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TWO KINDS OF CONVERGENCES IN HYPERBOLIC SPACES IN THREE-STEP ITERATIVE SCHEMES

Seung Hyun Kim [a] and Mee Kwang Kang [b,*]

Abstract. In this paper, we introduce a new three-step iterative scheme for three finite families of nonexpansive mappings in hyperbolic spaces. And, we establish a strong convergence and a Δ-convergence of a given iterative scheme to a common fixed point for three finite families of nonexpansive mappings in hyperbolic spaces. Our results generalize and unify the several main results of [1, 4, 5, 9].

1. Introduction and Preliminaries

In 1990, Reich and Shafrir [8] introduced the concept of hyperbolic spaces, which includes normed linear spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls. From then Kohlenbach [6] generalized the concept of hyperbolic spaces of Reich and Shafrir [8] with $CAT(0)$-spaces in 2004. On the other hand, Dhompongsa and Panyanak [3] investigated the concept of Δ-convergence in $CAT(0)$-sapces and Khan et al. [4] introduced the concept of Δ-convergence in the more general setup of hyperbolic spaces. Recently, Akbulut and Gunduz [1] introduced a two-step algorithm for two finite families of nonexapnsive self-mappings in a hyperbolic space and established a strong convergence result and a Δ-convergence result.

In this paper, we consider a strong convergence and a Δ-convergence of a new three-step iterative scheme for three finite families of nonexpansive mappings in hyperbolic spaces. Our results extend and unify the corresponding ones in [1, 4, 5, 9].

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*Corresponding author.

Definition 1.1 ([6]). (X, d, W) is called a *hyperbolic space* if (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a mapping satisfying
(i) $d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$
for $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

Definition 1.2 ([10]). A hyperbolic space (X, d, W) is said to be *uniformly convex* if for any $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$,

$$d(x, z) \leq r, \;\; d(y, z) \leq r \;\text{ and }\; d(x, y) \geq \varepsilon r \Rightarrow d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r.$$

A function $\eta : (0, \infty) \times (0, 1] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called a *modulus* of uniform convexity. We call η a *monotone function* if it decreases with r (for a fixed ε).

A set $A(\{x_n\}) := \{x \in X : \inf_{y \in X} \limsup_{n\to\infty} d(y, x_n) = \limsup_{n\to\infty} d(x, x_n)\}$ is called the *asymptotic center* of a given sequence $\{x_n\}$ in a hyperbolic space X with an asymptotic radius $r(\{x_n\}) := \inf_{y \in X} \limsup_{n\to\infty} d(y, x_n)$.

Definition 1.3 ([4]). A sequence $\{x_n\}$ in X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of its all subsequences. In this case, x is called the Δ-*limit* of $\{x_n\}$ and denoted as $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Lemma 1.4 ([4]). *Let (X, d, W) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq t$, $\limsup_{n\to\infty} d(y_n, x) \leq t$ and $\limsup_{n\to\infty} d(W(x_n, y_n, a_n), x) = t$ for some $t \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.5 ([7]). *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 1.6 ([4]). *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and*

$r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, where $r(y_m, \{x_n\}) = \limsup_{n\to\infty} d(y_m, x_n)$, then $\lim_{m\to\infty} y_m = y$.

Now, we introduce a new three-step iterative scheme for three finite families of nonexpansive mappings in hyperbolic spaces. A self-mapping T on K is said to be *nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for $x, y \in K$.

Algorithm 1.1. *Let $\{T_n\}$, $\{S_n\}$ and $\{R_n\}(n \in \overline{1, N}(:= \{1, \cdots, N\}))$ be three finite families of nonexpansive self mappings on K, and $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(T_n x_n, S_n y_n, \alpha_n) \\ y_n = W(R_n x_n, T_n z_n, \beta_n) \\ z_n = W(x_n, R_n x_n, \gamma_n) \ \text{ for } \ n \in \mathbb{N}, \end{cases}$$

where $T_n = T_{n(modN)}$, $S_n = S_{n(modN)}$ and $R_n = R_{n(modN)}$ are nonexpansive mappings, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ for $N \in \mathbb{N}$.

