J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2021.28.1.15 ISSN(Online) 2287-6081
Volume 28, Number 1 (February 2021), Pages 15–26

A NEW GENERALIZED CUBIC FUNCTIONAL EQUATION AND ITS STABILITY PROBLEMS

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Abstract. The purpose of this paper is to introduce a new type of a cubic functional equation and then investigate its stability problems in a convex modular space with a generalized $\triangle_a$-condition.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [20] concerning the stability of group homomorphisms at the Mathematics Club of the University of Wisconsin in 1940. In the next year, Hyers [6] gave a partial answer to the question of Ulam for additive groups under the assumption that groups are Banach spaces. Hyers's method used in [6], which is often called the direct method, has been applied for studying the stability of various functional equations. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [17] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference. Rassias' paper [17] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations.

The second most popular technique of proving the stability of functional equations is the fixed point methods. It was used for the first time in 1991 by Baker [2] who applied a variant of Banach's fixed point theorem to obtain the stability of a functional equation in a single variable. By using the fixed point method the stability problems of several functional equations over various normed spaces have been extensively investigated by a number of authors (see [4, 3, 14]). Most authors

Received by the editors September 19, 2019. Accepted January 14, 2021.
2010 *Mathematics Subject Classification.* 39B52.
Key words and phrases. cubic functional equation, modular space, convex modular, $\triangle_a$-condition, Hyers-Ulam-Rassias stability, fixed point theorem.

follow Radu's approach [16] and make use of a theorem of Margolis and Diaz [11]. In particular, Jun and Kim [8] considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

since it should be easy to see that a function $f(x)=cx^3$ is a solution of the equation (1.1). Since then the stability of cubic functional equations has been investigated by a number of authors (see [9, 5, 15] for details). In particular, Najati [15] investigated the following generalized cubic functional equation

$$f(sx+y)+f(sx-y)=sf(x+y)+sf(x-y)+2(s^3-s)f(x) \tag{1.2}$$

for a positive integer $s \geq 2$.

As we notice there are various definitions for the stability of the cubic functional equations, in this paper, we will introduce a new type of a generalized cubic functional equation as follows :

$$f(ax-by)-f(bx-ay)+ab(a-b)f(x+y)=(a+b)^2(a-b)[f(x)+f(y)] \tag{1.3}$$

where a and b are integers with $a \neq b$ and $a,\, b \neq 0, \pm 1$.

In this paper, we will obtain a general solution of the generalized cubic functional equation (1.3) and then investigate the stability problems by using both the direct method and the fixed point method for the given generalized cubic functional equation. To obtain the stability problems, we will introduce a convex modular space and some basic properties concerning the convex modular.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}\,(\mathbb{R}$ or $\mathbb{C})$. A generalized functional $\rho : X \to [0, \infty]$ is called a *modular* if for any $x\,, y \in X$,

(M1) $\rho(x)=0$ if and only if $x=0$.

(M2) $\rho(\alpha x)=\rho(x)$ for all scalar α with $|\alpha|=1$.

(M3) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ for all scalars $\alpha\,,\beta \geq 0$ with $\alpha+\beta=1$.

If (M3) is replaced by

(M4) $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ for all scalars $\alpha\,,\beta \geq 0$ with $\alpha+\beta=1$ then the functional ρ is said to be a *convex modular*.

A modular ρ defines the following vector space:

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

and we call X_ρ a modular space. A modular ρ is said to satisfy the Δ_2-*condition* if there exists $k>0$ such that $\rho(2x) \leq k\rho(x)$ for all $x \in X_\rho$. We call the constant

k a Δ_2-*constant* related to Δ_2-condition. Now, let $\{x_n\}$ be a sequence in X_ρ. The sequence $\{x_n\}$ is ρ-*convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ is called ρ-*Cauchy* if for any $\varepsilon > 0$ one has $\rho(x_n - x_m) < \varepsilon$ for sufficiently large n, $m \in \mathbb{N}$. Also, X_ρ is called ρ-*complete* if any ρ-Cauchy sequence is ρ-convergent to a point in X_ρ. The modular theory on linear spaces and the related modular theory on linear spaces have been established by Nakano [13]. Kim and Shin [10] investigated the stability problems of additive and quadratic functional equations in modular spaces.

