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CHENG-YAU OPERATOR AND GAUSS MAP OF TRANSLATION SURFACES

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Abstract. We study translation surfaces in the Euclidean 3-space $\mathbb{E}^3$ and the Gauss map N with respect to the so-called Cheng-Yau operator $\square$. As a result, we prove that the only translation surfaces with Gauss map N satisfying $\square N = AN$ for some 3×3 matrix A are the flat ones. We also show that the only translation surfaces with Gauss map N satisfying $\square N = AN$ for some nonzero 3×3 matrix A are the cylindrical surfaces.

1. Introduction

Suppose that M is a surface in the Euclidean 3-space $\mathbb{E}^3$ and S^2 denotes the unit sphere in $\mathbb{E}^3$ centered at the origin. The map $N : M \to S^2 \subset \mathbb{E}^3$ which sends each point p of M to the unit normal vector $N(p)$ to M at the point p is called the *Gauss map* of the surface M. Let us denote by Δ the Laplace operator on M corresponding to the induced metric on M from $\mathbb{E}^3$. Then it is well known that M has constant mean curvature if and only if $\Delta N = ||dN||^2 N$ ([10]).

Surfaces with Gauss map N which is an eigenfunction of Laplacian, that is, $\Delta N = \lambda N$ for some constant $\lambda \in R$, are the planes, circular cylinders and spheres ([3]). Generalizing this equation, F. Dillen and others ([5]) studied surfaces of revolution in the Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map N satisfies the condition

$$\Delta N = AN, \quad A \in R^{3\times 3}. \tag{1.1}$$

In particular, they established the following characterization theorem ([5]):

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Proposition 1.1. *A surface of revolution in $\mathbb{E}^3$ satisfies (1.1) for some 3×3 matrix A if and only if it is an open part of one of the planes, the spheres and the circular cylinders.*

For ruled surfaces in $\mathbb{E}^3$, the following characterization theorem was proved ([2]):

Proposition 1.2. *A ruled surface in $\mathbb{E}^3$ satisfies (1.1) for some 3×3 matrix A if and only if it is an open part of one of the planes and the circular cylinders.*

Generalized slant cylindrical surfaces (GSCS's) are natural extended notion of surfaces of revolution, cylindrical surfaces and tubes along a plane curve([8]). In [9], it was proved that among the GSCS's in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

The so-called Cheng-Yau operator $\Box$ (or, L_1) is a natural extension of the Laplace operator Δ (cf. [1], [4]). For the Cheng-Yau operator $\Box$, in [7] the following classification theorem was established: Let M be a surface of revolution in $\mathbb{E}^3$. Then the Gauss map N of M satisfies $\Box N = AN$ for some 3×3 matrix A if and only if M is an open part of the following surfaces: (1) a plane, (2) a right circular cone, (3) a circular cylinder, (4) a sphere.

Hence, it is quite reasonable to ask as follows.

Question 1.3. Among the translation surfaces in the Euclidean 3-space $\mathbb{E}^3$, which one satisfies the following condition?

$$\Box N = AN, \quad A \in R^{3\times 3}. \tag{1.2}$$

In this paper, we give a complete answer to the above question. We also show that the only translation surfaces with Gauss map N satisfying $\Box N = AN$ for some nonzero 3×3 matrix A are the cylindrical surfaces.

2. Cheng-Yau Operator and Examples

Suppose that M is a surface in E^3 with Gauss map N and S denotes the shape operator of M with respect to the Gauss map N. We put $P_0 = I, P_1 = tr(S)I - S$, where I denotes the identity operator acting on the tangent bundle of M. For each $k = 0, 1$, we define an operator $L_k : C^\infty(M) \to C^\infty(M)$ by $L_k(f) = -tr(P_k \circ \nabla^2 f)$, where $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator corresponding to the Hessian of f. Then, the operator L_0 is just the Laplace operator acting on M, i.e., $L_0 = \Delta$ and $L_1 = \Box$ is called the Cheng-Yau operator ([4]).

