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ESTIMATE OF THIRD ORDER HANKEL DETERMINANT FOR A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH CARDIOID DOMAIN

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Abstract. The present paper deals with the upper bound of third order Hankel determinant for a certain subclass of analytic functions associated with Cardioid domain in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. The results proved here generalize the results of several earlier works.

1. Introduction

Let us denote by $\mathcal{A}$, the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consists of univalent functions in E. The most remarkable result in the theory of univalent functions was Bieberbach's conjecture, established by Bieberbach [4]. It states that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [6], proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were come into existence and it gave rise to some new subclasses of $\mathcal{S}$.

Let f and g be two analytic functions in E. We say that f is subordinate to g (denoted as $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then this subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

We first present an overview of some basic classes, in order to introduce our class:
$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$, the class of starlike functions.

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$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}$, the class of convex functions.

Reade [25] introduced the concept of close-to-star functions. The class of close-to-star functions generally denoted by $\mathcal{CS}^*$, consists of functions $f \in \mathcal{A}$ such that

$$Re\left(\frac{f(z)}{g(z)}\right) > 0, \quad g \in \mathcal{S}^*.$$

$R = \{f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E\}$, the class of bounded turning functions introduced and studied by MacGregor [15].

$R^{'} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E \right\}$, the subclass of close-to-star functions studied by MacGregor [16].

Later on, Murugusundramurthi and Magesh [20] studied the following unified class:

$$\mathcal{R}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, z \in E \right\}.$$

Particularly, $\mathcal{R}(1) \equiv \mathcal{R}$ and $\mathcal{R}(0) \equiv \mathcal{R}^{'}$.

For $f \in \mathcal{A}$, the relation $f \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2$ means that f lies in the region bounded by the cardioid given by

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0.$$

Various subclasses of analytic functions have been studied by subordinating to different kind of functions. Malik et al. [17, 18], Sharma et al. [27] and Raza et al. [24] studied certain classes af analytic functions associated with cardioid domain. Shi et al. [28] studied the classes $\mathcal{S}^*_{car}$, $\mathcal{K}_{car}$ and $\mathcal{R}_{car}$ associated with cardioid domain.

Motivated by these works, we define the following class of analytic functions by subordinating to $1 + \frac{4}{3}z + \frac{2}{3}z^2$.

Definition 1.1 A function $f \in \mathcal{A}$ is said to be in the *class* $\mathcal{R}^{\alpha}_{car}$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

We have the following observations:

(i) $\mathcal{R}^0_{car} \equiv \mathcal{R}^{'}_{car}$.

(ii) $\mathcal{R}^1_{car} \equiv \mathcal{R}_{car}$.

In 1976, Noonan and Thomas [21] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q = 2, n = 1$ and $a_1 = 1$, the Hankel determinant reduces to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional. Fekete and Szegö [8] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \mathcal{S}$.

Also for $q = 2, n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is Hankel determinant of order 2.

One more very useful functional is $J_{n,m}(f) = a_n a_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which was investigated by Ma [14] and is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions as it is very useful in establishing the bounds for the third Hankel determinant.

Furthermore, for $q = 3, n = 1$, the Hankel determinant yields

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

which is the third order Hankel determinant.
For $f \in \mathcal{S}$ and $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1}$$

Extensive work has been done on the estimation of second Hankel determinant by various authors including Noor [22], Ehrenborg [7], Layman [11], Singh [29], Mehrok and Singh [19] and Janteng et al. [9]. It is little bit complicated to establish the upper bound for the third order Hankel determinant. It was Babalola [3], who firstly obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Later on, a few researchers including Shanmugam et al. [26], Bucur et al. [5], Altinkaya and Yalcin [1], Singh and Singh [30] have worked in the direction of third Hankel determinant for various subclasses of analytic functions

In the present paper, we establish the upper bounds for the initial coefficients, Fekete-Szegö inequality, Zalcman functional, second Hankel determinant and third hankel determinant, for the class $\mathcal{R}^{\alpha}_{car}$. Also various known results follow as particular cases.

Let $\mathcal{P}$ denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, the following lemmas have been used:

Lemma 1 ([23, 10]). *If $p \in \mathcal{P}$, then*

$$|p_k| \leq 2, k \in \mathbb{N},$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2},$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1,$$

$$|p_{n+2k} - \lambda p_n p_k^2| \leq 2(1 + 2\lambda), (\lambda \in \mathbb{R}),$$

$$|p_m p_n - p_k p_l| \leq 4, (m + n = k + l; m, n \in \mathbb{N}),$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2max\{1, |2\rho - 1|\}.$$

Lemma 2 ([2]). *Let $p \in \mathcal{P}$, then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [23] *that*

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

Lemma 3 ([12, 13]). *If $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \leq 1$ and $|z| \leq 1$.