2. A Solution for a Generalized Cubic Functional Equation

In this section let X and Y be real vector spaces and we will investigate the general solution of the functional equation (1.3). Before we proceed, we would like to introduce some basic definitions concerning 3-additive symmetric mappings and key concepts. More general version of n-additive symmetric mappings are also found in [19] and [21]. A mapping $A : X \to Y$ is said to be *additive* if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$. A mapping $A_3 : X^3 \to Y$ is called 3-*additive* if it is additive in each variable. A mapping A_3 is said to be *symmetric* if $A_3(x_1, x_2, x_3) = A_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for every permutation $\{\sigma(1), \sigma(2), \sigma(3)\}$ of $\{1, 2, 3\}$. If $A_3(x_1, x_2, x_3)$ is an 3-additive symmetric mapping, then $A^3(x)$ will denote the diagonal $A_3(x, x, x)$ and $A^3(rx) = r^3 A^3(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$. $A^3(x)$ will be called a *monomial function* of degree 3 (assuming $A^3 \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_s = x$ and $x_{s+1} = \cdots = x_3 = y$ in $A_3(x_1, x_2, x_3)$ will be denoted by $A^{s,3-s}(x, y)$.

Theorem 2.1. *A mapping $f : X \to Y$ is a solution of the functional equation (1.3) if and only if f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the* 3*-additive symmetric mapping $A_3 : X^3 \to Y$.*

Proof. Suppose f satisfies the functional equation (1.3). On letting $x = y = 0$ in the equation(1.3), we have

$$(a - b)(2a^2 + 3ab + 2b^2)f(0) = 0 .$$

Hence $f(0) = 0$. On putting $y = 0$ in the equation (1.3), we get

$$f(ax) - f(bx) = (a^3 - b^3)f(x) \tag{2.1}$$

for all $x \in X$. Also, on letting $x = 0$ in the equation (1.3), we get

$$f(-by) - f(-ay) + ab(a-b)f(-y) = (a+b)^2(a-b)f(y)$$

for all $y \in X$. Replacing y by x in the previous equation and using the equation (2.1), we have

$$-(a^3 - b^3)f(-x) = (a^3 - b^3)f(x)\,.$$

That is, $f(x) = -f(-x)$, for all $x \in X$. By Theorems 3.5 and 3.6 in [21], f is a generalized polynomial function of degree at most 3, that is, f is of the form

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.2}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of the i-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3$. By $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$, we get $A^2(x) + A^0(x) = 0$. Hence we have

$$f(x) = A^3(x) + A^1(x)\,,$$

for all $x \in X$. The equation (2.1) and $A^n(rx) = r^n A^n(x)$ ($n = 1$ or 3) for all $x \in X$ and all $r \in \mathbb{Q}$ imply that

$$\begin{aligned} & a^3A^3(x) + aA^1(x) - b^3A^3(x) - bA^1(x) = f(ax) - f(bx) \\ = \ & a^3f(x) - b^3f(x) = a^3(A^3(x) + A^1(x)) - b^3(A^3(x) + A^1(x)) \end{aligned}$$

for all $x \in X$. Hence we may conclude that $A^1(x) = 0$. Thus $f(x) = A^3(x)$ for all $x \in X$, as desired.