First, we give a lemma as follows ([1]):

Lemma 2.1. *Suppose that M is a surface in E^3 with Gaussian curvature K and mean curvature H. Then, for the Gauss map N of M one obtains*

$$\Box N = \nabla K + 2HKN, \tag{2.1}$$

where ∇K is the gradient of Gaussian curvature K.

Finally, using Lemma 2.1 we give some examples of surfaces with Gauss map N satisfying (1.2).

Examples 2.2.
(1) Flat surfaces. In this case, we have $\Box N = 0$, and hence flat surfaces satisfy $\Box N = AN$ for some 3×3 matrix A. The matrix A must be a singular matrix.
(2) Spheres: $|x - p|^2 = r^2$. In this case, we have $N(x) = \frac{1}{r}(x - p)$ and $\Box N = AN$ with $A = -\frac{2}{r^3}I$, where I denotes the identity matrix.
(3) Cylindrical surface $X(s,t) = (x(s), y(s), t), (s,t) \in I \times J$ over a unit speed curve $\alpha(s) = (x(s), y(s), 0), s \in I$. In this case, we have $N = (-y'(s), x'(s), 0)$ and $\Box N = 0$ so that the cylindrical surface satisfies $\Box N = AN$ with

$$A = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

3. Translation Surfaces

In this Section, we study translation surfaces with Gauss map N in the Euclidean 3-space $\mathbb{E}^3$. Let $X : M \to E^3$ be a translation surface in $\mathbb{E}^3$. We may assume that M is parametrized by

$$X(s,t) = (s, t, \tilde{f}(s) + \tilde{g}(t)), \quad (s,t) \in I \times J \tag{3.1}$$

for smooth functions $\tilde{f}$ and $\tilde{g}$ of the variables $s \in I$ and $t \in J$, respectively. Then, we have the natural frame $\{X_s, X_t\}$ given by

$$X_s = \frac{\partial X}{\partial s} = (1, 0, f), \quad X_t = \frac{\partial X}{\partial t} = (0, 1, g),$$

where $f = d\tilde{f}/ds, g = d\tilde{g}/dt$. Also we get the following:

$$X_{ss} = (0, 0, f'), \quad X_{st} = (0, 0, 0), \quad X_{tt} = (0, 0, g'),$$

$$X_s \times X_t = (-f, -g, 1), \quad Q = |X_s \times X_t| = \sqrt{1 + f^2 + g^2}$$

and

$$E =< X_s, X_s >= 1 + f^2, \quad F =< X_s, X_t >= fg, \quad G =< X_t, X_t >= 1 + g^2.$$

The unit normal vector field $N = \frac{1}{Q}(-f, -g, 1)$ to the surface M is called the Gauss map of the translation surface M. Since

$$l =< X_{ss}, N >= \frac{f'(s)}{Q}, \quad m =< X_{st}, N >= 0, \quad n =< X_{tt}, N >= \frac{g'(t)}{Q},$$

the Gaussian curvature K and mean curvature H are given by respectively,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f'(s)g'(t)}{Q^4} \tag{3.2}$$

and

$$2H = \frac{nE + lG - 2mF}{f^2 + g^2 + 1} = \frac{(1 + f^2)g'(t) + (1 + g^2)f'(s)}{Q^3}, \tag{3.3}$$

where $f'(s)$ and $g'(t)$ are the derivatives of $f(s)$ and $g(t)$, respectively ([6]). If we put

$$e_1 = \frac{1}{\sqrt{E}} X_s \tag{3.4}$$

and

$$e_2 = \frac{1}{Q\sqrt{E}}\{-F\ X_s + E\ X_t\}, \tag{3.5}$$

then $\{e_1, e_2, N = e_1 \times e_2\}$ is an orthonormal frame field on the translation surface M.

