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}} - 1\right). \end{aligned}$$

Since ψ is non-decreasing, we take

(3.2)

$$\psi\left(\max\left\{\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right),\ \left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)}-1\right)\right\}\right)$$
$$\leq \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}}-1\right)$$
$$-\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}}-1\right).$$

Thus, it follows from (3.2) that

$$\psi\left(\frac{1}{M_\delta(T_F(x,\ y),\ T_F(u,\ v),\ t)}-1\right)$$
$$=\psi\left(\frac{1}{\min\{M(F(x,\ y),\ F(u,\ v),\ t),\ M(F(y,\ x),\ F(v,\ u),\ t)\}}-1\right)$$
$$=\psi\left(\max\left\{\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right),\ \left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)}-1\right)\right\}\right)$$
$$\leq \theta\left(\frac{1}{\min\{M(Gx,\ Gu,\ t), M(Gy,\ Gv,\ t)\}}-1\right)$$
$$-\varphi\left(\frac{1}{\min\{M(Gx,\ Gu,\ t), M(Gy,\ Gv,\ t)\}}-1\right)$$
$$\leq \theta\left(\frac{1}{M_\delta(T_G(x,\ y),\ T_G(u,\ v),\ t)}-1\right)-\varphi\left(\frac{1}{M_\delta(T_G(x,\ y),\ T_G(u,\ v),\ t)}-1\right).$$

It is only need to apply Theorem 3.1 to the mappings $T = T_F$ and $G = T_G$ in the partially ordered metric space $(X^2,\ M_\delta,\ \sqsubseteq)$ with the help of Lemma 3.1. □

Corollary 3.2. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $G : X \to X$ are two mappings such that F has the mixed $G-$monotone property with respect to $\preceq$ on X for which there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying (3.1), for all $x,\ y,\ u,\ v \in X$, with $Gx \preceq Gu$ and $Gy \succeq Gv$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Suppose that $F(X^2) \subseteq G(X)$, G is continuous and monotone non-decreasing and the pair $\{F,\ G\}$ is commuting. Also suppose that either*

(a) F is continuous or

(b) $(X,\ M,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$Gx_0 \preceq F(x_0,\ y_0) \text{ and } Gy_0 \succeq F(y_0,\ x_0).$$

Then F and G have a coupled coincidence point. In addition, suppose that for every $(x,\ y)$, $(x^*, y^*) \in X^2$, there exists a $(u,\ v) \in X^2$ such that $(F(u,\ v),\ F(v,\ u))$ is comparable to $(F(x,\ y),\ F(y,\ x))$ and $(F(x^*,\ y^*),\ F(y^*,\ x^*))$, and also the pair $(F,\ G)$ is weakly compatible. Then F and G have a unique coupled common fixed point, that is, there exists a unique $(x,\ y) \in X^2$ such that $x = Gx = F(x,\ y)$ and $y = Gy = F(y,\ x)$.

Corollary 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that*

$$\begin{aligned} & \psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\ \leq\ & \theta\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}} - 1\right) \\ & -\varphi\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}} - 1\right). \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$, with $x \preceq u$ and $y \succeq v$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Also suppose that either

(a) F is continuous or

(b) $(X,\ M,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

In a similar way, we may state the results analog of Corollary 2.2, Corollary 2.3, Corollary 2.4, Corollary 2.5 and Corollary 2.6 for Theorem 3.1, Corollary 3.2 and Corollary 3.3.

4. Application to Ordinary Differential Equations

In this section, we study the existence of a solution for the following first-order periodic problem:

$$\text{(4.1)} \qquad \begin{cases} u'(t) = f(t,\ u(t),\ u(t)),\ t \in [0,\ T], \\ u(0) = u(T), \end{cases}$$

where $T > 0$ and $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Evidently the space $X = C(I, \mathbb{R})$ $(I = [0, T])$ of all continuous functions from I to $\mathbb{R}$ is a complete metric space with respect to the sup metric

$$d(x, y) = \sup_{t \in I} |x(t) - y(t)|, \text{ for all } x, y \in X.$$

Define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for all } x, y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a complete fuzzy metric space with $a * b = \min\{a, b\}$ for all a, $b \in [0, 1]$. Also X equipped with the following partial order:

$$x \preceq y \Longleftrightarrow x(t) \le y(t), \text{ for all } t \in I \text{ and for all } x, y \in X. \tag{4.2}$$