Conversely, assume that $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of a 3-additive symmetric mapping $A_3 : X^3 \to Y$. Note that

$$A^3(qx + ry) = q^3A^3(x) + 3q^2rA^{2,1}(x,y) + 3q^1r^2A^{1,2}(x,y) + r^3A^3(y)$$

where $q, r \in \mathbb{Q}$. Hence we have

$$\begin{aligned} & A^3(ax - by) - A^3(bx - ay) + ab(a-b)A^3(x+y) \\ & = (a+b)^2(a-b)[A^3(x) + A^3(y)] \end{aligned}$$

for all x, $y \in X$. Thus we may conclude that f satisfies the equation (1.3). $\square$

Now, we call the mapping f a *generalized cubic mapping* if f satisfies the equation (1.3).

3. The Direct Method Approach for the Stability Problem

Throughout this section let V be a linear space and X_ρ a ρ-complete convex modular space unless otherwise stated. Now, we will state some basic properties to be used in this section.

Remark 3.1. (1) $\rho(\alpha x) \le \alpha \rho(x)$ for all $0 \le \alpha \le 1$.
(2) $\rho(\sum_{j=1}^n \alpha_j x_j) \le \sum_{j=1}^n \alpha_j \rho(x_j)$, where $\sum_{j=1}^n \alpha_j \le 1$ for all $\alpha_j \ge 0$.

Lemma 3.2. *Let X be a linear space over a field $\mathbb{K}\,(\mathbb{R}$ or $\mathbb{C})$. Suppose X satisfies the $\triangle_2$-condition with $\triangle_2$-constant k. Then for each $a \in \mathbb{K}$ with $|a| > 1$ there exists a constant k_a such that $\rho(ax) \le k_a \rho(x)$, for all $x \in X$.*

Proof. Since $|a| > 1$ then there exists $n \in \mathbb{N}$ such that $2^{n-1} < |a| \le 2^n$. Hence

$$\begin{aligned} \rho(ax) = \rho(\frac{a}{|a|}|a|x) = \rho(|a|x) &= \rho(\frac{|a|}{2^n}2^n x) \\ &\le \frac{|a|}{2^n}\rho(2^n x) \le \frac{|a|}{2^n}k^n \rho(x) = k_a \rho(x) \end{aligned}$$

where $k_a = \frac{|a|}{2^n}k^n$. □

Definition 3.3. Let $a \ge 2$ be an integer number. A modular ρ is said to satisfy the Δ_a-*condition* if there exists $k_a > 0$ such that $\rho(ax) \le k_a \rho(x)$ for all $x \in X_\rho$. We call the constant k_a a Δ_a-*constant* related to Δ_a-condition and a.

We note that it is easy to show that the $\triangle_a$-constant k_a is greater than equal to a, for any integer $a \ge 2$. For a given function $f : V \to X_\rho$ and a fixed integer $a \ge 2$ let

$$D_a f(x,\, y) := f(ax-y) - f(x-ay) + a(a-1)f(x+y) - (a+1)^2(a-1)[f(x)+f(y)]$$

for all $x\,, y \in V$.

Theorem 3.4. *Let $a \ge 2$ be a integer number. Suppose X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a. If there exists a function $\phi : V^2 \to [0,\, \infty)$ for which a mapping $f : V \to X_\rho$ satisfy*

$$\rho(D_a f(x,\, y)) \le \phi(x,\, y) \tag{3.1}$$

$$\lim_{n\to\infty} k_a^{3n} \phi\Big(\frac{x}{a^n},\, \frac{y}{a^n}\Big) = 0 \;\; and \;\; \sum_{j=1}^{\infty} \Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, \frac{y}{a^j}\Big) < \infty \tag{3.2}$$

for all $x,\, y \in V$, then there exists a unique generalized cubic mapping $C : V \to X_\rho$ defined by $C(x) = \rho - \lim_{n\to\infty} a^{3n} f\Big(\frac{x}{a^n}\Big)$ and

$$\rho\Big(f(x) - C(x)\Big) \le \frac{1}{a\,k_a^2}\sum_{j=1}^{\infty}\Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\,0\Big) \tag{3.3}$$

for all $x \in V$.