2. Δ-convergence Result

In this section, we establish a Δ-convergence of Algorithm 1.1. Denote by $F(T)$, the set of fixed points of T. First of all, we prove the following lemmas needed in our results.

Lemma 2.1. *Let K be a nonempty closed convex subset of a hyperbolic space X and $\{T_i\}$, $\{S_i\}$ and $\{R_i\}$ be three finite families of nonexpansive self-mappings on K with $F := (\bigcap_{i=1}^{N} F(T_i)) \bigcap (\bigcap_{i=1}^{N} F(S_i)) \bigcap (\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by Algorithm 1.1. Then, for any $p \in F$, $\lim_{n\to\infty} d(x_n, p)$ exsits.*

Proof. For any $p \in F$,

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T_n x_n, S_n y_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(T_n x_n, p) + \alpha_n d(S_n y_n, p) \\ &= (1-\alpha_n) d(T_n x_n, T_n p) + \alpha_n d(S_n y_n, S_n p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n d(y_n, p) \\ &= (1-\alpha_n) d(x_n, p) + \alpha_n d(W(R_n x_n, T_n z_n, \beta_n), p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n \{(1-\beta_n) d(R_n x_n, p) + \beta_n d(T_n z_n, p)\} \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)d(x_n,p)+\alpha_n\{(1-\beta_n)d(R_nx_n,R_np)+\beta_n d(T_nz_n,T_np)\}\\
&\le (1-\alpha_n)d(x_n,p)+\alpha_n\{(1-\beta_n)d(x_n,p)+\beta_n d(z_n,p)\}\\
&= (1-\alpha_n\beta_n)d(x_n,p)+\alpha_n\beta_n d(z_n,p)\\
&= (1-\alpha_n\beta_n)d(x_n,p)+\alpha_n\beta_n d(W(x_n,R_nx_n,\gamma_n),p)\\
&\le (1-\alpha_n\beta_n)d(x_n,p)+\alpha_n\beta_n\{(1-\gamma_n)d(x_n,p)+\gamma_n d(R_nx_n,p)\}\\
&\le (1-\alpha_n\beta_n)d(x_n,p)+\alpha_n\beta_n\{(1-\gamma_n)d(x_n,p)+\gamma_n d(x_n,p)\}\\
&= d(x_n,p),
\end{aligned}
$$

which implies that $\lim_{n\to\infty} d(x_n,p)$ exists for $p\in F$. □

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with a monotone modulus of uniform convexity η and let $\{T_i\}$, $\{S_i\}$, $\{R_i\}$, $\{x_n\}$ and F be the same as cited in Lemma 2.1. Then $\lim_{n\to\infty} d(x_n,T_ix_n)=\lim_{n\to\infty} d(x_n,S_ix_n)=\lim_{n\to\infty} d(x_n,R_ix_n)=0$ for $i\in\overline{1,N}$.*

Proof. From Lemma 2.1, $\lim_{n\to\infty} d(x_n,p)$ exists for $p\in F$, say $\lim_{n\to\infty} d(x_n,p)=c$. In the case of $c=0$, the proof is trivial. Now, we deal with the case of $c>0$.

$$
\begin{aligned}
d(z_n,p) &= d(W(x_n,R_nx_n,\gamma_n),p)\\
&\le (1-\gamma_n)d(x_n,p)+\gamma_n d(R_nx_n,p)\\
&\le (1-\gamma_n)d(x_n,p)+\gamma_n d(x_n,p)\\
&= d(x_n,p),
\end{aligned}
$$

