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WEIERSTRASS SEMIGROUPS AT PAIRS OF NON-WEIERSTRASS POINTS ON A SMOOTH PLANE CURVE OF DEGREE 5

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Abstract. We classify all semigroups each of which arises as a Weierstrass semigroup at a pair of non-Weierstrass points on a smooth plane curve of degree 5. First we find the candidates of semigroups by computing the dimensions of linear series on the curve. Then, by constructing examples of smooth plane curves of degree 5, we prove that each of the candidates is actually a Weierstrass semigroup at some pair of points on the curve. We need to study the systems of quadratic curves, which cut out the canonical series on the plane curve of degree 5.

1. Introduction and Preliminaries

Let C be a smooth projective curve of genus $g \geq 2$ over the complex field $\mathbb{C}$, $\mathcal{M}(C)$ the field of meromorphic functions on C and $\mathbb{N}_0$ the set of all nonnegative integers. For two distinct points P, $Q \in C$, we define the Weierstrass semigroup $H(P) \subseteq \mathbb{N}_0$ at a point and the Weierstrass semigroup at a pair of points $H(P,Q) \subseteq \mathbb{N}_0^2$ by

$$\begin{aligned} H(P) &= \{\alpha \in \mathbb{N}_0 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}, \\ H(P,Q) &= \{(\alpha,\beta) \in \mathbb{N}_0^2 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}, \end{aligned}$$

where $(f)_\infty$ means the divisor of poles of $f \in \mathcal{M}(C)$. Indeed, $H(P)$ and $H(P,Q)$ form sub-semigroups of $\mathbb{N}_0$ and $\mathbb{N}_0^2$, respectively.

The cardinality of the set $G(P) = \mathbb{N}_0 \setminus H(P)$ is exactly g. We call P a Weierstrass point if $G(P) \neq \{1, 2, \ldots, g\}$.

The set $G(P,Q) = \mathbb{N}_0^2 \setminus H(P,Q)$ is also finite, but its cardinality is dependent on the points P and Q. In [5], the upper and lower bound of cardinality of $G(P,Q)$ are given as $\binom{g+2}{2} - 1 \leq \text{card } G(P,Q) \leq \binom{g+2}{2} - 1 - g + g^2$.

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We review some basic facts concerning the Weierstrass semigroups at a pair of points on a curve. ([4], [5]).

Lemma 1.1 ([5])**.** *For each* $\alpha \in G(P)$*, let* $\beta_\alpha = \min\{\beta \mid (\alpha, \beta) \in H(P,Q)\}$*. Then* $\alpha = \min\{\gamma \mid (\gamma, \beta_\alpha) \in H(P,Q)\}$*. Moreover, we have*

$$\{\beta_\alpha \mid \alpha \in G(P)\} = G(Q).$$

The above lemma shows that the set $H(P,Q)$ defines a bijective mapping $\sigma = \sigma(P,Q)$ from $G(P)$ to $G(Q)$ which is defined by $\alpha \mapsto \beta_\alpha$. Homma [4] obtained the formula for the cardinality of $G(P,Q)$ using the cardinality of the set of pairs (α, α') which are reversed by σ. We use the following notations;

$$\begin{aligned} \Gamma = \Gamma(P,Q) &:= \{(\alpha, \beta_\alpha) \mid \alpha \in G(P)\} \\ &= \{(p_i, q_{\sigma(i)}) \mid i = 1, 2, \ldots, g\}, \\ \widetilde{\Gamma} = \widetilde{\Gamma}(P,Q) &:= \Gamma(P,Q) \cup \big(H(P) \times \{0\}\big) \cup \big(\{0\} \times H(Q)\big). \end{aligned}$$

The above set $\Gamma(P,Q)$ is called the *generating subset* of the Weierstrass semigroup $H(P,Q)$. For given distinct two points P, Q, the set $\Gamma(P,Q)$ determines not only $\widetilde{\Gamma}(P,Q)$ but also the sets $H(P,Q)$ and $G(P,Q)$ completely, as described in the lemma below. To state the lemma we use the natural partial order on the set $\mathbb{N}_0^2$ defined as

$$(\alpha, \beta) \geq (\gamma, \delta) \text{ if and only if } \alpha \geq \gamma \text{ and } \beta \geq \delta,$$

and the least upper bound of two elements (α_1, β_1), (α_2, β_2) is defined as

$$\text{lub}\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} = (\max\{\alpha_1, \alpha_2\},\ \max\{\beta_1, \beta_2\}).$$

Lemma 1.2 ([5, 6])**.** (1) *The subset* $H(P,Q)$ *of* $\mathbb{N}_0^2$ *is closed under the* lub(*least upper bound*) *operation.* (2) *Every element of* $H(P,Q)$ *is expressed as the* lub *of one or two elements of the set* $\widetilde{\Gamma}(P,Q)$*.* (3) *The set* $G(P,Q) = \mathbb{N}_0^2 \setminus H(P,Q)$ *is expressed as*

$$G(P,Q) = \bigcup_{l \in G(P)} \big(\{(l, \beta) | \beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l)) | \alpha = 0, 1, \ldots, l - 1\}\big).$$

We can characterize the elements of $\Gamma(P,Q)$ and $H(P,Q)$ using the dimensions of divisors. We denote $\dim(\alpha, \beta)$ the dimension of the complete linear series $|\alpha P + \beta Q|$.

Lemma 1.3 ([5])**.** *Let* (α, β) *be an element in* $\mathbb{N}_0^2$ *with* $\beta \geq 1$ [*resp.* $\alpha \geq 1$]*. Then*

$$\dim(\alpha, \beta) = \dim(\alpha, \beta - 1) + 1 [\textit{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1]$$

if and only if there exists $(\gamma, \beta) \in \widetilde{\Gamma}$ *[resp.* $(\alpha, \delta) \in \widetilde{\Gamma}$ *] with* $0 \leq \gamma \leq \alpha$ *[resp.* $0 \leq \delta \leq \beta$ *].*

Lemma 1.4. *For* $\alpha \geq 1$ *and* $\beta \geq 1$*, the pair* (α, β) *is an element of* $\Gamma(P, Q)$ *[resp.* $H(P, Q)$*] if and only if*

$$\begin{aligned} \dim(\alpha, \beta) &= \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1 \\ &= \dim(\alpha - 1, \beta - 1) + 1 \end{aligned}$$

$$[\text{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1].$$

Proof. By Lemma 1.3, since $(\alpha, \beta) \in \Gamma(P, Q)$ implies that there is no element (α', β) [resp. (α, β')] $\in H(P, Q)$ with $0 \leq \alpha' \leq \alpha$ [resp. $0 \leq \beta' \leq \beta$], the lemma holds. □

The following two theorems are well-known.