Proof. On letting $x = y = 0$ in the inequality (3.1), we have

$$\rho\Big((2a^3 + a^2 - a - 2)f(0)\Big) \le \phi(0,\,0)\,.$$

Hence $f(0) = 0$ because of the equation (3.2). On taking $x = \frac{x}{a}$ and $y = 0$ in the inequality (3.1), we get

$$\rho\Big(f(x) - a^3 f\Big(\frac{x}{a}\Big)\Big) \le \phi\Big(\frac{x}{a}, 0\Big) \tag{3.4}$$

for all $x \in V$. By using the property of the $\triangle_a$-condition and the Remark 3.1, we have

$$\begin{aligned}\rho\Big(f(x) - a^{3n} f\Big(\frac{x}{a^n}\Big)\Big) &= \rho\Big(\sum_{j=1}^{n}\frac{1}{a^j}\Big(a^{4j-3} f\Big(\frac{x}{a^{j-1}}\Big) - a^{4j} f\Big(\frac{x}{a^j}\Big)\Big)\Big)\\ &\le \frac{1}{k_a^3}\sum_{j=1}^{n}\Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\,0\Big)\end{aligned}$$

for any positive integer n and all $x \in V$. Also, for all positive integers n and m with $n \ge m$, we get

$$\begin{aligned}\rho\Big(a^{3n} f\Big(\frac{x}{a^n}\Big) - a^{3m} f\Big(\frac{x}{a^m}\Big)\Big) &\le k_a^{3m}\rho\Big(a^{3(n-m)} f\Big(\frac{x}{a^n}\Big) - f\Big(\frac{x}{a^m}\Big)\Big)\\ &\le \frac{1}{k_a^3}k_a^{3m}\sum_{j=1}^{n-m}\Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^{m+j}},\,0\Big)\\ &\le \frac{1}{k_a^3}\Big(\frac{a}{k_a}\Big)^m \sum_{j=m+1}^{n}\Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\,0\Big)\end{aligned}$$

for all $x \in V$. The last part of the above inequalities tends to zero as $m \to \infty$. Hence the sequence $\{a^{3n} f\Big(\frac{x}{a^n}\Big)\}$ is a ρ-Cauchy sequence in the ρ-complete convex modular space. This means that the sequence $\{a^{3n} f\Big(\frac{x}{a^n}\Big)\}$ is ρ-convergent in X_ρ. Hence we may define a mapping $C : V \to X_\rho$ by

$$C(x) = \lim_{n\to\infty} a^{3n} f\Big(\frac{x}{a^n}\Big)$$

for all $x \in V$. In fact, this means that

$$\lim_{n\to\infty} \rho\Big(a^{3n} f\Big(\frac{x}{a^n}\Big) - C(x)\Big) = 0 \tag{3.5}$$

for all $x \in V$. By using $\triangle_a$-condition with $\triangle_a$-constant k_a, we have

$$\begin{aligned}
&\rho(f(x) - C(x)) = \rho\Big(f(x) - a^{3n} f\Big(\frac{x}{a^n}\Big) + a^{3n} f\Big(\frac{x}{a^n}\Big) - C(x)\Big) \\
\leq\ & \rho\Big(\frac{1}{a}\Big(af(x) - a^{3n+1} f\Big(\frac{x}{a^n}\Big)\Big) + \frac{1}{a}\Big(a^{3n+1} f\Big(\frac{x}{a^n}\Big) - aC(x)\Big)\Big) \\
\leq\ & \frac{k_a}{a}\, \rho\Big(f(x) - a^{3n} f\Big(\frac{x}{a^n}\Big)\Big) + \frac{k_a}{a}\, \rho\Big(a^{3n} f\Big(\frac{x}{a^n}\Big) - Q(x)\Big) \\
\leq\ & \frac{k_a}{a}\, \frac{1}{k_a^3} \sum_{j=1}^{n} \Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, 0\Big) + \frac{k_a}{a}\, \rho\Big(a^{3n} f\Big(\frac{x}{a^n}\Big) - C(x)\Big)
\end{aligned}$$