which implies that

$$\limsup_{n\to\infty} d(z_n,p)\le c \tag{2.1}$$

and

$$
\begin{aligned}
d(y_n,p) &= d(W(R_nx_n,T_nz_n,\beta_n),p)\\
&\le (1-\beta_n)d(R_nx_n,p)+\beta_n d(T_nz_n,p)\\
&\le (1-\beta_n)d(x_n,p)+\beta_n d(z_n,p)\\
&\le (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)\\
&= d(x_n,p),
\end{aligned}
$$

which implies that

$$\limsup_{n\to\infty} d(y_n,p)\le c. \tag{2.2}$$

Since $d(T_nx_n,p) \le d(x_n,p)$, $d(S_ny_n,p) \le d(y_n,p)$ and $d(R_nx_n,p) \le d(x_n,p)$ for $n \in \mathbb{N}$, from (2.1) and (2.2), we get

$$\limsup_{n\to\infty} d(T_nx_n,p) \le c, \ \limsup_{n\to\infty} d(S_ny_n,p) \le c \text{ and } \limsup_{n\to\infty} d(R_nx_n,p) \le c. \tag{2.3}$$

Moreover, since $c = \lim_{n\to\infty} d(x_{n+1},p) = \lim_{n\to\infty} d(W(T_nx_n,S_ny_n,\alpha_n),p)$, by Lemma 1.4, we have

$$\lim_{n\to\infty} d(T_nx_n,S_ny_n) = 0. \tag{2.4}$$

Now

$$\begin{aligned} d(x_{n+1},p) &= d(W(T_nx_n,S_ny_n,\alpha_n),p) \\ &\le (1-\alpha_n)d(T_nx_n,p) + \alpha_n d(S_ny_n,p) \\ &\le (1-\alpha_n)d(T_nx_n,p) + \alpha_n d(S_ny_n,T_nx_n) + \alpha_n d(T_nx_n,p) \\ &= d(T_nx_n,p) + \alpha_n d(S_ny_n,T_nx_n), \end{aligned}$$

which implies that

$$c \le \liminf_{n\to\infty} d(T_nx_n,p).$$

From (2.3), we have

$$\lim_{n\to\infty} d(T_nx_n,p) = c. \tag{2.5}$$

On the other hand,

$$\begin{aligned} d(T_nx_n,p) &\le d(T_nx_n,S_ny_n) + d(S_ny_n,p) \\ &\le d(T_nx_n,S_ny_n) + d(y_n,p), \end{aligned}$$

which implies that $c \le \liminf_{n\to\infty} d(y_n,p)$. Thus, from (2.2),
$c = \lim_{n\to\infty} d(y_n,p) = \lim_{n\to\infty} d(W(R_nx_n,T_nz_n,\beta_n),p)$. By Lemma 1.4, we have

$$\lim_{n\to\infty} d(R_nx_n,T_nz_n) = 0. \tag{2.6}$$

Now

$$\begin{aligned} d(y_n,p) &= d(W(R_nx_n,T_nz_n,\beta_n),p) \\ &\le (1-\beta_n)d(R_nx_n,p) + \beta_n d(T_nz_n,p) \\ &\le (1-\beta_n)d(R_nx_n,p) + \beta_n\{d(T_nz_n,p) + d(R_nx_n,p)\} \\ &= d(R_nx_n,p) + \beta_n d(T_nz_n,R_nx_n), \end{aligned}$$

which implies that $c \le \liminf_{n\to\infty} d(R_n x_n, p)$.
From (2.3), we have $\lim_{n\to\infty} d(R_n x_n, p) = c$. By (2.6), we obtain

$$\begin{aligned} d(R_n x_n, p) &\le d(R_n x_n, T_n z_n) + d(T_n z_n, p) \\ &\le d(R_n x_n, T_n z_n) + d(z_n, p), \end{aligned}$$

which implies that $c \le \liminf_{n\to\infty} d(z_n, p)$.
Thus, $c = \lim_{n\to\infty} d(z_n, p) = \lim_{n\to\infty} d(W(x_n, R_n x_n, \gamma_n), p)$. By Lemma 1.4, we have

$$\lim_{n\to\infty} d(x_n, R_n x_n) = 0. \tag{2.7}$$

Now

$$\begin{aligned} d(z_n, x_n) &= d(W(x_n, R_n x_n, \gamma_n), x_n) \\ &\le (1-\gamma_n) d(x_n, x_n) + \gamma_n d(R_n x_n, x_n) \\ &= \gamma_n d(R_n x_n, x_n), \end{aligned}$$

which implies by (2.7) that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{2.8}$$