The gradient ∇K of the Gaussian curvature K of M can be computed as follows:

$$\nabla K = e_1(K)e_1 + e_2(K)e_2 = VX_s + WX_t\ , \tag{3.6}$$

where we put

$$V = \frac{1}{Q^2}\{GX_s(K) - FX_t(K)\}, \quad W = \frac{1}{Q^2}\{-FX_s(K) + EX_t(K)\}, \tag{3.7}$$

respectively. By a straightforward computation, we get the following:

$$X_s(K) = Q^{-4}f''(s)g'(t) - 4Q^{-6}f(s)f'(s)^2 g'(t), \tag{3.8}$$

$$X_t(K) = Q^{-4}f'(s)g''(t) - 4Q^{-6}f'(s)g(t)g'(t)^2. \tag{3.9}$$

4. Translation Surfaces with Gauss Map N satisfying $\Box N = AN$

In this section, we suppose that the Gauss map N of the translation surface M parametrized by (3.1) satisfies for a 3×3 matrix $A = (a_{ij})$

$$\Box N = AN. \tag{4.1}$$

Recall that the Gauss map N of the translation surface M is given by

$$N = \frac{1}{Q}(-f(s), -g(t), 1). \tag{4.2}$$

Then, it follows from (2.1), (3.6) and (4.1) that

$$VX_s + WX_t + 2HKN = AN. \tag{4.3}$$

Substituting X_s, X_t in Section 3 and N in (4.2) into the equation (4.3), we have the following:

$$QV - 2HKf = A_1, \tag{4.4}$$

$$QW - 2HKg = A_2 \tag{4.5}$$

and

$$QVf + QWg + 2HK = A_3, \tag{4.6}$$

where we put

$$A_1 = -a_{11}f - a_{12}g + a_{13},\ A_2 = -a_{21}f - a_{22}g + a_{23},\ A_3 = -a_{31}f - a_{32}g + a_{33}.$$

From (3.2), (3.3) and (3.7-9), we get the following:

$$\begin{aligned} Q^2V &= GX_s(K) - FX_t(K) \\ &= (1+g^2)\{Q^{-4}f''g' - 4Q^{-6}f(f')^2g'\} - fg\{Q^{-4}f'g'' - 4Q^{-6}f'g(g')^2\} \\ &= Q^{-6}[\{Q^2f''g' - 4f(f')^2g'\}(1+g^2) + fg\{-Q^2f'g'' + 4f'g(g')^2\}], \end{aligned} \tag{4.7}$$

(4.8)

$$\begin{aligned} Q^2W &= -FX_s(K) + EX_t(K) \\ &= -fg\{Q^{-4}f''g' - 4Q^{-6}f(f')^2g'\} + (1+f^2)\{Q^{-4}f'g'' - 4Q^{-6}f'g(g')^2\} \\ &= Q^{-6}[\{4f(f')^2g' - Q^2f''g'\}(fg) + \{Q^2f'g'' - 4f'g(g')^2\}(1+f^2)] \end{aligned}$$

and

$$2HK = Q^{-7}f'g'\{(1+f^2)g' + (1+g^2)f'\}. \tag{4.9}$$

First, note that (4.6) can be rewritten as the following form:

$$Q^8Vf + Q^8Wg + 2HKQ^7 = A_3Q^7. \tag{4.10}$$

Substituting (4.7-9) into (4.10), we obtain the following:

$$A_3Q^7 = Q^2ff''g' + (f')^2\{-4f^2g' + (1+g^2)g'\} + f'\{Q^2gg'' - 4g^2(g')^2 + (1+f^2)(g')^2\},$$

and hence we have

$$Q^2ff''g' = D_1(f')^2 + D_2f' + A_3Q^7, \tag{4.11}$$

where we put

$$D_1 = (4f^2 - g^2 - 1)g', \quad D_2 = (4g^2 - f^2 - 1)(g')^2 - Q^2gg''.$$

Second, (4.4) implies that

$$Q^8V - 2HKQ^7f = A_1Q^7.$$

Hence, in the same manner as above we get the following:

$$Q^2f''(1+g^2)g' = B_1(f')^2 + B_2f' + A_1Q^7, \tag{4.12}$$

where

$$B_1 = 5f(1+g^2)g', \quad B_2 = Q^2fgg'' + f(g')^2(1+f^2-4g^2).$$

Third, (4.5) implies similarly that

$$Q^2ff''gg' = C_1(f')^2 + C_2f' - A_2Q^7, \tag{4.13}$$

where

$$C_1 = gg'(4f^2 - 1 - g^2), \quad C_2 = (1+f^2)\{Q^2g'' - 5g(g')^2\}.$$

Let us combine (4.11) with (4.13). From (4.13) − (4.11) × g one has

$$R_2f' = R_3Q^7, \tag{4.14}$$

where we put

$$R_2 = C_2 - D_2g, \quad R_3 = A_2 + A_3g.$$

Also, it follows from (4.12) × f − (4.13) × g − (4.11) that

$$S_1(f')^2 + S_2f' = S_3Q^7, \tag{4.15}$$

where

$$S_1 = B_1f - C_1g - D_1, \quad S_2 = B_2f - C_2g - D_2, \quad S_3 = A_3 - A_2g - A_1f.$$

Together with (4.14), (4.15) × R_2^2 gives

$$S_1R_3^2Q^7 = R_2(R_2S_3 - R_3S_2). \tag{4.16}$$

Note that we have

$$S_1 = B_1 f - C_1 g - D_1 = Q^2(1+g^2)g' \tag{4.17}$$

and

$$R_2 = C_2 - D_2 g = Q^2\{Q^2 g'' - 4g(g')^2\}. \tag{4.18}$$

Hence, it follows from (4.16) that the function $f(s)$ satisfies

$$p(f) := (1+g^2)^2(g')^2 R_3^4 (Q^2)^7 - \{Q^2 g'' - 4g(g')^2\}^2 (R_2 S_3 - R_3 S_2)^2 = 0, \tag{4.19}$$

which is a polynomial in f of degree 18. The coefficients are functions of $g(t), g'(t)$ and $g''(t)$. Let us denote the polynomial $p(f)$ as follows:

$$p(f) = \sum_{j=0}^{18} b_j f^j. \tag{4.20}$$

Finally, we prove the following lemma which plays a key role in the proof of our theorems.

Lemma 4.1. *Let M be a translation surface in the Euclidean 3-space $\mathbb{E}^3$ parametrized by (3.1) which satisfies $\square N = AN$ for a 3×3 matrix A. Suppose that M is non-flat. Then the matrix A is the zero matrix.*

Proof. Since the translation surface M is not flat, it follows from (3.2) that the sets $I_1 = \{s \in I | f'(s) \neq 0\}$ and $J_1 = \{t \in J | g'(t) \neq 0\}$ are nonempty, respectively. Note that the coefficients $b_j, j = 0, 1, \cdots, 18$ are functions of $g(t)$, $g'(t)$ and $g''(t)$. If the polynomial $p(f)$ is nontrivial at some point $t = t_0$, that is, one of b_j at $t = t_0$ is nonzero, then the function $f(s)$ must be constant on I. This contradiction shows that the coefficients $b_j, j = 0, 1, \cdots, 18$ must vanish on the whole domain J.

On the open set $J_1 = \{t \in J | g'(t) \neq 0\}$, we have

$$b_{18} = (1+g^2)^2(g')^2(a_{21} + a_{31} g)^4 = 0, \tag{4.21}$$

which shows that

$$a_{21} = a_{31} = 0. \tag{4.22}$$

Hence we get $b_{17} = 0$ and

$$b_{16} = -(a_{11})^2 (g'')^4 = 0. \tag{4.23}$$

First of all, we consider the following case.