Definition 4.1. A coupled lower-upper solution for (4.1) is a function $(p, q) \in C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ such that

$$\begin{aligned} p'(t) &\le f(t, p(t), q(t)) \text{ and } q'(t) \ge f(t, q(t), p(t)) \text{ for } t \in I, \\ p(0) &= p(T) = q(0) = q(T) = 0. \end{aligned}$$

Theorem 4.1. *Consider problem* (4.1) *with $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous and for $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$,*

$$0 \le f(t, x, y) + \lambda x - f(t, u, v) - \lambda u \le \frac{\lambda}{6}\left((x - u) + (y - v)\right),$$

Then the existence of a coupled upper-lower solution of (4.1) *provides the existence of a solution of* (4.1).

Proof. (4.1) *reduces to the following integral equation*

$$u(t) = \int_0^T G(t, s)[f(s, u(s), u(s)) + \lambda u(s)]ds,$$

where $G(t, s)$ is the Green function given by

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s < t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases}$$

Define $F : X^2 \to X$ by

$$F(x, y)(t) = \int_0^T G(t, s)[f(s, x(s), y(s)) + \lambda x(s)]ds.$$

If $x_1 \succeq x_2$, then by using our assumption, we have

$$f(t, x_1, y) + \lambda x_1 \ge f(t, x_2, y) + \lambda x_2.$$

Since $G(t,\ s) > 0$, *that for* $t \in I$, *it implies*

$$\begin{aligned} F(x_1,\ y)(t) &= \int_0^T G(t,\ s)[f(s,\ x_1(s),\ y(s)) + \lambda x_1(s)]ds \\ &\geq \int_0^T G(t,\ s)[f(s,\ x_2(s),\ y(s)) + \lambda x_2(s)]ds \\ &= F(x_2,\ y)(t). \end{aligned}$$

Also, if $y_1 \succeq y_2$, *then by using our assumption, we have*

$$f(t,\ x,\ y_1) \leq f(t,\ x,\ y_2).$$

Since $G(t,\ s) > 0$, *that for* $t \in I$, *it implies*

$$\begin{aligned} F(x,\ y_1)(t) &= \int_0^T G(t,\ s)[f(s,\ x(s),\ y_1(s)) + \lambda x(s)]ds \\ &\leq \int_0^T G(t,\ s)[f(s,\ x(s),\ y_2(s)) + \lambda x(s)]ds \\ &= F(x,\ y_2)(t). \end{aligned}$$

Therefore F *has mixed monotone property. Now, for* $x \succeq y$ *and* $y \preceq v$, *we have*

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ &= \sup_{t\in I} |F(x,\ y)(t) - F(u,\ v)(t)| \\ &= \sup_{t\in I} \left| \int_0^T G(t,\ s)[f(s,\ x(s),\ y(s)) + \lambda x(s) - f(s,\ u(s),\ v(s)) - \lambda u(s)]ds \right| \\ &\leq \sup_{t\in I} \left| \int_0^T G(t,\ s) \cdot \frac{\lambda}{6} \left((x(s) - u(s)) + (y(s) - v(s)) \right) ds \right| \\ &\leq \frac{\lambda}{6}(d(x,\ u) + d(y,\ v)) \sup_{t\in I} \left| \int_0^T G(t,\ s)ds \right| \\ &\leq \frac{\lambda}{6}(d(x,\ u) + d(y,\ v)) \sup_{t\in I} \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right| \\ &\leq \frac{d(x,\ u) + d(y,\ v)}{6}. \end{aligned}$$

Thus

$$\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1 \leq \frac{1}{3} \left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}} - 1 \right).$$

Put $\psi(t) = \theta(t) = t$ *and* $\varphi(t) = \frac{2t}{3}$ *for* $t \geq 0$. *Obviously* ψ *is an altering distance function,* $\psi(t)$, $\theta(t)$ *and* $\varphi(t)$ *satisfy the condition of* $\psi(t) - \theta(t) + \varphi(t) > 0$ *for all* $t > 0$. *Thus for* $x \succeq u$ *and* $y \preceq v$, *we get*

$$\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \ \leq\ \theta\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}} - 1\right) -\varphi\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}} - 1\right).$$