From (2.6), (2.7) and (2.8), we have

$$\begin{aligned} d(x_n, T_n x_n) &\le d(x_n, R_n x_n) + d(R_n x_n, T_n z_n) + d(T_n z_n, T_n x_n) \\ &\le d(x_n, R_n x_n) + d(R_n x_n, T_n z_n) + d(z_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Next

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(T_n x_n, S_n y_n, \alpha_n), x_n) \\ &\le (1-\alpha_n) d(T_n x_n, x_n) + \alpha_n d(S_n y_n, x_n) \\ &\le (1-\alpha_n) d(T_n x_n, x_n) + \alpha_n \{d(S_n y_n, T_n x_n) + d(T_n x_n, x_n)\} \\ &= d(T_n x_n, x_n) + \alpha_n d(S_n y_n, T_n x_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $\lim_{n\to\infty} d(x_{n+i}, x_n) = 0$ for $i \in \overline{1, N}$. Further, observe that

$$\begin{aligned} d(x_n, T_{n+i} x_n) &\le d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i} x_{n+i}) + d(T_{n+i} x_{n+i}, T_{n+i} x_n) \\ &\le d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i} x_{n+i}) + d(x_{n+i}, x_n) \\ &= 2d(x_n, x_{n+i}) + d(x_{n+i}, T_{n+i} x_{n+i}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $\{d(x_n, T_i x_n)\}$ is a subsequence of $\bigcup_{i=1}^{N} \{d(x_n, T_{n+i} x_n)\}$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for $i \in \overline{1, N}$. Similarly, we can obtain $\lim_{n\to\infty} d(x_n, S_i x_n) = \lim_{n\to\infty} d(x_n, R_i x_n) = 0$ for $i \in \overline{1, N}$. $\square$

Now, we prove the following Δ-convergence result.

Theorem 2.3. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with a monotone modulus of uniform convexity η and $\{T_i\}$, $\{S_i\}$ and $\{R_i\}$ be three finite families of nonexpansive self-mappings on K with $F := (\bigcap_{i=1}^{N} F(T_i))\bigcap(\bigcap_{i=1}^{N} F(S_i))\bigcap(\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by Algorithm 1.1. Then $\{x_n\}$ Δ-converges to an element of F.*

Proof. From Lemma 2.1, $\{x_n\}$ is bounded. Therefore by Lemma 1.5, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$ for some $x \in K$. Assume that $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{u\}$. Then by Lemma 2.2, we get $\lim_{k\to\infty} d(x_{n_k}, T_i x_{n_k}) = \lim_{k\to\infty} d(x_{n_k}, S_i x_{n_k}) = \lim_{k\to\infty} d(x_{n_k}, R_i x_{n_k}) = 0$ for $i \in \overline{1, N}$. We claim that $u \in F$. Now, we define a sequence $\{v_m\}$ in K by $v_m = T_m u$ for $m \in \mathbb{N}$, where $T_m = T_{m(modN)}$. On the other hand,

$$\begin{aligned} d(v_m, x_{n_k}) &\leq d(T_m u, T_m x_{n_k}) + d(T_m x_{n_k}, T_{m-1} x_{n_k}) + \cdots + d(T_1 x_{n_k}, x_{n_k}) \\ &\leq d(u, x_{n_k}) + 2\sum_{i=1}^{m} d(x_{n_k}, T_i x_{n_k}). \end{aligned}$$