Theorem 1.5 (Riemann-Roch Theorem)**.** *Let* C *be a nonsingular curve of genus* g*, and* P, Q *points on* C*. Then*

$$\dim(\alpha P + \beta Q) = \alpha + \beta - g + h^0(K - (\alpha P + \beta Q)),$$

where K *is the canonical series on the curve* C*.*

Theorem 1.6. *The canonical series on a nonsingular curve of degree* $d \geq 4$ *is cut out by the system of curves of degree* $d - 3$*.*

Theorem 1.7 ([2, 3])**.** *Let* C *be a smooth plane curve of degree* $d \geq 4$*. For* $e \in \mathbb{N}$*, there is no base point free pencil* g^1_e *if and only if* $(n - 1)d + 1 \leq e \leq nd - (n^2 + 1)$ *for some* $n \in \mathbb{N}$ *with* $1 \leq n \leq \sqrt{d - 2}$*.*

In Section 2, we find all candidates of the Weierstrass semigroup at a pair of non-Weierstrass points on a smooth plane curve of degree 5. In Section 3, we prove the existence of curves and points having such semigroups as their Weierstrass semigroups.

To construct smooth plane curves satisfying our condition, we use the following theorems frequently.

Theorem 1.8 ([8, Bertini's Theorem])**.** *The generic element of a linear system is smooth away from the base locus of the system.*

Theorem 1.9 ([7, Namba's Lemma])**.** *Let C, C_1 and C_2 be plane curves. If P is a nonsingular point of C, then we have*

$$I(C_1 \cap C_2; P) \geq \min\{I(C \cap C_1; P), I(C \cap C_2; P)\}.$$

Theorem 1.10 ([1, Bezout's Theorem])**.** *Let C_m and C_n be smooth plane curves of degree m and n, respectively. If they have no common component, then we have*

$$\sum_{P \in C_m \cap C_n} I(C_m \cap C_n; P) = mn.$$

2. A Weierstrass Semigroup at a Pair of Non-Weierstrass Points

In this section, we let C be a smooth plane curve of degree 5, hence of genus 6. For a point $P \in C$, let L_P denote the tangent line to C at P, and M_p denote the quadratic curve which meets C at P with the highest multiplicity. Since the dimension of the system of quadratic curves is five, we have $M_P \centerdot C \geq 5P$. For a curve F, $I(F \cap C; P)$ denotes the intersection multiplicity of F and C at P, and $F \centerdot C = \sum_{P \in F \cap C} I(F \cap C; P)P$ denotes the divisor on C cut out by F.

We have the following lemma.

Lemma 2.1. *Let P be a non-Weierstrass point on a smooth plane curve C of degree 5, i.e., $G(P) = \{1, 2, \ldots, 6\}$. Then there is no quadratic curve which meets the curve C with multiplicity ≥ 6, and hence $I(L_P \cap C; P) = 2$, $I(M_P \cap C; P) = 5$, and M_P is a conic* (*an irreducible quadratic curve*).

Proof. Obvious. □

From now on, we mean P and Q two distinct non-Weierstrass points on the given curve C. By Lemma 2.1,

$$I(L_P \cap C; P) = I(L_Q \cap C; Q) = 2,$$

and M_P and M_Q are conics with

$$I(M_P \cap C; P) = I(M_Q \cap C; Q) = 5.$$

For two tangent lines L_P and L_Q one of the following holds;

I. $L_P = L_Q$.

II. $L_P \cap L_Q = \{Q\}$. [or $L_P \cap L_Q = \{P\}$.]

III. $L_P \cap L_Q = \{R\}$ with $R \neq P, Q$.

For M_P and M_Q, we may let $M_P \,.\, C = 5P + aQ + D_1$, $M_Q \,.\, C = bP + 5Q + D_2$, where $0 \leq a, b \leq 5$ and the divisors D_1 and D_2 are disjoint from $\{P, Q\}$. If $a + b \geq 5$ then $M_P = M_Q$ and $a = b = 5$ since $M_P \,.\, M_Q \geq bP + aQ$. Thus, if $M_P \neq M_Q$, then $a + b \leq 4$.

We find the possible Weierstrass semigroups of the pair (P, Q) case by case. First, we compute the $\dim(a, b) := \dim(aP + bQ)$ for each $(a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Lemma 2.2. *We have the following:*

(1) $\dim(a, 0) = 0$ *for any* a *with* $0 \leq a \leq 6$ *and* $\dim(0, b) = 0$ *for any* b *with* $0 \leq b \leq 6$.

(2) $\dim(a, b) = 0$ *for any* (a, b) *with* $a + b \leq 3$.

Proof. Since P and Q are non-Weierstrass point, (1) is obvious. By Theorem 1.7, (2) holds. □

As corollaries of Lemma 1.4, we obtain the following two lemmas which we use frequently to find an element of Γ. Recall that (α, β) is said to be special if there exists a canonical divisor $\geq \alpha P + \beta Q$, which is equivalent to the fact that there is a quadratic curve M such that $M \,.\, C \geq \alpha P + \beta Q$.

