J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2022.29.4.353 ISSN(Online) 2287-6081
Volume 29, Number 4 (November 2022), Pages 353–368

A WEIERSTRASS SEMIGROUP AT A PAIR OF INFLECTION POINTS WITH HIGH MULTIPLICITIES

Seon Jeong Kim [a] and Eunju Kang [b,*]

Abstract. In the previous paper [4], we classified the Weierstrass semigroups at a pair of inflection points of multiplicities d and $d-1$ on a smooth plane curve of degree d. In this paper, as a continuation of those results, we classify all semigroups each of which arises as a Weierstrass semigroup at a pair of inflection points of multiplicities d, $d-1$ and $d-2$ on a smooth plane curve of degree d.

1. Introduction and Preliminaries

Let C be a smooth projective curve of genus $g \geq 2$, $\mathcal{M}(C)$ the field of rational functions on C and $\mathbb{N}_0$ the set of all nonnegative integers.

For a point P on C, there are exactly g integers $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_g < 2g$ such that there is no rational function f on C with a pole of order α_k at P. The integer α_k is called a gap at P and the sequence $\{\alpha_k \mid k = 1, 2, \cdots, g\}$ is called as the Weierstrass gap sequence at P. By the Riemann-Roch Theorem, we get

$$\begin{aligned} G(P) &= \{\alpha \in \mathbb{N}_0 \mid \nexists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\} \\ &= \{\alpha \in \mathbb{N}_0 \mid \exists \text{ holomorphic differential on } C \text{ of order } \alpha - 1 \text{ at } P\} \\ &= \{\alpha \in \mathbb{N}_0 \mid \exists \text{ canonical divisor on } C \text{ of order } \alpha - 1 \text{ at } P\} \end{aligned}$$

where $(f)_\infty$ means the divisor of poles of the rational function f. For a smooth plane curve C of degree $d \geq 4$, the canonical series is cut out by the system of all curves of degree $d-3$. So the order sequence of canonical divisors at P can be obtained as the set $\{I(C \cap f_{d-3}, P) \mid f_{d-3} \text{ is a polynomial of degree } d-3\}$.

Received by the editors September 07, 2022. Accepted November 08, 2022.
2010 *Mathematics Subject Classification.* 14H55, 14H51, 14H45, 14G50.
Key words and phrases. Weierstrass semigroup at a pair, Weierstrass semigroup at a point, inflection point.
*Corresponding author.

We call that P is a Weierstrass point if $G(P) \neq \{1, 2, \cdots, g\}$ or equivalently the order sequence of canonical divisors at P is not $\{0, 1 \longrightarrow g-1\}$. There are only finite number of Weierstrass points on C, which means that the order sequence of canonical divisors at a point is exactly $\{0, 1 \longrightarrow g-1\}$ except for a finite number of points.

The non-gaps at P form a semigroup under addition and we call it as the Weierstrass semigroup $H(P)$. So $H(P) = \mathbb{N}_0 \backslash G(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P\}$. We extend the Weierstrass semigroup at P to a Werierstrass semogroup at two distinct points P, $Q \in C$ as $H(P, Q) = \{(\alpha, \beta) \in \mathbb{N}_0^2 \mid \exists f \in \mathcal{M}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}$ and let $G(P, Q) = \mathbb{N}_0^2 \setminus H(P, Q)$.

As the cardinality of the set $G(P)$ is finite, in fact exactly g, the set $G(P, Q)$ is also finite, but its cardinality is dependent on the points P and Q. In [5], the first author proved that the upper and lower bound of such sets are given as $\binom{g+2}{2} - 1 \leq \text{card } G(P, Q) \leq \binom{g+2}{2} - 1 - g + g^2$, and that $H(P, Q)$ induces a bijection $\sigma = \sigma(P, Q)$ between $G(P)$ and $G(Q)$ which is defined by $\sigma(\alpha) = \beta_\alpha := \min\{\beta \mid (\alpha, \beta) \in H(P, Q)\}$. Homma [2] obtained the same formula for the cardinality of $G(P, Q)$ using the cardinality of the set $\{(\alpha, \alpha') \mid \alpha, \alpha \in G(P),\ (\alpha - \alpha')(\sigma(\alpha) - \sigma(\alpha')) < 0\}$ i.e., the set of pairs (α, α') which are reversed by σ. We use the following notations;