2. Bounds of $|H_3(1)|$ for the Class $\mathcal{R}^{\alpha}_{car}$

Theorem 2.1 *If $f \in \mathcal{R}^{\alpha}_{car}$, then*

$$|a_2| \leq \frac{4}{3(1+\alpha)}, \tag{2}$$

$$|a_3| \leq \frac{4}{3(1+2\alpha)}, \tag{3}$$

$$|a_4| \leq \frac{4}{3(1+3\alpha)}, \tag{4}$$

and

$$|a_5| \leq \frac{3}{1+4\alpha}. \tag{5}$$

The estimates are sharp.

Proof. Since $f \in \mathcal{R}^{\alpha}_{car}$, by the principle of subordination, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2. \tag{6}$$

Define $p(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$, which implies $w(z) = \dfrac{p(z)-1}{p(z)+1}$.
On expanding, we have

(7)

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ...$$

Also

$1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2 = 1 + \frac{2}{3}p_1 z$

$$+\left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right)z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1p_2\right)z^3 + \left(\frac{2}{3}p_4 + \frac{1}{24}p_1^4 - \frac{1}{6}p_2^2 - \frac{1}{3}p_1p_3\right)z^4 + ... \tag{8}$$

Using (7) and (8), (6) yields

$1 + (1+\alpha)a_2 z + (1+2\alpha)a_3 z^2 + (1+3\alpha)a_4 z^3 + (1+4\alpha)a_5 z^4 + ...$

(9)

$$= 1 + \frac{2}{3}p_1 z + \left(\frac{2}{3}p_2 - \frac{p_1^2}{6}\right)z^2 + \left(\frac{2}{3}p_3 - \frac{1}{3}p_1p_2\right)z^3 + \left(\frac{2}{3}p_4 + \frac{1}{24}p_1^4 - \frac{1}{6}p_2^2 - \frac{1}{3}p_1p_3\right)z^4 + ...$$

Equating the coefficients of z, z^2, z^3 and z^4 in (9) and on simplification, we obtain

$$a_2 = \frac{2}{3(1+\alpha)}p_1, \tag{10}$$

$$a_3 = \frac{1}{1+2\alpha}\left[\frac{2}{3}p_2 - \frac{p_1^2}{6}\right], \tag{11}$$

$$a_4 = \frac{1}{3(1+3\alpha)}\left[2p_3 - p_1p_2\right], \tag{12}$$

and

$$a_5 = \frac{1}{24(1+4\alpha)}\left[16p_4 + p_1^4 - 4p_2^2 - 8p_1p_3\right]. \tag{13}$$

Using first inequality of Lemma 1 in (10), the result (2) is obvious.
From (11), we have

$$|a_3| = \frac{2}{3(1+2\alpha)}\left|p_2 - \frac{1}{4}p_1^2\right|. \tag{14}$$

Using sixth inequality of Lemma 1 in (14), the result (3) can be easily obtained.
(12) can be expressed as

$$a_4 = \frac{2}{3(1+3\alpha)}\left[p_3 - \frac{1}{2}p_1p_2\right]. \tag{15}$$

On applying inequality 3 of Lemma 1 in (15), the result (4) is obvious.
Further, (13) can be re-written as

$$a_5 = \frac{1}{24(1+4\alpha)}\left[16\left(p_4 - \frac{1}{4}p_2^2\right) - 8p_1\left(p_3 - \frac{1}{8}p_1^3\right)\right]. \tag{16}$$

On applying triangle inequality and using third inequality of Lemma 1, the result (5) is obvious from (16).
The results (2), (3), (4) and (5) are sharp for the function f given by

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

□

On putting $\alpha = 0$, Theorem 2.1 yields the following result:

Corollary 2.1 *If $f \in \mathcal{R}'_{car}$, then*

$$|a_2| \leq \frac{4}{3}, |a_3| \leq \frac{4}{3}, |a_4| \leq \frac{4}{3}, |a_5| \leq 3.$$

For $\alpha = 1$, Theorem 2.1 gives the following result due to Shi et al. [28]:

Corollary 2.2 *If $f \in \mathcal{R}_{car}$, then*

$$|a_2| \leq \frac{2}{3}, |a_3| \leq \frac{4}{9}, |a_4| \leq \frac{1}{3}, |a_5| \leq \frac{3}{5}.$$

Theorem 2.2 *If $f \in \mathcal{R}^{\alpha}_{car}$, then*

$$|a_3 - a_2^2| \le \frac{4}{3(1+2\alpha)}. \tag{17}$$

Proof. From (10) and (11), we have

$$|a_3 - a_2^2| = \frac{2}{3(1+2\alpha)} \left| p_2 - \frac{3\alpha^2 + 22\alpha + 11}{12(1+\alpha)^2} p_1^2 \right|. \tag{18}$$