for all $x \in V$. As $n \to \infty$, the last part of the above inequalities implies that

$$\rho(f(x) - C(x)) \leq \frac{1}{a\, k_a^2} \sum_{j=1}^{\infty} \Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j},\, 0\Big)$$

for all $x \in V$, that is, this implies the inequality (3.3). Next, we will show the mapping C is a generalized cubic mapping, that is, it satisfies the equality (1.3) when $b = 1$. We note that

$$\rho\Big(a^{3n} D_a f\Big(\frac{x}{a^n}\,,\, \frac{y}{a^n}\Big)\Big) \leq k_a^{3n} \phi\Big(\frac{x}{a^n}\,,\, \frac{y}{a^n}\Big) \to 0$$

for all $x,\, y \in V$, as $n \to \infty$.

By an integer number $a \geq 2$ and the Remark 3.1, we have

$$\begin{aligned}
& \rho(D_a C(x,\, y)) \\
=\ & \rho\Big(D_a C(x,\, y) - a^{3n} D_a f\Big(\frac{x}{a^n}\,,\, \frac{y}{a^n}\Big) + a^{3n} D_a f\Big(\frac{x}{a^n}\,,\, \frac{y}{a^n}\Big)\Big) \\
\leq\ & \frac{k_a^3}{a^3}\Big[\rho\Big(C(ax - y) - a^{3n} f\Big(\frac{ax - y}{a^n}\Big)\Big) - \rho\Big(C(x - ay) - a^{3n} f\Big(\frac{x - ay}{a^n}\Big)\Big) \\
& + \frac{k_a(k_a - 1)^2}{2} \rho\Big(C(x + y) - a^{3n} f\Big(\frac{x + y}{a^n}\Big)\Big) \\
& - (k_a^2 - 1)^2 \rho\Big(C(x) - a^{3n} f\Big(\frac{x}{a^n}\Big)\Big) - (k_a^2 - 1)^2 \rho\Big(C(y) - a^{3n} f\Big(\frac{y}{a^n}\Big)\Big) \\
& + \rho\Big(a^{3n} D_a f\Big(\frac{x}{a^n}\,,\, \frac{y}{a^n}\Big)\Big)\Big]
\end{aligned}$$

for all $x,\, y \in V$. The note and the equation (3.5) imply that $\rho(D_a C(x,\, y)) = 0$ for all $x,\, y \in V$. Hence the mapping C is a generalized cubic mapping, as desired. Finally, we have to show that the mapping C is unique. To show the uniqueness, we may assume that there is another generalized cubic mapping $\widetilde{C} : V \to X_\rho$ satisfies the inequality (3.3). We note that when $b = 1$, the equation (2.1) implies that $f(ax) =$

$a^3 f(x)$ for all $x \in X$. Hence we have $C(x) = a^{3n} C(\frac{x}{a^n})$ and $\widetilde{C}(x) = a^{3n} \widetilde{C}(\frac{x}{a^n})$ for all $x \in V$. Hence we get

$$\begin{aligned} \rho(C(x) - \widetilde{C}(x)) &= \rho\Big(a^{3n} C(\frac{x}{a^n}) - a^{3n} \widetilde{C}(\frac{x}{a^n})\Big) \\ &\leq \frac{k_a^{3n}}{a} \Big[\rho\Big(C\Big(\frac{x}{a^n}\Big) - f\Big(\frac{x}{a^n}\Big) + \widetilde{C}\Big(\frac{x}{a^n}\Big) - f\Big(\frac{x}{a^n}\Big)\Big)\Big] \\ &\leq \frac{2}{a^2 k_a^2} \Big(\frac{a}{k_a}\Big)^n \sum_{j=n+1}^{\infty} \Big(\frac{k_a^4}{a}\Big)^j \phi\Big(\frac{x}{a^j}, 0\Big) \end{aligned}$$

for all $x \in V$. On taking the limit as $n \to \infty$, the uniqueness is proved. □