$$\begin{aligned} \Gamma = \Gamma(P, Q) &:= \{(\alpha, \beta_\alpha) \mid \alpha \in G(P)\} = \{(p_i, q_{\sigma(i)}) \mid i = 1, 2, \cdots, g\}, \\ \widetilde{\Gamma} = \widetilde{\Gamma}(P, Q) &:= \Gamma(P, Q) \cup \big(H(P) \times \{0\}\big) \cup \big(\{0\} \times H(Q)\big). \end{aligned}$$

The above set $\Gamma(P, Q)$ is called the *generating subset* of the Weierstrass semigroup $H(P, Q)$. Indeed, for given distinct points P and Q, the set $\Gamma(P, Q)$ determines not only $\widetilde{\Gamma}(P, Q)$ but also the sets $H(P, Q)$ and $G(P, Q)$ completely, as described below. We use the natural partial order on the set $\mathbb{N}_0^2$ as $(\alpha, \beta) \geq (\gamma, \delta)$ if and only if $\alpha \geq \gamma$ and $\beta \geq \delta$. Also we define the least upper bound of two elements (α_1, β_1), (α_2, β_2) is defined as $\text{lub}\{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} = (\max\{\alpha_1, \alpha_2\},\ \max\{\beta_1, \beta_2\})$. In [5] and [6], the following are proved: (1) The subset $H(P, Q)$ of $\mathbb{N}_0^2$ is closed under the lub(least upper bound) operation. (2) Every element of $H(P, Q)$ is expressed as the lub of one or two elements of the set $\widetilde{\Gamma}(P, Q)$. (3) The set $G(P, Q) = \mathbb{N}_0^2 \setminus H(P, Q)$ is expressed as $G(P, Q) = \bigcup_{l \in G(P)} \big(\{(l, \beta) | \beta = 0, 1, \ldots, \sigma(l) - 1\} \cup \{(\alpha, \sigma(l)) | \alpha = 0, 1, \ldots, l-1\}\big)$.

We can characterize the elements of $\Gamma(P, Q)$ and $H(P, Q)$ using the dimensions of divisors. We denote $\dim(\alpha, \beta) := \dim |\alpha P + \beta Q|$, the dimension of the complete linear series $|\alpha P + \beta Q|$.

Lemma 1.1. *For* $\alpha \geq 1$ *and* $\beta \geq 1$*, the pair* (α, β) *is an element of* $\Gamma(P,Q)$ [resp. $H(P,Q)$] *if and only if*

$$\dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1 = \dim(\alpha - 1, \beta - 1) + 1$$

$$[\text{resp. } \dim(\alpha, \beta) = \dim(\alpha - 1, \beta) + 1 = \dim(\alpha, \beta - 1) + 1].$$

Proof. See [3]. □

Theorem 1.2. *Let* $m \geq 1$*,* $m' \geq 0$*,* $n' \geq n \geq 1$ *and* $a \geq 0$ *be integers. Suppose that* $\dim(s+m, t-n) = \dim(s,t) + a$ *for all* $s \geq m'$*,* $t \geq n'$*. Let* $\alpha \geq m'+1$ *and* $\beta \geq n'+1$*. Then* $(\alpha+m, \beta-n) \in \Gamma(P,Q)$ [resp. $(\alpha+m, \beta-n) \in H(P,Q)$] *if and only if* $(\alpha, \beta) \in \Gamma(P,Q)$ [resp. $(\alpha, \beta) \in H(P,Q)$].

Proof. It follows from Lemma 1.1. □

Theorem 1.3. *Suppose that* mP *is linearly equivalent to* mQ*. If* $(\alpha, \beta), (\alpha+m, \beta') \in \Gamma(P,Q)$*, then* $\beta' = \beta - m$*.*

Proof. It follows from Theorem 1.2. □

When we prove the existence of a smooth plane curve with aligned inflection points of given intersection multiplicities, we use the following theorem. Here $\mathbb{P}_d$ denotes the set of all smooth plane curves of degree d, and $i(T, C; P)$ denotes the intersection multiplicity of two curves T and C at the point P.