Using sixth inequality of Lemma 1, (18) takes the form

$$|a_3 - a_2^2| \le \frac{4}{3(1+2\alpha)} \max\left\{1, \frac{5 + 10\alpha - 3\alpha^2}{6(1+\alpha)^2}\right\}. \tag{19}$$

But $\dfrac{5 + 10\alpha - 3\alpha^2}{6(1+\alpha)^2} \le 1$ for $0 \le \alpha \le 1$.
Hence, the result (17) is obvious from (19). □

Substituting for $\alpha = 0$, Theorem 2.2 yields the following result:

Corollary 2.3 *If $f \in \mathcal{R}'_{car}$, then*

$$|a_3 - a_2^2| \le \frac{4}{3}.$$

Putting $\alpha = 1$, Theorem 2.2 yields the following result:

Corollary 2.4 *If $f \in \mathcal{R}_{car}$, then*

$$|a_3 - a_2^2| \le \frac{4}{9}.$$

Theorem 2.3 *If $f \in \mathcal{R}^{\alpha}_{car}$, then*

$$|a_2 a_3 - a_4| \le \frac{4}{3(1+3\alpha)}. \tag{20}$$

Proof. Using (10), (11), (12) and after simplification, we have

$$\begin{aligned} |a_2 a_3 - a_4| = \frac{1}{9(1+\alpha)(1+2\alpha)(1+3\alpha)} \big| &(1+3\alpha)p_1^3 - (7 + 21\alpha + 6\alpha^2)p_1 p_2 \\ &+ 6(1+\alpha)(1+2\alpha)p_3 \big|. \end{aligned} \tag{21}$$

On applying Lemma 2 in (21), it yields (20). □

For $\alpha = 0$, the following result is a consequence of Theorem 2.3:

Corollary 2.5 *If $f \in \mathcal{R}'_{car}$, then*

$$|a_2 a_3 - a_4| \le \frac{4}{3}.$$

For $\alpha = 1$, we can obtain the following result from Theorem 2.3:

Corollary 2.6 If $f \in \mathcal{R}_{car}$, then

$$|a_2a_3 - a_4| \leq \frac{1}{3}.$$

Theorem 2.4 *If $f \in \mathcal{R}_{car}^{\alpha}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{16}{9(1+2\alpha)^2}. \tag{22}$$

The bound is sharp.

Proof. Using (10), (11) and (12), we have

$$\begin{aligned}|a_2a_4 - a_3^2| =& \bigg|\frac{4p_1p_3}{9(1+\alpha)(1+3\alpha)} - \frac{2\alpha^2p_1^2p_2}{9(1+\alpha)(1+2\alpha)^2(1+3\alpha)} - \frac{p_1^4}{36(1+2\alpha)^2}\\ &- \frac{4}{9(1+2\alpha)^2}p_2^2\bigg|.\end{aligned}$$

Substituting for p_2 and p_3 from Lemma 3 and letting $p_1 = p$, we get

$$\begin{aligned}|a_2a_4 - a_3^2| = &\frac{1}{36(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\big| - (3\alpha^2+4\alpha+1)p^4 + 4\alpha^2p^2(4-p^2)x\\ &- 4(1+2\alpha)^2p^2(4-p^2)x^2 - 4(1+\alpha)(1+3\alpha)(4-p^2)^2x^2\\ &+ 8(1+2\alpha)^2p(4-p^2)(1-|x|^2)z\big|.\end{aligned}$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0, 2]$. Then by using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0, 1]$, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| \leq &\frac{1}{36(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\big[(3\alpha^2+4\alpha+1)p^4 + 4\alpha^2p^2(4-p^2)t\\ &+ 4(1+2\alpha)^2p^2(4-p^2)t^2 + 4(1+\alpha)(1+3\alpha)(4-p^2)^2t^2\\ &+ 8(1+2\alpha)^2p(4-p^2) - 8(1+2\alpha)^2p(4-p^2)t^2\big] = F(p,t).\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial t} = &\frac{1}{36(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\big[4\alpha^2p^2(4-p^2) + 8(4-p^2)(2-p)t[\alpha^2(6-p)\\ &+ 8\alpha + 2]\big] \geq 0.\end{aligned}$$

Therefore $F(p,t)$ is an increasing function of t and so

$$\begin{aligned}\max\{F(p,t)\} = F(p,1) = &\frac{1}{36(1+\alpha)(1+2\alpha)^2(1+3\alpha)}\big[(3\alpha^2+4\alpha+1)p^4\\ &+4\alpha^2p^2(4-p^2) + 4(1+2\alpha)^2p^2(4-p^2) + 4(1+\alpha)(1+3\alpha)(4-p^2)^2\big] = H(p).\end{aligned}$$

$H'(p) = 0$ gives $p = 0$. Also $H''(p) < 0$ for $p = 0$.