Lemma 2.3. *If* $(\alpha - 1, \beta - 1)$ *is a maximal special element, then* $(\alpha, \beta) \in \Gamma$*.*

Proof. Since $(\alpha, \beta - 1)$, $(\alpha - 1, \beta)$ and (α, β) are nonspecial, by Riemann-Roch Theorem and Lemma 1.4, the lemma holds. □

Lemma 2.4. *If* $\dim(\alpha - 1, \beta) = 0 = \dim(\alpha, \beta - 1)$ *and* $\dim(\alpha, \beta) = 1$*, then* $(\alpha, \beta) \in \Gamma$*.*

Proof. Since $\dim(\alpha - 1, \beta - 1) = 1$, it is obvious by Lemma 1.4. □

For a curve F, we use the notation $r(F.C) = F.C - (I(F \cap C; P)P + I(F \cap C; Q)Q)$, the remaining divisor of degree $(\deg F)(\deg C) - (I(F \cap C; P) + I(F \cap C; Q))$, with the support disjoint from $\{P, Q\}$.

Case I. $L_P = L_Q$ Then $L_P \,.\, C = 2P + 2Q + r(L_P \,.\, C)$. Note that $r(L_P \,.\, C)$ is a point distinct from P and Q. In this case, by Bezout's Theorem, we have

$$M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C),$$

$$M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C).$$

Using these, we prove the theorem below.

Theorem 2.5. *We have*

$$\Gamma = \{(1,6),(2,2),(3,4),(4,3),(5,5),(6,1)\}. \tag{1}$$

Proof. Since there is no quadratic curve cut out the divisor $6Q$ or $P+5Q$, by Riemann-Roch Theorem, we have $\dim(0,6) = 0 = \dim(1,5)$. By Lemma 1.4 and 2.2, we have $(1,6) \in \Gamma$. Similarly, we can prove $(6,1) \in \Gamma$.

Since $L_P = L_Q$ cut out $2P+2Q$, any quadratic curve passing through $2P+2Q$ contains the line L_P as a component, by Bezout's Theorem. Thus the dimension of quadratic curves passing through $2P+2Q$ is equal to the dimension of the systems of all lines, which is 3. By Riemann-Roch Theorem, we have $\dim(2,2) = 1$. By Lemma 2.2, $(2,2) \in \Gamma$.

Any quadratic curves passing through $(2+c)P + (2+d)Q$, $c+d \geq 2$, $0 \leq c,d \leq 2$ is unique. By Riemann-Roch Theorem, we have $\dim(2,4) = 1$, $\dim(3,3) = 1$, $\dim(3,4) = 2$. On the other hand, the dimension of quadratic curves passing through $2P+3Q$ is equal to that of lines passing through Q, by Bezout's Theorem. By Riemann-Roch Theorem, $\dim(2,3) = 1$. By Lemma 1.4, $(3,4) \in \Gamma$. Similarly, we can prove $(4,3) \in \Gamma$.

L_P^2 is the unique quadratic curve cut out $4P+4Q$. Thus there is no quadratic curve passing through $4P+5Q$ or $5P+4Q$. By Riemann-Roch Theorem, $\dim(4,4) = \dim(4,5) = \dim(5,4) = 3$ and $\dim(5,5) = 4$. By Lemma 1.4, $(5,5) \in \Gamma$. □

Case II. $L_P \cap L_Q = \{Q\}$

Then $L_P \centerdot C = 2P + Q + r(L_P \centerdot C)$ and $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$. In this case, by Bezout's Theorem, we have

$$M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C),$$

$$M_Q \centerdot C = aP + 5Q + r(M_Q \centerdot C) \text{ for } a = 0,1.$$

Theorem 2.6. *We obtain*

$$\Gamma = \{(1,5),(2,6),(3,4),(4,2),(5,3),(6,1)\} \textit{ for } a = 1; \tag{2}$$

and

$$\Gamma = \{(1,6),(2,5),(3,4),(4,2),(5,3),(6,1)\} \textit{ for } a = 0. \tag{3}$$

Proof. Let $a = 1$, i.e. $M_Q \cdot C = P + 5Q + r(M_Q \cdot C)$. Then, by Riemann-Roch Theorem, $\dim(1, 5) = 1$. Since M_Q is irreducible, $\dim(0, 5) = 1 = \dim(1, 4)$, by Bezout's Theorem. Thus $(1, 5) \in \Gamma$ by Lemma 1.4. Since $2P + 5Q$ and $P = 6Q$ are nonspecial (i.e., there is no quadratic polynomial passing through such divisors), we have $(2, 6) \in \Gamma$. The quadratic curves L_P^2 is the unique quadratic curve passing through $3P + 2Q$ or $4P + Q$ or $4P + 2Q$. Thus $\dim(4, 2) = 1$ and $(4, 2) \in \Gamma$. Since $5P+2Q$ and $4P+3Q$ are nonspecial, we have $(5, 3) \in \Gamma$. Since $3P+4Q$ and $6P+Q$ are nonspecial and of dimension 1, we get $(3, 4)$, $(6, 1) \in \Gamma$.

If $a = 0$, then $M_Q \cdot C = 0P+5Q+r(M_Q \cdot C)$. Then $P+5Q$ and $6Q$ are nonspecial and $\dim(1, 6) = 1$ and hence $(1, 6) \in \Gamma$. Similarly we can prove $(2, 5) \in \Gamma$. The other elements are obtained similarly as in (2) □

Case III. $L_P \cap L_Q = \{R\}$ with $R \neq P, Q$

We have $L_P \cdot C = 2P + 0Q + r(L_P \cdot C)$ and $L_Q \cdot C = 0P + 2Q + r(L_Q \cdot C)$. In this case, we have

$$M_P \cdot C = 5P + bQ + r(M_P \cdot C) \text{ for } 0 \leq b \leq 5,$$

$$M_P \cdot C = aP + 5Q + r(M_P \cdot C) \text{ for } 0 \leq a \leq 5.$$

If $a + b \geq 5$, then $M_P = M_Q$ by Bezout's Theorem. Thus, if $M_P \neq M_Q$, then $a + b \leq 4$.