Theorem 1.4 ([1])**.** *Fix a line* L *in* $\mathbb{P}^2$ *and different points* $P_0, P_1, \ldots, P_{d-e}$ *on* L *with integers* $0 \leq e \leq d$*. Fix lines* $T_1, \ldots, T_{d-e}$ *passing through* $P_1, \ldots, P_{d-e}$ *different from* L*. For a sequence* $\underline{m} = (m_1, \ldots, m_{d-e})$ *with* $d \geq m_1 \geq \cdots \geq m_{d-e}$*, let*

$$\mathcal{P}_{(e,\underline{m})} = \{C \in \mathbb{P}_d \,|\, C \text{ is smooth, } i(L, C; P_0) = e,$$
$$i(T_j, C; P_j) = m_j \text{ for } 1 \leq j \leq d-e\}.$$

Then $\mathcal{P}_{(e,\underline{m})}$ *is not empty if and only if the following condition holds:*

For every j*,* $1 \leq j < d-e$*, if* $m_{j+1} < m_j$ *then* $m_{j+1} \leq d-j$*.*

Let C be a smooth plane curve of degree $d \geq 4$ and P a point on C. From now on, $T_P C$ denotes the tangent line to C at a point $P \in C$ and $T_P C \centerdot C$ denotes the divisor on C cut out by the line $T_P C$. Also we use the notation $i_P C = i(T_P C, C; P)$ to denote the intersection multiplicity of the tangent line and C at P on C, which satisfies that $2 \leq i_P C \leq d$. Recall that an *inflection point* P of a curve C means a simple point with $i_P C \geq 3$.

In [4], we completed the classification of the Weierstrass semigroups each of which occurs at a pair of inflection points P, Q with $i_P C \geq d-1$ and $i_Q C \geq d-1$.

In this paper, we will complete the classification of the Weierstrass semigroups at pairs (P, Q) with $i_P C \geq d-2$ and $i_Q C \geq d-2$. We find all candidates of the Weierstrass semigroups at such a pair, and then prove the existence of curves and points having such semigroups as their Weierstrass semigroups.

Considering the results of [4], we only need to deal with the following cases:

(1) $i_P C = d$ and $i_Q C = d-2$.

(2) $i_P C = d-1$ and $i_Q C = d-2$.

(3) $i_P C = d-2$ and $i_Q C = d-2$.

Recall that, for a point P with $i_P C \geq d-2$, the Weierstrass gap sequence $G(P)$ at P is uniquely determined as;

$$G(P) = \cup_{k=0}^{d-3}\{k(d-t)+r \mid r = 1, \ldots, d-2-k\}, \qquad t = 0, 1, 2$$

where $i_P C = d-t$ (See [1]). In the following sections, to obtain $\Gamma(P, Q)$, we find a bijection between $G(P)$ and $G(Q)$. To do so, it is convenient to arrange the numbers of $G(P)$ in a triangle shape as follows:

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
	$2+(d-1)$	$3+(d-1)$	$\cdots$	$\cdots$	$d-3+(d-1)$	$d-2+(d-1)$
		$3+2(d-1)$	$\cdots$	$\cdot$	$d-3+2(d-1)$	$d-2+2(d-1)$
			$.$	$\vdots$	$\vdots$	$\vdots$
				$.$	$\vdots$	$\vdots$
					$d-3+(d-4)(d-1)$	$d-2+(d-4)(d-1)$
						$d-2+(d-3)(d-1)$

Table 1. $G(P)$ with $i_P C = d$

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
$1+d$	$2+d$	$3+d$	$\cdots$	$\cdots$	$d-3+d$	
$\vdots$	$\vdots$	$\vdots$	$.$			
$\vdots$	$\vdots$	$.$				
$1+(d-4)d$	$2+(d-4)d$					
$1+(d-3)d$						

Table 2. $G(P)$ with $i_P C = d$

Even though the shapes of arrays are different, we notice that

(the set of numbers in Table 1) = (the set of numbers in Table 2),

(the set of numbers in Table 3) = (the set of numbers in Table 4),

(the set of numbers in Table 5) = (the set of numbers in Table 6).