This implies $\max\{H(p)\} = H(0) = \dfrac{16}{9(1+2\alpha)^2}$, which proves (22).
The result is sharp for $p_1 = 0$, $p_2 = \pm 2$ and $p_3 = 0$. □

Putting $\alpha = 0$, Theorem 2.4 gives the following result:

Corollary 2.7 *If $f \in \mathcal{R}'_{car}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{16}{9}.$$

Substituting for $\alpha = 1$, the following result is obvious from Theorem 2.4:

Corollary 2.8 *If $f \in \mathcal{R}_{car}$, then*

$$|a_2a_4 - a_3^2| \leq \frac{16}{81}.$$

Theorem 2.5 *If $f \in \mathcal{R}^{\alpha}_{car}$, then*

$$|H_3(1)| \leq \frac{4(55 + 550\alpha + 1959\alpha^2 + 2868\alpha^3 + 1356\alpha^4)}{27(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}. \tag{23}$$

Proof. By using (3), (4), (5), (17), (20) and (22) in (1), the result (23) can be easily obtained. □

For $\alpha = 0$, Theorem 2.5 yields the following result:

Corollary 2.9 *If $f \in \mathcal{R}'_{car}$, then*

$$|H_3(1)| \leq \frac{220}{27}.$$

For $\alpha = 1$, Theorem 2.3 yields the following result:

Corollary 2.10 *If $f \in \mathcal{R}_{car}$, then*

$$|H_3(1)| \leq \frac{1697}{3645}.$$

3. Bounds of $|H_3(1)|$ for Two-fold and Three-fold Symmetric Functions

A function f is said to be *n-fold symmetric* if is satisfy the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.

By $\mathcal{S}^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class $\mathcal{S}$.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{24}$$

An analytic function f of the form (24) belongs to the family $\mathcal{R}_{car}^{\alpha(n)}$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2, p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^n = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E \right\}. \tag{25}$$

Theorem 3.1 *If $f \in \mathcal{R}_{car}^{\alpha(2)}$, then*

$$|H_3(1)| \leq \frac{16}{9(1+2\alpha)(1+4\alpha)}. \tag{26}$$

Proof. If $f \in \mathcal{R}_{car}^{\alpha(2)}$, so there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2. \tag{27}$$

Using (24) and (25) for $n = 2$, (27) yields

$$a_3 = \frac{2}{3(1+2\alpha)} p_2, \tag{28}$$

$$a_5 = \frac{1}{1+4\alpha}\left(\frac{2}{3}p_4 - \frac{1}{6}p_2^2\right). \tag{29}$$

Also

$$H_3(1) = a_3 a_5 - a_3^3. \tag{30}$$

Using (28) and (29) in (30), it yields

$$H_3(1) = \frac{4}{9(1+2\alpha)(1+4\alpha)} p_2 \left[p_4 - \frac{3(1+2\alpha)^2 + 8(1+4\alpha)}{12(1+2\alpha)^2} p_2^2 \right]. \tag{31}$$

On applying triangle inequality and using third inequality of Lemma 1, we can easily get the result (26). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 3.1:

Corollary 3.1 *If* $f \in \mathcal{R}_{car}^{'(2)}$, *then*

$$|H_3(1)| \leq \frac{16}{9}.$$

For $\alpha = 1$, Theorem 3.1 agrees with the following result:

Corollary 3.2 *If* $f \in \mathcal{R}_{car}^{(2)}$, *then*

$$|a_3 - a_2^2| \leq \frac{16}{135}.$$

Theorem 3.2 *If* $f \in \mathcal{R}_{car}^{\alpha(3)}$, *then*

$$|H_3(1)| \leq \frac{16}{9(1+3\alpha)^2}. \tag{32}$$

Proof. If $f \in \mathcal{R}_{car}^{\alpha(3)}$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2. \tag{33}$$

Using (24) and (25) for $n = 3$, (33) gives

$$a_4 = \frac{2}{3(1+3\alpha)}p_3. \tag{34}$$

Also

$$H_3(1) = -a_4^2. \tag{35}$$

Using (34) in (35), it yields

$$H_3(1) = -\frac{4}{9(1+3\alpha)^2}p_3^2. \tag{36}$$

On applying triangle inequality and using first inequality of Lemma 1, (32) can be easily obtained. □

For $\alpha = 0$, Theorem 3.2 yields the following result:

Corollary 3.3 *If* $f \in \mathcal{R}_{car}^{'(3)}$, *then*

$$|H_3(1)| \leq \frac{16}{9}.$$

For $\alpha = 1$, Theorem 3.2 yields the following result:

Corollary 3.4 *If* $f \in \mathcal{R}_{car}^{(3)}$, *then*

$$|H_3(1)| \leq \frac{1}{9}.$$

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