Subcase III-1. $M_P \cdot C = 5P + 5Q$, i.e., $M_P = M_Q$ and $r(M_P \cdot C) = 0$

Theorem 2.7. *We have*

$$\Gamma = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\}. \tag{4}$$

Proof. For any $a = 1, \ldots, 5$, M_P is the unique quadratic curve passing through $aP+(6-a)Q$, hence $\dim(a, 6-a) = 1$. By Bezout's Theorem, M_P is also the unique quadratic curve passing through $(a-1)P+(6-a)Q$ [resp. $aP+(6-a-1)Q$]. Thus $\dim(a-1, 6-a) = 0 = \dim(a, 6-a-1)$. By Lemma 2.4, we have $(a, 6-a) \in \Gamma$ for all $a = 1, \ldots, 5$.

By Lemma 2.3, we get $(6, 6) \in \Gamma$. □

Subcase III-2. $M_P \cdot C = 5P + 4Q + r(M_P \cdot C)$

Then $M_Q \cdot C = 0P + 5Q + r(M_P \cdot C)$.

Theorem 2.8. *We have*

$$\Gamma = \{(1,6),(2,4),(3,3),(4,2),(5,1),(6,5)\}. \tag{5}$$

Proof. By Lemma 2.4, we have $(a, 6-a) \in \Gamma$ for all $a = 2, \dots, 5$.
By Lemma 2.3, we have $(1,6)$, $(6,5) \in \Gamma$. □

Subcase III-3. $M_P \cdot C = 5P + 3Q + r(M_P \cdot C)$ and $M_Q \cdot C = 1P + 5Q + r(M_Q \cdot C)$

Theorem 2.9. *We have*

$$\Gamma = \{(1,5),(2,6),(3,3),(4,2),(5,1),(6,4)\}. \tag{6}$$

Proof. The pairs $(1,5),(3,3),(4,2),(5,1)$ are elements of Γ by Lemma 2.4. The pairs $(2,6)$ and $(6,4)$ are elements of Γ by Lemma 2.3. □

Subcase III-4. $M_P \cdot C = 5P + 3Q + r(M_P \cdot C)$ and $M_Q \cdot C = 0P + 5Q + r(M_Q \cdot C)$

Theorem 2.10. *We have*

$$\Gamma = \{(1,6),(2,5),(3,3),(4,2),(5,1),(6,4)\}. \tag{7}$$

Proof. Using Riemann-Roch Theorem and Lemma 2.4, the pairs $(1,6)$, $(2,5)$, $(3,3)$, $(4,2)$, $(5,1)$ are elements of Γ. The pair $(6,4)$ is an element of Γ by Lemma 2.3. □

Subcase III-5. $M_P \cdot C = 5P + 2Q + r(M_P \cdot C)$ and $M_Q \cdot C = 2P + 5Q + r(M_Q \cdot C)$

Theorem 2.11. *We have*

$$\Gamma = \{(1,5),(2,4),(3,6),(4,2),(5,1),(6,3)\}. \tag{8}$$

Proof. Using Lemma 2.4, the pairs $(1,5)$, $(2,4)$, $(4,2)$, $(5,1)$ are elements of Γ. The pair $(3,6)$, $(6,3)$ is an element of Γ by Lemma 2.3. □

Subcase III-6. $M_P \cdot C = 5P + 2Q + r(M_P \cdot C)$ and $M_Q \cdot C = P + 5Q + r(M_Q \cdot C)$

Theorem 2.12. *We have*

$$\Gamma = \{(1,5),(2,6),(3,4),(4,2),(5,1),(6,3)\}. \tag{9}$$

Proof. Using Lemma 2.4, the pairs $(1,5),(3,4),(4,2),(5,1)$ are elements of Γ. The pair $(2,6)$ and $(6,3)$ are elements of Γ by Lemma 2.3. □

Subcase III-7. $M_P \cdot C = 5P + 2Q + r(M_P \cdot C)$ and $M_Q \cdot C = 0P + 5Q + r(M_Q \cdot C)$

Theorem 2.13. *We obtain two* Γ*'s. If* $(2,4)$ *is nonspecial.*

$$\Gamma = \{(1,6),(2,5),(3,4),(4,2),(5,1),(6,3)\}. \tag{10}$$

If $(2,4)$ *is special.*

$$\Gamma = \{(1,6),(2,4),(3,5),(4,2),(5,1),(6,3)\}. \tag{11}$$

Proof. Let $(2,4)$ be nonspecial. Using Riemann-Roch Theorem and Lemma 2.4, the pairs $(1,6),(2,5),(3,4),(4,2),(5,1)$ are elements of Γ. The pair $(6,3)$ is an element of Γ by Lemma 2.3.

Let $(2,4)$ be special. Using Riemann-Roch Theorem and Lemma 2.4, the pairs $(1,6),(2,4),(4,2),(5,1)$ are elements of Γ. The pair $(3,5),(6,3)$ is an element of Γ by Lemma 2.3. □

Subcase III-8. $M_P \cdot C = 5P + Q + r(M_P \cdot C)$ and $M_Q \cdot C = P + 5Q + r(M_Q \cdot C)$

Theorem 2.14. *We obtain two* Γ*'s. If* $(3,3)$ *is nonspecial,*

$$\Gamma = \{(1,5),(2,6),(3,4),(4,3),(5,1),(6,2)\}. \tag{12}$$

If $(3,3)$ *is special,*

$$\Gamma = \{(1,5),(2,6),(3,3),(4,4),(5,1),(6,2)\}. \tag{13}$$

Proof. Let $(3,3)$ be nonspecial. Using Riemann-Roch Theorem and Lemma 2.4, the pairs $(1,5),(3,4),(4,3),(5,1)$ are elements of Γ. The pairs $(2,6),(6,2)$ are elements of Γ by Lemma 2.3.

Let $(3,3)$ be special. Using Riemann-Roch Theorem and Lemma 2.4, the pairs $(1,5),(3,3),(5,1)$ are elements of Γ. The pairs $(2,6),(4,4),(6,2)$ are elements of Γ by Lemma 2.3. □

Subcase III-9. $M_P \cdot C = 5P + Q + r(M_P \cdot C)$ and $M_Q \cdot C = 0P + 5Q + r(M_Q \cdot C)$

Theorem 2.15. *In this case,* Γ *is one of the following 4 sets:*

$$\Gamma = \{(1,6),(2,4),(3,3),(4,5),(5,1),(6,2)\}. \tag{14}$$

$$\Gamma = \{(1,6),(2,5),(3,3),(4,4),(5,1),(6,2)\}. \tag{15}$$

$$\Gamma = \{(1,6),(2,4),(3,5),(4,3),(5,1),(6,2)\}. \tag{16}$$

$$\Gamma = \{(1,6),(2,5),(3,4),(4,3),(5,1),(6,2)\}. \tag{17}$$

Proof. Lemma 2.3 implies $(1,6),\ (6,2) \in \Gamma$, and Lemma 2.4 implies $(5,1) \in \Gamma$.