Therefore, we have

$$r(v_m, \{x_{n_k}\}) = \limsup_{k\to\infty} d(v_m, x_{n_k}) \leq \limsup_{k\to\infty} d(u, x_{n_k}) = r(u, \{x_{n_k}\}),$$

which implies that $|r(v_m, \{x_{n_k}\}) - r(u, \{x_{n_k}\})| \to 0$ as $k \to \infty$. By Lemma 1.6, we have $T_{m(modN)} u = u$, so u is a common fixed point of $\{T_i\}$. By the same argument, we can show that u is a common fixed point of $\{S_i\}$ and $\{R_i\}$. Therefore, $u \in F$. Moreover, by Lemma 2.1, $\lim_{n\to\infty} d(x_n, u)$ exists.

Assume that $x \neq u$. By the uniqueness of the asymptotic center,

$$\begin{aligned} \limsup_{k\to\infty} d(x_{n_k}, u) &< \limsup_{k\to\infty} d(x_{n_k}, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) = \limsup_{k\to\infty} d(x_{n_k}, u), \end{aligned}$$

which is a contradiction, so $x = u$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{x_{n_k}\}) = \{x\}$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to an element of F. □

Remark 2.1. Lemma 2.1, Lemma 2.2 and Theorem 2.1 generalize Lemma 2.1, Lemma 2.2 and Theorem 2.3 in [1], respectively.

3. Strong Convergence Result

In this section, we establish a strong convergence of Algorithm 1.1.

Recall that a sequence $\{x_n\}$ in a metric space X is said to be Fejér monotone with respect to a subset K of X if $d(x_{n+1}, p) \leq d(x_n, p)$ for $p \in K$ and $n \in \mathbb{N}$.

Lemma 3.1 ([2]). *Let K be a nonempty closed subset of a complete metric space (X, d) and let a sequence $\{x_n\}$ in X be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if $\lim_{n\to\infty} d(x_n, K) = 0$.*

Definition 3.2. Three finite families $\{T_i\}$, $\{S_i\}$, $\{R_i\}(i \in \overline{1, N})$ of self-mappings on K are said to satisfy *condition* $(\mathcal{A})$ if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for $t > 0$ such that

$$\max_{i\in\overline{1,N}}\{d(x, S_i x) + d(x, T_i x) + d(x, R_i x)\} \geq f(d(x, F)) \text{ for } x \in K,$$

where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Now, we prove the following strong convergence result.

Theorem 3.3. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with a monotone modulus of uniform convexity η and $\{T_i\}$, $\{S_i\}$ and $\{R_i\}$ be three finite families of nonexpansive self-mappings on K satisfy the condition $(\mathcal{A})$ with $F := (\bigcap_{i=1}^{N} F(T_i)) \bigcap (\bigcap_{i=1}^{N} F(S_i)) \bigcap (\bigcap_{i=1}^{N} F(R_i)) \neq \emptyset$. Suppose that $\{x_n\}$ is generated by Algorithm 1.1. Then a sequence $\{x_n\}$ in K converges strongly to an element of F.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists for $p \in F$ and from the Proof of Lemma 2.1, $\{x_n\}$ is Fejér monotone with respect to F. By Lemma 2.2, $\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = \lim_{n\to\infty} d(x_n, R_i x_n) = 0$ for $i \in \overline{1, N}$. From the condition $(\mathcal{A})$, we have

$$f(d(x_n, F)) \leq \max_{i\in\overline{1,N}}\{d(x, S_i x) + d(x, T_i x) + d(x, R_i x)\} \to 0 \text{ as } n \to \infty.$$

Since f is non-decreasing with $f(0) = 0$, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Hence, from Lemma 3.1, $\{x_n\}$ converges strongly to an element of F. $\square$

Remark 3.1. Theorem 3.1 generalizes Theorem 4.5 in [5].

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[a]Department of Mathematics, Kyungsung University, Busan 48434, Republic of Korea
Email address: jiny0610@hotmail.com

[b]Department of Mathematics, Dongeui University, Busan 47340, Republic of Korea
Email address: mee@deu.ac.kr