Corollary 3.5. *Let $a \geq 2$ be a integer number and θ and $p > \log_a \frac{k_a^4}{a}$ be real numbers. Suppose V is a normed space with norm $||\cdot||$ and X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a . If $f : V \to X_\rho$ such that*

$$\rho(D_a f(x, y)) \leq \theta(||x||^p + ||y||^p) \tag{3.6}$$

for all $x, y \in V$, then there exists a unique generalized cubic mapping $C : V \to X_\rho$ such that

$$\rho\Big(f(x) - C(x)\Big) \leq \frac{\theta\, k_a^2}{a\,(a^{p+1} - k_a^4)} ||x||^p \tag{3.7}$$

for all $x \in V$.

Proof. On taking $\phi(x, y) = \theta(||x||^p + ||y||^p)$ in the Theorem 3.4, we know that the inequality (3.6) holds. Also, it satisfies the inequalities (3.2). According to Theorem 3.4, we have the result as in the inequality (3.7). □

4. THE FIXED POINT METHOD APPROACH FOR THE STABILITY PROBLEM

In this section we shall study the generalized Hyers-Ulam stability for the generalized cubic functional equation (1.3) in a modular space by using the fixed point method. Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 4.1. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity. Now, we will introduce one of fundamental results of fixed point theory. For the proof, refer to [11].

Theorem 4.2 ([11, 18, The alternative of fixed point])**.** *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant* $0 < L < 1$. *Then for each given* $x \in X$, *either*

$$d(T^n x,\, T^{n+1} x) = \infty \textit{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

(1) $d(T^n x,\, T^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(2) The sequence $\{T^n x\}$ *is convergent to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* T *in the set*

$$Y = \{y \in X \mid d(T^{n_0} x, y) < \infty\}\,;$$

(4) $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in Y$.

Theorem 4.3. *Let* $a \geq 2$ *be a integer number. Suppose* X_ρ *satisfies the* $\triangle_a$*-condition with* $\triangle_a$*-constant* k_a *and* $\phi : V^2 \to [0, \infty)$ *be a function such that there exists an constant* $0 < L < 1$ *with*

$$\phi(\frac{x}{a}, \frac{y}{a}) \leq \frac{L}{k_a^3}\, \phi(x,\, y) \tag{4.1}$$

for all $x,\, y \in V$. *If* $f : V \to X_\rho$ *is a mapping satisfying*

$$\rho(D_a f(x, y)) \leq \phi(x, y) \tag{4.2}$$

for all $x,\, y \in V$, *then there exists a unique generalized cubic mapping* $C : V \to X_\rho$ *defined by* $C(x) = \rho - \lim_{n\to\infty} a^{3n} f\Big(\frac{x}{a^n}\Big)$ *and*

$$\rho\Big(f(x) - C(x)\Big) \leq \frac{L}{k_a^3(1-L)}\phi(x,\, 0) \tag{4.3}$$

for all $x \in V$.

Proof. First of all, we note that the inequality (4.1) implies that $\phi(0,\, 0) = 0$. We put $x = y = 0$ in the equation (4.2) to obtain $f(0) = 0$. Substituting $x = \frac{x}{a}$ and $y = 0$ in the inequality (4.2) and using the inequality (4.1), we have

$$\rho\Big(f(x) - a^3 f\Big(\frac{x}{a}\Big)\Big) \leq \phi\Big(\frac{x}{a}, 0\Big) \leq \frac{L}{k_a^3}\phi(x, 0) \tag{4.4}$$

for all $x \in V$.