If $(3,4)$ is special, then it should be maximal special. Lemma 2.3 implies that $(4,5) \in \Gamma$, Lemma 2.4 implies that $(2,4),\ (3,3) \in \Gamma$. Thus we get the set (14).

Suppose that $(3,4)$ is nonspecial. If $(3,3)$ is special, then $(4,4) \in \Gamma$ by Lemma 2.3, and $(3,3) \in \Gamma$ by Lemma 2.4. By Lemma 1.1, we have $(2,5) \in \Gamma$, and hence we get the set (15).

Suppose that $(3,4)$ is nonspecial and $(2,4)$ is special, then $(3,5) \in \Gamma$ by Lemma 2.3, and $(2,4) \in \Gamma$ by Lemma 2.4. By Lemma 1.1, we have $(4,3) \in \Gamma$, and hence the set (16).

Finally, suppose that $(3,3)$ and $(2,4)$ are nonspecial. Since $\dim(2,4) = \dim(3,3) = 0$ and $\dim(3,4) = 1$ by Riemann-Roch Theorem. By Lemma 2.4, we have $(3,4) \in \Gamma$. Since $(5,1),\ (6,2) \in \Gamma$, Lemma 1.1 implies that $\dim(4,2) = 0$. Thus $(4,3) \in \Gamma$ by Lemma 2.4. By Lemma 1.1, we have $(2,5) \in \Gamma$, and hence the set (17). □

Subcase III-10. $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$

For our convenience, we use a terminolgoy "the inverse relation". For a set Γ, we let $\Gamma^{-1} = \{(\beta,\alpha) \mid (\alpha,\beta) \in \Gamma\}$ and we call it the inverse relation of the set Γ. Note that $\Gamma(Q,P) = \Gamma^{-1}(P,Q)$.

Theorem 2.16. *In this case, Γ is one of the following 6 sets or inverse relations of them:*

$$\Gamma = \{(1,6),(2,4),(3,3),(4,2),(5,5),(6,1)\}. \tag{18}$$

$$\Gamma = \{(1,6),(2,4),(3,3),(4,5),(5,2),(6,1)\}. \tag{19}$$

$$\Gamma = \{(1,6),(2,4),(3,5),(4,2),(5,3),(6,1)\}. \tag{20}$$

$$\Gamma = \{(1,6),(2,4),(3,5),(4,3),(5,2),(6,1)\}. \tag{21}$$

$$\Gamma = \{(1,6),(2,5),(3,3),(4,4),(5,2),(6,1)\}. \tag{22}$$

$$\Gamma = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}. \tag{23}$$

Proof. Lemma 2.3 implies $(1,6),\ (6,1) \in \Gamma$. We consider case by case.

(a) Suppose that $(4,4)$ is special. Lemma 2.3 implies $(5,5) \in \Gamma$. By Bezout's Theorem, there is a conic cut out the divisor $4P + 4Q$ which is the unique conic passing through the divisors $aP + bQ$, for all a, b with $1 \leq a, b \leq 4$ and $a + b \geq 5$.

Thus, by Riemann-Roch Theorem and Lemma 2.4, we have $(2,4),\ (3,3),\ (4,2) \in \Gamma$ and we get the set (18).

(b) Suppose that $(4,4)$ is nonspecial and $(3,4)$ is special. Lemma 2.3 implies $(4,5) \in \Gamma$. Since $(2,4)$ and $(3,3)$ are also special, by Lemma 2.4, we have $(2,4),\ (3,3) \in \Gamma$. By Lemma 1.1, we have $(5,2) \in \Gamma$, and hence the set (19).

(c) Suppose that $(4,4),\ (3,4)$ are nonspecial and $(2,4),\ (4,2)$ are special. Lemma 2.3 implies $(3,5),\ (5,3) \in \Gamma$. Lemma 2.4 implies that $(2,4),\ (4,2) \in \Gamma$, hence we get the set (20).

(d) Suppose that $(4,4),\ (3,4),\ (4,2)$ are nonspecial and $(2,4)$ is special. Lemma 2.3 implies $(3,5) \in \Gamma$. Lemma 2.4 implies that $(2,4),\ (4,3) \in \Gamma$. Lemma 1.1 implies $(5,2) \in \Gamma$, hence we get the set (21).

(e) Suppose that $(4,3),\ (3,4)$ are nonspecial and $(3,3)$ is special. Then $(2,4)$ and $(4,2)$ are nonspecial. Indeed, if $(2,4)$ or $(4,2)$ is special, by Bezout's Theorem, the same conic cut out $3P+3Q$ and $2P+4Q$ (or $4P+2Q$). Then $(4,3)$ (or $(3,4)$) is special which contradicts the assumption. Lemma 2.3 implies $(4,4) \in \Gamma$ and Lemma 2.4 implies $(3,3) \in \Gamma$. Again, Lemma 2.4 implies $(2,5),\ (5,2) \in \Gamma$, hence we get the set (22).

(f) Suppose that $(2,4),\ (3,3),\ (4,2)$ are nonspecial. Since $(1,6),\ (6,1) \in \Gamma$, $(1,5)$ and $(5,1)$ are also nonspecial. Lemma 2.4 implies $(2,5),\ (3,4),\ (4,3),\ (5,2) \in \Gamma$, hence we get the set (23). □

3. Construction of Curves

In this section, each semigroup listed in the previous section is actually appeared as a Weierstrass semigroup at an pair of points on some smooth plane curve of degree 5. We use the following lemma frequently in examples.

