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NON-EXISTANCE REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM WITH CODAZZI TYPE OF STRUCTURE TENSOR FIELD

Dong Ho Lim [a] and Hwa Soo Kim [b, *]

Abstract. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper we prove that if the structure tensor field is Codazzi type, then M is a Hopf hypersurface. We characterize such Hopf hypersurfaces of $M_n(c)$.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. It is well-known that a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$, respectively.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([2]) and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbf{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary groups $PU(n+1)$. R. Takagi ([10]) completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. On

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*Corresponding author.

the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero ([6]) and so on. Berndt ([1]) classified all homogeneous Hopf hyersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B.

A real hypersurface of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, then M is said to be a type A for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

Theorem A ([6, 8])**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

For the structure tensor field ϕ on M, we define the Lie derivative $\mathcal{L}_\xi$ by $(\mathcal{L}_\xi\phi)X = [\xi, \phi X] - \phi[\xi, X]$, and $\nabla_\xi\phi$ is the covariant derivative with respect to a unit vector field X. We call the Lie derivative and covariant derivative in the Reeb vector field ξ direction of the structure tensor field as ξ-Lie parallel and ξ-parallel. Many geometricians have studied real hypersurfaces from certain conditions and obtained some results on the classification of real hypersurfaces in complex space form $M_n(c)$.

As for the derivatives of structure tensor field, Lim ([4]) has proved the following Theorem.

Theorem B ([4])**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. It satisfies $\mathcal{L}_\xi\phi = \nabla_\xi\phi$ if and only if M is a locally congruent to one of the model space of type A.*

In this paper we shall study a real hypersurface in a nonflat complex space form $M_n(c)$, with Codazzi type of structure tensor field, and give some characterizations of such a real hypersurface in $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2. PRELIMINARIES

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings :

$$\nabla_X \xi = \phi AX, \tag{2.2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss, Codazzi equations and operator of Lie derivative respectively :

$$\begin{aligned} R(X,Y)Z = \ &\tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &-2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.5}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\} \tag{2.6}$$

where $\alpha = \eta(A\xi)$. We put

$$A\xi = \alpha\xi + \mu W, \tag{2.7}$$

where W be a unit vector field orthogonal to ξ and μ does not vanish on Ω.

3. Some Lemmas.

In this section, we assume that Ω is not empty, then we shall prove Theorem 1 and 2. If the vector field ξ is a principal curvature vector in a nonflat complex space form i.e. $A\xi=\alpha\xi$ then M is called a Hopf hypersurface of $M_n(c)$. For such a Hopf hypersurface, we now recall some well known results which will be used to prove our results (see [7])

Lemma 3.1 ([7]). *Let be a Hopf hypersurface in a nonflat complex space form $M_n(c)$. If X is a unit vector such that $AX = \lambda X$, Then*

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{3.1}$$

Lemma 3.2 ([7]). *The B type hypersurface $H_n\boldsymbol{C}$ in has three distinct principal curvatures, $\frac{1}{r}cothu$, $\frac{1}{r}tanhu$ of multiplicity $n-1$ and $\alpha = \frac{2}{r}tanh2u$ of multiplicity 1. On the other hand, In $P_n\boldsymbol{C}$, type B hypersurface also has three distinct principal curvatures, $-\frac{1}{r}tanu$ of multiplicity $2p$, $\frac{1}{r}cotu$ of multiplicity $2q$ and $\alpha = \frac{2}{r}cot2u$ of multiplicity 1, where $p > 0, q > 0, andp + q = n - 1$.*

Lemma 3.3. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M satisfies $(\nabla_X\phi)Y = (\nabla_Y\phi)X$, Then M is a Hopf hypersurface in $M_n(c)$.*

Proof. We assume that $(\nabla_X\phi)Y = (\nabla_Y\phi)X$ for any vector fields X and Y. Then, by using (2.3) and symmetric properties of the shape operator, we have

$$\begin{aligned}(\nabla_X\phi)Y - (\nabla_Y\phi)X &= \eta(Y)AX - g(AX,Y)\xi - (\eta(X)AY - g(AY,X)\xi)\\ &= \eta(Y)AX - \eta(X)AY\end{aligned}$$

Under the our assumption, it follows from the above equation that

$$\eta(Y)AX - \eta(X)AY = 0. \tag{3.2}$$

If we put $Y = \xi$ into (3.2) and make use of (2.7), then we have

$$AX = \alpha\eta(X)\xi + \mu\eta(X)W. \tag{3.3}$$

If we substitute $X = W$ into (3.3) and then we obtain

$$AW = 0. \tag{3.4}$$

Taking inner product of (3.4) with ξ and using (2.7), we have $\mu = 0$ on Ω and it is a contradiction.

Thus the set Ω is empty and hence M is a Hopf hypersurface. □

4. Non-existence of Real Hypersurfaces

In this section, we will discuss non-existence of the real hypersurface for Lemma 3.2 in the complex space form, that is, we shall prove Theorem 4.1

Theorem 4.1. *There exist no real hypersurface of $M_n(c)$, $c \neq 0$, whose structure tensor field is Codazzi type.*

Proof. By Lemma 3.3, the real hypersurface M satisfying $(\nabla_X \phi)Y = (\nabla_Y \phi)X$ is a Hopf hypersurface in $M_n(c)$, that is, $A\xi = \alpha\xi$. Since ξ is a Reeb vector field, the assumption $(\nabla_X \phi)Y = (\nabla_Y \phi)X$ is given by

$$\eta(Y)AX - \eta(X)AY = 0. \tag{4.1}$$

If we put $Y = \xi$ into (4.1), then we have

$$AX = \alpha\eta(X). \tag{4.2}$$

For any vector field $X \perp \xi$ on M such that $AX = \lambda X$, it follows from (4.2) that the principal value $\lambda = 0$. From the equation (3.1), we obtain

$$-\frac{\alpha}{2}A\phi X = \frac{c}{4}\phi X. \tag{4.3}$$

If $\alpha = 0$, then $c = 0$, and there is no real hypersurface. Thus, the constant value $\alpha \neq 0$, it follows from (3.1) that ϕX is also a principal direction, say $A\phi X = -\frac{c}{2\alpha}\phi X$. From this results, real hypersurface M has at most 3 distinct principal curvatures, that is, $\{\lambda, \mu\} = \{0, -\frac{c}{2\alpha}\}$ in $M_n(c)$. To classify the real hypersurface, let M be locally congruent to one of type A. then we have $\phi AX = 0 \neq -\frac{c}{2\alpha}\phi X = A\phi X$. Therefore, by the Theorem A, M is not locally congruent to one of type A. Now, we assume that real hypersurface M is locally congruent to model space of type B. Then, we can get the principal curvature $\lambda\mu = 0$. By the Lemma 3.3, this is a contradiction and such a Hopf hypersurfaces M does not exists. Therefore, we conclude that M is not locally congruent to one of type A or$B$ and the proof is completed. □

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[a]DEPARTMENT OF MATHEMATICS EDUCATION, SEHAN UNIVERSITY, JEOLLANAM-DO 58447, REPUBLIC OF KOREA
Email address: dhlim@sehan.ac.kr

[b]DEPARTMENT OF MATHEMATICS EDUCATION, SEHAN UNIVERSITY, JEOLLANAM-DO 58447, REPUBLIC OF KOREA
Email address: hskim@sehan.ac.kr