Case 1. $g''(t) \equiv 0$ on J_1. In this case, we have

$$R_2 = -4g(g')^2 Q^2, \quad S_2 = (1+f^2)(g')^2 Q^2. \tag{4.24}$$

Hence one obtains

$$p(f) = (g')^2 Q^4 q(f), \tag{4.25}$$

where we put

$$q(f) = (1+g^2)^2 R_3^4 Q^{10} - 16g^2(g')^6\{4gS_3 + (1+f^2)R_3\}^2. \tag{4.26}$$

Note that in this case, it follows from (4.22) that

$$\begin{aligned} R_3 &= -\{a_{32}g^2 + (a_{22}-a_{33})g - a_{23}\}, \\ S_3 &= a_{11}f^2 + \{(a_{12}+a_{21})g - (a_{13}+a_{31})\}f + \{a_{22}g^2 - (a_{23}+a_{32})g + a_{33}\}. \end{aligned} \tag{4.27}$$

Hence, using the expression $q(f) = \sum_{j=0}^{10} c_j f^j$ we have

$$c_{10} = (1+g^2)^2 R_3^4 = 0, \tag{4.28}$$

which implies $R_3 = 0$, that is,

$$a_{32} = a_{23} = 0, \quad a_{22} = a_{33}. \tag{4.29}$$

It follows from (4.26) with $R_3 = 0$ that $S_3 = 0$, which implies

$$a_{11} = a_{22} = a_{33} = 0, \quad a_{12}+a_{21} = a_{13}+a_{31} = a_{23}+a_{32} = 0. \tag{4.30}$$

Together with (4.22) and (4.29), (4.30) shows that the matrix A is the zero matrix.

Second, we consider the following case.

CASE 2. $g''(t) \neq 0$ on a nonempty open set $J_2 \subset J_1$. In this case, it follows from (4.23) that

$$a_{11} = 0. \tag{4.31}$$

Using (4.22) and (4.31), we obtain

$$\begin{aligned} b_{14} &= (1+g^2)^2(g')^2\{a_{32}g^2 + (a_{22}-a_{33})g - a_{23}\}^4 \\ &\quad - (g'')^4(a_{12}g - a_{13})^2 = 0, \end{aligned} \tag{4.32}$$

$$\begin{aligned} b_{13} &= -2(g'')^3(a_{12}g - a_{13})[\{a_{22}g^2 - (a_{23}+a_{32})g + a_{33}\}g'' \\ &\quad + \{a_{32}g^2 + (a_{22}-a_{33})g - a_{23}\}(g')^2] = 0 \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} b_{12} &= 7(1+g^2)^3(g')^2\{a_{32}g^2 + (a_{22}-a_{33})g - a_{23}\}^4 \\ &\quad - (g'')^2[\{a_{22}g^2 - (a_{23}+a_{32})g + a_{33}\}g'' \\ &\quad + \{a_{32}g^2 + (a_{22}-a_{33})g - a_{23}\}(g')^2]^2 \\ &\quad - 2(g'')^3(a_{12}g - a_{13})^2\{3(1+g^2)g'' - 8g(g')^2\} = 0. \end{aligned} \tag{4.34}$$

With the help of (4.33), we consider the following two subcases:

SUBCASE 2-1. $a_{12}g - a_{13} \equiv 0$ on J_2. In this case we have

$$a_{12} = a_{13} = 0, \tag{4.35}$$

and hence from (4.32) one obtains

$$a_{32} = a_{23} = a_{22} - a_{33} = 0. \tag{4.36}$$

Furthermore, together with (4.35) and (4.36) it follows from (4.34) that

$$a_{22} = a_{33} = a_{23} + a_{32} = 0. \tag{4.37}$$

Hence, we see that the matrix A is the zero matrix.