Lemma 3.1. *Let C be a smooth plane curve of degree 5, and P, Q points on C. Suppose that there is a unique conic M such that $M \centerdot C = \alpha P + \beta Q + r(M \centerdot C)$. If $(\alpha', \beta') \not\leq (\alpha, \beta)$ and $\min\{\alpha, \alpha'\} + \min\{\beta, \beta'\} \geq 5$, then (α', β') is nonspecial.*

Proof. If (α', β') is special, there exists a quadratic curve M' such that $M' \centerdot C \geq \alpha' P + \beta' Q$. Then the sum of intersection multiplicities of M and M' is ≥ 5. By Bezout's Theorem, we have $M = M'$, which contradicts the assumptions. □

Example 3.2 (An example for Case I)**.** Let $M_1 := yz - x^2$, $M_2 = yz - (x-z)^2$, $L_1 = y$. Consider the family of curves defined by the equation $c_1 M_1 M_2 \ell + c_2 L_1^2 \ell_{11} \ell_{21} \ell'$

for general $c_1, c_2 \in \mathbb{C}$, general lines ℓ, ℓ', a general line ℓ_{11} through P, a general line ℓ_{21} through Q. Using Bertini's Theorem, we can prove that general members of the family are smooth. Let C be such a curve. Then C is a smooth plane curve of degree 5. Let $P = (0,0,1)$ and $Q = (1,0,1)$. Then $L_P = L_Q = L_1 = y$, $M_P = yz - x^2$ and $M_Q = yz - (x-z)^2$. We can check that $L_P \centerdot C = 2P + 2Q + r(L_P \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$,$M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Now by Theorem 2.5, we obtain the the set (1).

Example 3.3 (Examples for Case II)**.** (a) Let C be a curve defined by $c_1(yz - x^2)(xy - z^2 + xz)(x - 3z) + c_2(xy - z^2)y^2(y + 2x)$ for general c_1, c_2. Let $P = (0,1,0)$, $Q = (0,0,1)$, points on C. Then $L_P = x$, $L_Q = y$, $M_P = xy - z^2$, $M_Q = yz - x^2$. We have $L_P \centerdot C = 2P + Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$, $M_Q \centerdot C = P + 5Q + r(M_Q \centerdot C)$. By Theorem 2.6, $\Gamma(P,Q)$ is equal to the set (2).

(b) Let C be defined by $c_1(yz-x^2)(xz-z^2-y^2)(z+x+y)+c_2y^2x(x-z)(y-x+z)$ for general c_1, c_2. Let $P = (0,0,1)$, $Q = (1,0,1)$. Then $L_P = y$, $L_Q = x - z$, $M_P : yz-x^2$, and $M_Q : xz-(z^2+y^2)$. We can check that $L_P \centerdot C = 2P+Q+r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$, $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. By Theorem 2.6, $\Gamma(P,Q)$ is equal to the set (3).

Example 3.4 (An example for Subcase III-1)**.** Let C be defined by $c_1(yz - x^2)(y - z)(y + z)(y - 2z) + c_2x^5$ for general c_1, c_2. Let $P = (0,0,1)$, $Q = (0,1,0)$. Then $L_P = y$, $L_Q = z$, $M_P = M_Q : yz-x^2$. We can check that $L_P \centerdot C = 2P+0Q+r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 5Q + r(M_P \centerdot C)$. By Theorem 2.7, $\Gamma(P,Q)$ is equal to the set (4).

Example 3.5 (An example for Subcase III-2)**.** Let C be defined by $c_1(yz - x^2)(y - z)(y + z)(y - 2z) + c_2x^4(y + x)$ for general c_1, c_2. Let $P = (0,0,1)$, $Q = (0,1,0)$. Then $L_P = y$, $L_Q = z$, $M_P : yz-x^2$, $M_Q : yz-x^2-2z^2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 4Q + r(M_P \centerdot C)$, and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. By Theorem 2.8, $\Gamma(P,Q)$ is equal to the set (5).

Example 3.6 (An example for Subcase III-3)**.** Let C be defined by $c_1(yz - x^2)(y - a_1x)(y - a_2x)\ell + c_2(yz - x^2 - xy + y^2)(-2x + y)(y - z - b_1(x - z)(y - 1 - b_2(x - z))$ for general a_1, a_2, b_1, b_2, c_1, c_2 and a general line ℓ. Let $P = (1,1,1)$, $Q = (0,0,1)$.

Then $L_P = y$, $L_Q = z$, $M_P : yz - x^2$, $M_Q : yz - x^2 - 2z^2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 3Q + r(M_P \centerdot C)$, and $M_Q \centerdot C = P + 5Q + r(M_Q \centerdot C)$. By Theorem 2.9, $\Gamma(P, Q)$ is equal to the set (6).

Example 3.7 (An example for Subcase III-4)**.** Let $M_1 = yz - x^2$, $M_2 = yz - x^2 - xy$, $L_1 = -2x + y$. Let C be defined by $c_1 M_2 L_1^2 \ell_{11} + c_2 M_1 \ell_{21} \ell_{22} \ell$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21}, ℓ_{22} through Q, a general ℓ, where $P = (1, 1, 1)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = y$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 3Q + r(M_P \centerdot C)$, and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. By Theorem 2.10, $\Gamma(P, Q)$ is equal to the set (7).

Example 3.8 (An example for Subcase III-5)**.** Let $M_1 = (x - z)^2 + y^2 - z^2$, $M_2 = (x - z)^2 + 4y^2 - z^2$. Let C be defined by $c_1 M_2 (y - x)(y - 2x)(y - 3x) + c_2 M_1 (y - x + 2)(y - 2x + 4)(y - 3x + 6)$ for general c_1, c_2. Let $P = (0, 0, 1)$, $Q = (2, 0, 1)$. Then $L_P = x$, $L_Q = x - 2z$, $M_P = M_1$ and $M_Q = M_2$. Then we can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 2Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 2P + 5Q + r(M_Q \centerdot C)$. Then by Theorem 2.11, $\Gamma(P, Q)$ is equal to the set (8).