Now we define a set S as

$$S := \{g : V \to X_\rho \text{ with } g(0) = 0\}$$

and then a mapping d on $S \times S$ by

$$d(g,h) = \inf\{\mu \in [0,\, \infty] : \rho(g(x) - h(x)) \le \mu\phi(x,0), \forall x \in V\}$$

where $\inf \emptyset = \infty$ as a definition. Then (S,d) is a complete generalized metric space; see Lemma 2.1 in [12]. Now we also define a linear mapping $T : S \to S$ by

$$Tg(x) := a^3 g\left(\frac{x}{a}\right)$$

for all $x \in V$. Let $g,h \in S$ be given such that $d(g,h) \le \mu$. Then we have

$$\rho\left(g(x) - h(x)\right) \le \mu\phi(x,0)$$

for all $x \in V$. Hence we get

$$\begin{aligned} \rho((Tg)(x) - (Th)(x)) &= \rho\Big(a^3 g\left(\frac{x}{a}\right) - a^3 h\left(\frac{x}{a}\right)\Big) \\ &\le k_a^3 \mu\phi\Big(\frac{x}{a},\, 0\Big) \le L\mu\phi(x,\, 0) \end{aligned}$$

for all $x \in V$. Hence we have the fact that $d(g,h) \le \mu$ implies $d(Tg,Th) \le L\mu$ to obtain

$$d(Tg,Th) \le Ld(g,h)$$

for all $g,h \in S$. Thus T is strictly contractive because L is a constant with $0 < L < 1$. Now from the observation of the inequality (4.4) we have

$$d(f,Tf) \le \frac{L}{k_a^3} < \infty\,. \tag{4.5}$$

According to the (2) of the Theorem 4.2, there exists a function $C : V \to X_\rho$ which is a fixed point of T such that $\rho(T^n f,\, C) \to 0$ as $n \to \infty$. By using mathematical induction, we can show that

$$T^n f(x) := a^{3n} f\Big(\frac{x}{a^n}\Big)$$

for all $n \in \mathbb{N}$. Since $d(T^n f,\, C) \to 0$ ad $n \to \infty$, for each fixed $x \in V$, we have

$$\lim_{n\to\infty} \rho\Big(a^{3n} f\Big(\frac{x}{a^n}\Big) - C(x)\Big) = 0\,. \tag{4.6}$$

Hence we may conclude that

$$C(x) = \rho - \lim_{n\to\infty} a^{3n} f\Big(\frac{x}{a^n}\Big)$$

for all $x \in V$. By the (4) of the Theorem 4.2 and the inequality (4.5), we get

$$d(f,\, C) \le \frac{1}{1-L} d(Tf,\, f) \le \frac{L}{k_a^3(1-L)}\,. \tag{4.7}$$

This means that the inequality (4.3) holds for all $x \in V$. Also, the uniqueness of the mapping C follows from the (3) of the Theorem 4.2. We also knew that the mapping C was a generalized cubic mapping as in the proof of Theorem 3.4 such as $\rho(D_a C(x, y)) = 0$ for all $x, y \in V$. □

Corollary 4.4. *Let $a \geq 2$ be a integer number and θ and $p > \log_a \frac{k_a^3}{a}$ be real numbers. Suppose V is a normed space with norm $||\cdot||$ and X_ρ satisfies the $\triangle_a$-condition with $\triangle_a$-constant k_a . If $f : V \to X_\rho$ such that*

$$\rho(D_a f(x, y)) \leq \theta(||x||^p + ||y||^p) \tag{4.8}$$

for all $x, y \in V$, then there exists a unique generalized cubic mapping $C : V \to X_\rho$ such that

$$\rho\Big(f(x) - C(x)\Big) \leq \frac{L\theta}{k_a^3(1-L)}||x||^p \tag{4.9}$$

for all $x \in V$.

Proof. On taking $\phi(x, y) = \theta(||x||^p + ||y||^p)$ in the Theorem 4.3, we know that the inequality (4.8) holds. Now, if we take $L = \frac{k_a^3}{a^p}$, then the assumption $p > \log_a \frac{k_a^3}{a}$ implies that $0 < L < 1$. Also, it satisfies the inequalities (4.1). According to Theorem 4.3, we have the result as in the inequality (4.9). □

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