Finally, assume that $(p,\ q) \in X^2$ *be a coupled upper-lower solution of* (4.1), *then*

$$p'(s) + \lambda p(s) \leq f(s,\ p(s),\ q(s)) + \lambda p(s),\ \textit{for } t \in I.$$

Multiplying by $G(t,\ s)$, *we get*

$$\int_0^T p'(s)G(t,\ s)ds + \lambda\int_0^T p(s)G(t,\ s)ds \leq F(p,\ q)(t),\ \textit{for } t \in I.$$

Then, for all for $t \in I$, *we have*

$$\int_0^t p'(s)\frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}ds + \int_t^T p'(s)\frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}ds + \lambda\int_0^T p(s)G(t,\ s)ds \leq F(p,\ q)(t).$$

Using an integration by parts and since $p(0) = p(T) = 0$, *for all* $t \in I$, *we get*

$$p(t) \leq F(p,\ q)(t).$$

This implies that $p \preceq F(p,\ q)$. *Similarly, one can show that* $q \succeq F(q,\ p)$. *Thus hypothesis of Corollary 3.3 holds. Consequently,* F *has a coupled fixed point* $(x,\ y) \in X^2$ *which is the solution to* (4.1) *in* $X = C(I,\ \mathbb{R})$.

Now, we study the existence and uniqueness of solution to the two-point boundary value problem.

$$\text{(4.3)} \qquad \begin{cases} -x''(t) = f(t,\ x(t),\ x(t)),\ x \in (0,\ +\infty),\ t \in [0,\ 1], \\ x(0) = x(1) = 0. \end{cases}$$

where $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The space $X = C(I,\ \mathbb{R})$ $(I = [0,\ 1])$ denote the set of all continuous functions from I to $\mathbb{R}$.

Theorem 4.2. *Under the assumptions*

(*a*) $f : [0,\ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(*b*) *Suppose that there exists* $0 \leq \gamma \leq 6$ *such that for all* $t \in I$, $x \succeq u$ *and* $y \preceq v$,

$$0 \leq f(t,\ x,\ y) - f(t,\ u,\ v) \leq \frac{\gamma}{6}(\zeta(x-u) + \zeta(y-v)),$$

where $\zeta(t) : [0, +\infty) \to [0, +\infty)$ *is a right upper semi-continuous and non-decreasing function with* $\zeta(0) = 0$, $\zeta(t) \leq t$, *for all* $t > 0$.

(*c*) *There exists* $(\alpha,\ \beta) \in C^2(I,\ \mathbb{R}) \times C^2(I,\ \mathbb{R})$ *solution to*

$$\left\{ \begin{array}{c} -p''(t) \leq f(t,\ p(t),\ q(t)),\ t \in [0,\ 1], \\ -q''(t) \geq f(t,\ q(t),\ p(t)),\ t \in [0,\ 1], \\ p(0) = p(1) = q(0) = q(1) = 0. \end{array} \right. \tag{4.4}$$

Then (4.3) *has one and only one solution in* $C^2(I,\ \mathbb{R})$.

Proof. Clearly the solution (in $C^2(I,\ \mathbb{R})$) of (4.3) is equivalent to the solution (in $C(I,\ \mathbb{R})$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ x(s))ds \text{ for } t \in [0,\ 1],$$

where $G(t,\ s)$ is the Green function of differential operator $-\frac{d^2}{dt^2}$ with Dirichlet boundary condition $x(0) = x(1) = 0$, that is,

$$G(t,\ s) = \left\{ \begin{array}{l} t(1-s),\ 0 \leq t \leq s \leq 1, \\ s(1-t),\ 0 \leq s \leq t \leq 1. \end{array} \right. \tag{4.5}$$

Define $F : X^2 \to X$ by

$$F(x,\ y)(t) = \int_0^1 G(t,\ s) f(s,\ x(s),\ y(s))ds,\ t \in [0,\ 1] \text{ and } x,\ y \in X.$$

From (b), F has the mixed monotone property with respect to $\preceq$ in X. Let x, y, u, $v \in X$ such that $x \succeq u$ and $y \preceq v$. From (b), we have