SUBCASE 2-2. $a_{12}g - a_{13} \neq 0$ on a nonempty open set $J_3 \subset J_2$. In this case, on J_3 we have from (4.33)

$$\begin{aligned} &\{a_{22}g^2 - (a_{23} + a_{32})g + a_{33}\}g'' \\ &+ \{a_{32}g^2 + (a_{22} - a_{33})g - a_{23}\}(g')^2 = 0. \end{aligned} \tag{4.38}$$

It follows from (4.32) and (4.34) that on J_3 we get

$$7(1 + g^2)g'' = 2\{3(1 + g^2)g'' - 8g(g')^2\}, \tag{4.39}$$

which implies

$$(1 + g^2)g'' + 16g(g')^2 = 0. \tag{4.40}$$

Together with (4.38), this shows that on J_3 one has

$$\begin{aligned} &(1 + g^2)\{a_{32}g^2 + (a_{22} - a_{33})g - a_{23}\} \\ &- 16g\{a_{22}g^2 - (a_{23} + a_{32})g + a_{33}\} = 0, \end{aligned} \tag{4.41}$$

which implies

$$a_{22} = a_{33} = a_{23} = a_{32} = 0. \tag{4.42}$$

Hence (4.32) also shows that

$$a_{12} = a_{13} = 0, \tag{4.43}$$

which is a contradiction. Therefore this subcase can not occur.

Thus, summarizing the above discussions shows that A is the zero matrix. This completes the proof of Lemma 4.1. $\square$

With the aid of Lemma 4.1, we establish the following classification theorem for translation surfaces with Gauss map N satisfying $\square N = AN$ for some 3×3 matrix A as follows.

Theorem 4.2. *Let M be a translation surface in the Euclidean 3-space $\mathbb{E}^3$. Then the only translation surfaces with Gauss map N satisfying $\Box N = AN$ for some 3×3 matrix A are the flat ones.*

Proof. We consider a translation surface M satisfying $\Box N = AN$ for some 3×3 matrix A. Suppose that M is non-flat. Then, it follows from Lemma 4.1 that A is the zero matrix. Hence we have $\Box N = 0$, which together with Lemma 2.1 implies M is flat. This contradiction shows that the translation surface M is flat.

The converse is obvious. This completes the proof of Theorem 4.2. □

When A is a nonzero matrix, we obtain the following.

Theorem 4.3. *Let M be a translation surface in the Euclidean 3-space $\mathbb{E}^3$. Then the only translation surfaces with Gauss map N satisfying $\Box N = AN$ for some nonzero 3×3 matrix A are the cylindrical surfaces.*

Proof. Suppose that $\Box N = AN$ for some nonzero 3×3 matrix A. Then, it follows from Theorem 4.2 that the surface M is flat, hence we have $AN = \Box N = 0$. We denote by $Ker(A)$ the kernel space of the matrix A, that is,

$$Ker(A) = \{x \in \mathbb{E}^3 | Ax = 0\}.$$

Then the image of the Gauss map N lies in the space $Ker(A)$.

Since A is nonzero, $Ker(A)$ is of at most 2-dimensional. Hence there exists a unit vector $a = (a_1, a_2, a_3)$ which is orthogonal to $Ker(A)$. Since $N = \frac{1}{Q}(-f, -g, 1)$, we obtain

$$a_1 f(s) + a_2 g(t) - a_3 = 0. \tag{4.43}$$

Since one of a_1 and a_2 is nonzero, (4.43) shows that one of $f(s)$ and $g(t)$ is constant. Therefore, without loss of generality we may assume that $f(s) = b$ for some constant b and hence we have $\tilde{f}(s) = bs + c$ with $c \in \mathbb{R}$.

It follows from (3.1) that the translation surface M is parametrized by

$$\begin{aligned} X(s,t) &= (s, t, bs + c + \tilde{g}(t)) \\ &= s(1, 0, b) + (0, t, c + \tilde{g}(t)), \end{aligned} \tag{4.44}$$

which shows that M is a cylindrical surface.

The converse is obvious. This completes the proof of Theorem 4.3. □

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