Example 3.9 (An example for Subcase III-6)**.** Let $M_1 = yz - x^2$, $M_2 = yz - 2x^2 + xy$, $L_1 = -2x + y$. Let C be defined by $c_1 M_2 L_1 \ell_{11} \ell_{12} + c_2 M_1 \ell_{21} \ell_{22} \ell_{23}$ for general c_1, c_2, general lines ℓ_{11}, ℓ_{12} through P, general lines $\ell_{21}, \ell_{22}, \ell_{23}$ through Q, where $P = (1, 1, 1)$, $Q = (0, 0, 1)$. Then $L_P = L_1$, $L_Q = y$, $M_P = M_1$, and $M_Q = M_2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 2Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = P + 5Q + r(M_Q \centerdot C)$. Then by Theorem 2.12, $\Gamma(P, Q)$ is equal to the set (9).

Example 3.10 (Examples for Subcase III-7)**.** (a) Let $M_1 = yz - x^2$, $M_2 = yz - 2x^2$, $L_1 = -2x + y$. Let C be defined by $c_1 M_2 L_1^2 \ell_{11} + c_2 M_1 \ell_{21} \ell_{22} \ell_{23}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines $\ell_{21}, \ell_{22}, \ell_{23}$ through Q, where $P = (1, 1, 1)$, $Q = (0, 0, 1)$. Then $L_P = L_1$, $L_Q = y$, $M_P = M_1$, and $M_Q = M_2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 2Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Let $M_3 := yz - 2x^2 + 2xy - y^2$. Then we have $M_3 \centerdot C = 2P + 3Q + r(M_3 \centerdot C)$, which implies that $(2, 4)$ is nonspecial by Lemma 3.1. By Theorem 2.13, $\Gamma(P, Q)$ is equal to the set (10).

(b) Let $M_1 = yz - 2x^2$, $M_2 = yz - x^2 + y^2$, $M = yz - x^2$ and $L_1 = z$. Let C be defined by $c_1M_2L_1^2\ell_{11} + c_2M_1\ell_{21}\ell_{22}\ell_{23}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines $\ell_{21}, \ell_{22}, \ell_{23}$ through Q, where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $L_P = L_1$, $L_Q := y$, $M_P = M_1$, and $M_Q = M_2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 2Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 2P + 4Q + r(M \centerdot C)$, which implies that $(2, 4)$ is special. By Theorem 2.13, $\Gamma(P, Q)$ is equal to the set (11).

Example 3.11 (Examples for Subcase III-8)**.** (a) Let $M_1 = (x-z)^2 + y^2 - z^2$, $M_2 = yz - x^2$, $L_1 = y - z$, $L_2 = y$. Let C be defined by $c_1M_2L_1\ell_{11}\ell_{12} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11}, ℓ_{12} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (1, 1, 1)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $M_P \centerdot C = 5P + Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = P + 5Q + r(M_Q \centerdot C)$. If we let $M_3 = yz - x^2 + 2xy - 2y^2$, then $M_3 \centerdot C = 3P + 2Q + r(M_3 \centerdot C)$, which implies that $(3, 3)$ is nonspecial by Lemma 3.1. Then by Theorem 2.14, $\Gamma(P, Q)$ is equal to the set (12).

(b) Let $M_1 = xz - y^2$, $M_2 = yz - x^2$, $L_1 = x - 2y + z$ and $L_2 = y$. Let C be defined by $c_1M_2L_1\ell_{11}\ell_{12} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11}, ℓ_{12} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (1, 1, 1)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $M_P \centerdot C = 5P + Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = P + 5Q + r(M_Q \centerdot C)$. If we let $M_3 = yz - x^2 + 3xy - 3y^2$, then $M_3 \centerdot C = 3P + 3Q + r(M_3 \centerdot C)$ which implies that $(3, 3)$ is special. Then by Theorem 2.14, $\Gamma(P, Q)$ is equal to the set (13).

Example 3.12 (Examples for Subcase III-9)**.** (a) Let $M_1 = yz - x^2 + xz$, $M_2 = yz - x^2 + y^2$, $M = yz - x^2$, $L_1 = z$, $L_2 = y$. Let C be defined by $c_1M_2L_1^2\ell_{11} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 3P + 4Q + r(M \centerdot C)$ which implies that $(3, 4)$ is special. By Theorem 2.15, $\Gamma(P, Q)$ is equal to the set (14).

(b) Let $M_1 = yz - x^2 + xz$, $M_2 = yz - x^2 - xy + y^2$, $M = yz - x^2$, $L_1 = z$ and $L_2 = y$. Let C be defined by $c_1M_2L_1^2\ell_{11} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (0, 1, 0)$,

$Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $L_P \centerdot C = 2P+0Q+r(L_P \centerdot C)$, $L_Q \centerdot C = 0P+2Q+r(L_Q \centerdot C)$, $M_P \centerdot C = 5P+Q+r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 3P + 3Q + r(M \centerdot C)$ which implies that $(3, 3)$ is special and $(3, 4)$ is nonspecial by Lemma 3.1. Then by Theorem 2.15, $\Gamma(P, Q)$ is equal to the set (15).

(c) Let $M_1 = yz - 2x^2 + xz$, $M_2 = yz - x^2 + y^2$, $M = yz - x^2$, $L_1 = z$ and $L_2 = y$. Let C be defined by $c_1M_2L_1^2\ell_{11} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $L_P \centerdot C = 2P+0Q+r(L_P \centerdot C)$, $L_Q \centerdot C = 0P+2Q+r(L_Q \centerdot C)$, $M_P \centerdot C = 5P+Q+r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 2P + 4Q + r(M \centerdot C)$ which implies that $(2, 4)$ is special and $(3, 4)$ is nonspecial by Lemma 3.1. Then by Theorem 2.15, $\Gamma(P, Q)$ is equal to the set (16).