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ = & \sup_{t \in I} |F(x,\ y)(t) - F(u,\ v)(t)| \\ = & \sup_{t \in I} \int_0^1 G(t,\ s)[f(s,\ x(s),\ y(s)) - f(s,\ u(s),\ v(s))]ds \\ \leq & \frac{\gamma}{6} \sup_{t \in I} \int_0^1 G(t,\ s) \cdot (\zeta(x(s) - u(s)) + \zeta(y(s) - v(s)))ds \\ \leq & \frac{\gamma}{3} \left(\frac{\zeta(d(x,\ u)) + \zeta(d(y,\ v))}{2} \right) \sup_{t \in I} \int_0^1 G(t,\ s)ds. \end{aligned}$$

Now, since G is non-decreasing, we have

$$\begin{aligned}\zeta(d(x,\ u)) &\leq \zeta(d(x,\ u)+d(y,\ v)),\\ \zeta(d(y,\ v)) &\leq \zeta(d(x,\ u)+d(y,\ v)),\end{aligned}$$

which implies

$$\frac{\zeta(d(x,\ u))+\zeta(d(y,\ v))}{2} \leq \zeta(d(x,\ u)+d(y,\ v)).$$

Therefore, we take

$$\begin{aligned}&d(F(x,\ y),\ F(u,\ v))\\ \leq\ &\frac{\gamma}{3}(\zeta(d(x,\ u)+d(y,\ v)))\sup_{t\in I}\int_0^1 G(t,\ s)ds.\end{aligned} \tag{4.6}$$

It is evident that

$$\int_0^1 G(t,\ s)ds = -\frac{t^2}{2}+\frac{t}{2},$$

and

$$\sup_{t\in[0,\ 1]}\int_0^1 G(t,\ s)ds = \frac{1}{8}.$$

Thus the inequality (4.6) and $0<\gamma\leq 6$ gives

$$\begin{aligned}d(F(x,\ y),\ F(u,\ v)) &\leq \frac{\gamma}{24}(\zeta(d(x,\ u)+d(y,\ v)))\\ &\leq \frac{1}{4}(\zeta(d(x,\ u)+d(y,\ v)))\\ &\leq \frac{d(x,\ u)+d(y,\ v)}{4}.\end{aligned}$$

Thus

$$\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1 \leq \frac{1}{2}\left(\frac{1}{\min\{M(Gx,\ Gu,\ t),\ M(Gy,\ Gv,\ t)\}}-1\right).$$

Put $\psi(t)=\theta(t)=t$ and $\varphi(t)=\frac{t}{2}$ for $t\geq 0$. It is evident that ψ is an altering distance function, $\psi(t)$, $\theta(t)$ and $\varphi(t)$ satisfy the condition of $\psi(t)-\theta(t)+\varphi(t)>0$ for all $t>0$. From the above inequality, for $x\succeq u$ and $y\preceq v$, we obtain

$$\begin{aligned}&\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right)\\ \leq\ &\theta\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}}-1\right)\\ &-\varphi\left(\frac{1}{\min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\}}-1\right).\end{aligned}$$

which is the contractive condition of Corollary 3.3. Assume $(p, q) \in C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R})$ be a solution to (4.3). Then

$$-p''(s) \leq f(s,\ p(s),\ q(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t,\ s)$, we get

$$\int_0^1 -p''(s)G(t,\ s)ds \leq F(p,\ q)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0,\ 1]$, we have

$$-(1-t)\int_0^t sp''(s)ds - t\int_t^1 (1-s)p''(s)ds \leq F(p,\ q)(t).$$

Since $p(0) = p(1) = 0$, for all $t \in [0,\ 1]$, we get

$$-(1-t)(tp'(t) - p(t)) - t(-(1-t)p'(t) - p(t)) \leq F(p,\ q)(t).$$

Thus, we have

$$p(t) \preceq F(p,\ q)(t),\ \text{for } t \in [0,\ 1].$$

It means that $p \preceq F(p, q)$. Similarly, one can prove that $q \succeq F(q, p)$. Thus hypothesis of Corollary 3.3 holds. Consequently, F has a coupled fixed point $(x,\ y) \in X^2$ which is the solution to (4.3) in $X = C(I,\ \mathbb{R})$. □

Remark. Applying the same techniques, it is possible to find tripled, quadruple and in general, multidimensional coincidence point theorems from Theorem 2.1.

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