(d) Let $M_1 = yz - 2x^2 + xz$, $M_2 = yz - x^2 - xy + y^2$, $M = yz - x^2$, $L_1 = z$ and $L_2 = y$. Let C be defined by $c_1M_2L_1^2\ell_{11} + c_2M_1L_2\ell_{21}\ell_{22}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21}, ℓ_{22} through Q, where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1$ and $L_Q = L_2$. We can check that $L_P \centerdot C = 2P+0Q+r(L_P \centerdot C)$, $L_Q \centerdot C = 0P+2Q+r(L_Q \centerdot C)$, $M_P \centerdot C = 5P+Q+r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 2P + 3Q + r(M \centerdot C)$ which implies that $(3, 3)$ and $(2, 4)$ are nonspecial by Lemma 3.1. Then by Theorem 2.15, $\Gamma(P, Q)$ is equal to the set (17).

Example 3.13 (Examples for Subcase III-10)**.** (a) Let $M_1 = yz - x^2 + z^2$, $M_2 = yz - x^2 + y^2$, $M = yz - x^2$. Let C be defined by $c_1M_1M_2\ell + c_2M\ell_{11}\ell_{21}\ell'$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21} through Q, general lines ℓ, ℓ' where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = z$ and $L_Q = y$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 4P + 4Q + r(M \centerdot C)$ which implies that $(4, 4)$ is special. Then by Theorem 2.16, $\Gamma(P, Q)$ is equal to the set (18).

(b) Let $M_1 = yz - x^2 + xz + z^2$, $M_2 = yz - x^2 + y^2$, $M = yz - x^2$. Let C be defined by $c_1M_1M_2\ell + c_2M\ell_{11}\ell_{12}\ell_{21}$ for general c_1, c_2, a general line ℓ_{11}, ℓ_{12} through P, general lines ℓ_{21} through Q, general lines ℓ where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = z$ and $L_Q = y$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$

and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M \centerdot C = 3P + 4Q + r(M \centerdot C)$ which implies that $(3, 4)$ is special. Then $(4, 4)$ is nonspecial by Lemma 3.1. Then by Theorem 2.16, $\Gamma(P, Q)$ is equal to the set (19).

(c) Let $M_1 = yz - 2x^2 + z^2$, $M_2 = yz - x^2 + y^2$, $M_3 = yz - x^2$, $M_4 = yz - 2x^2$, $L_1 = z$ and $L_2 = y$. Let C be defined by $c_1M_1M_2\ell + c_2M_3L_1\ell_{11}\ell_{21}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ_{21} through Q, general lines ℓ where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = L_1 = z$ and $L_Q = L_2 = y$. We can prove the following: $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$, $M_Q \centerdot C = 0P+5Q+r(M_Q \centerdot C)$, $M_3 \centerdot M_P = 2P+0Q+r(M_3 \centerdot M_P)$, $M_3 \centerdot M_Q = 0P+4Q+r(M_3 \centerdot M_Q)$, $M_4 \centerdot M_P = 4P + 0Q + r(M_4 \centerdot M_P)$, and $M_4 \centerdot M_Q = 0P + 2Q + r(M_4 \centerdot M_Q)$. Thus $(2, 4)$ and $(4, 2)$ are special. By (c) in the proof of Theorem 2.16, $\Gamma(P, Q)$ is equal to the set (20).

(d) Let $M_1 = yz-2x^2+xz+z^2$, $M_2 = yz-x^2+y^2$, $M_3 = yz-x^2$, $M_4 = yz-2x^2$, $L_1 = z$ and $L_2 = y$. Let C be defined by $c_1M_1M_2\ell + c_2M_3x\ell_{11}\ell_{12}$ for general c_1, c_2, a general line ℓ_{11}, ℓ_{12} through P, general lines ℓ where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = z$ and $L_Q = y$. We can check that $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$, $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$, $M_3 \centerdot C = 2P + 4Q + r(M_3 \centerdot C)$, and $M_4 \centerdot C = 3P + 2Q + r(M_4 \centerdot C)$. Thus $(2, 4)$ is special and $(4, 2)$ is nonspecial by Lemma 3.1. Thus $\Gamma(P, Q)$ is equal to the set (21) by Theorem 2.16.

(e) Let $M_1 = yz - x^2 - xz + z^2$, $M_2 = yz - x^2 - xy + y^2$, $M = yz - x^2$. Let C be defined by $c_1M_1M_2\ell + c_2ML_2\ell_{11}\ell_{12}$ for general c_1, c_2, a general line ℓ_{11}, ℓ_{12} through P, general lines ℓ where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = z$ and $L_Q = y$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P+2Q+r(L_Q \centerdot C)$, $M_P \centerdot C = 5P+0Q+r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P+5Q+r(M_Q \centerdot C)$. Also we have $M \centerdot C = 3P + 3Q + r(M \centerdot C)$ which implies that $(3, 3)$ is special. Then $(4, 3)$, $(3, 4)$, $(4, 2)$, $(2, 4)$ are nonspecial by Lemma 3.1. Then by Theorem 2.16, $\Gamma(P, Q)$ is equal to the set (22).

(f) Let $M_1 = yz - 2x^2 - xz + z^2$, $M_2 = yz - x^2 - xy + y^2$, $M_3 = yz - x^2$, $M_4 = yz - 2x^2$. Let C be defined by $c_1M_1M_2\ell + c_2M_3L_1L_2\ell_{11}$ for general c_1, c_2, a general line ℓ_{11} through P, general lines ℓ where $P = (0, 1, 0)$, $Q = (0, 0, 1)$. Then $M_P = M_1$, $M_Q = M_2$, $L_P = z$ and $L_Q = y$. We can check that $L_P \centerdot C = 2P + 0Q + r(L_P \centerdot C)$, $L_Q \centerdot C = 0P + 2Q + r(L_Q \centerdot C)$, $M_P \centerdot C = 5P + 0Q + r(M_P \centerdot C)$ and $M_Q \centerdot C = 0P + 5Q + r(M_Q \centerdot C)$. Also we have $M_3 \centerdot C = 2P + 3Q + r(M_3 \centerdot C)$ and $M_3 \centerdot C = 3P + 2Q + r(M_3 \centerdot C)$ which implies that $(3, 3)$, $(4, 2)$ and $(2, 4)$ are nonspecial by Lemma 3.1. Then by Theorem 2.16, $\Gamma(P, Q)$ is equal to the set (23).

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