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
	$2+(d-2)$	$3+(d-2)$	$\cdots$	$\cdots$	$d-3+(d-2)$	$d-2+(d-2)$
			.	$\vdots$	$\vdots$	$\vdots$
				.	$\vdots$	$\vdots$
					$d-3+(d-4)(d-2)$	$d-2+(d-4)(d-2)$
						$d-2+(d-3)(d-2)$

Table 3. $G(P)$ with $i_PC = d-1$

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
$1+(d-1)$	$2+(d-1)$	$3+(d-1)$	$\cdots$	$\cdots$	$d-3+(d-1)$	
$\vdots$	$\vdots$	$\vdots$	.			
$\vdots$	$\vdots$	.				
$1+(d-4)(d-1)$	$2+(d-4)(d-1)$					
$1+(d-3)(d-1)$						

Table 4. $G(P)$ with $i_PC = d-1$

1	2	3	$\cdots$	$\cdots$	$d-3$	$d-2$
	$2+(d-3)$	$3+(d-3)$	$\cdots$	$\cdots$	$d-3+(d-3)$	$d-2+(d-3)$
			.	$\vdots$	$\vdots$	$\vdots$
				.	$\vdots$	$\vdots$
					$d-3+(d-4)(d-3)$	$d-2+(d-4)(d-3)$
						$d-2+(d-3)(d-3)$

Table 5. $G(P)$ with $i_PC = d-2$

1	2	3	$\cdots$	$d-3$	$d-2$
$1+(d-2)$	$2+(d-2)$	$3+(d-2)$	$\cdots$	$d-3+(d-2)$	
$1+2(d-2)$	$2+2(d-2)$	$3+2(d-2)$	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	.		
$\vdots$	$\vdots$	.			
$1+(d-4)(d-2)$	$2+(d-4)(d-2)$				
$1+(d-3)(d-2)$					

Table 6. $G(P)$ with $i_PC = d-2$

2. At a Pair (P, Q) with $i_PC = d$ and $i_QC = d-2$

Let $i_PC = d$ and $i_QC = d-2$. Then we have $T_QC \cdot C = dP$ and $T_QC \cdot C = (d-2)Q + R_1 + R_2$ for some (not necessarily distinct) points R_1, R_2 different from Q. There are two possibilities: either $\{R_1, R_2\}$ contains P or not. If $\{R_1, R_2\}$ contains P, then $T_QC \cdot C = (d-2)Q + P + R$ with $R \neq P, Q$, since $T_PC \neq T_QC$.

Case 2-1. $T_QC \cdot C = (d-2)Q + P + R$ with $R \neq P, Q$

In this case, we have $|dP| = |(d-2)Q + P + R|$, which is the linear series cut out by the system of lines. Thus $|(d-1)P| = |(d-2)Q + R|$, which we donote $(d-1)P \sim (d-2)Q + R$.

Theorem 2.1. (i) *For $\alpha \geq 0, \beta \geq d-2$,*

$$\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1.$$

(ii) *For $\alpha \geq 1, \beta \geq d-1$,*

$$(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P,Q) \iff (\alpha, \beta) \in \Gamma(P,Q).$$

(iii) *Such a curve and points exist.*

Proof. Since $(d-1)P \sim (d-2)Q + R$, we have

$$\begin{aligned} & (\alpha + (d-1))P + (\beta - (d-2))Q \\ \sim \quad & \alpha P + (d-2)Q + R + (\beta - (d-2))Q = \alpha P + \beta Q + R. \end{aligned}$$

Thus R is not a base point of $|\alpha P + \beta Q + R|$. Hence $\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1$ and (i) is proved.

By Theorem 1.2, (ii) holds.

In Theorem 1.4, let $e = d-2$, $\underline{m} = (d,d)$. Then $\mathcal{P}_{(d-2,\underline{m})}$ is not empty and let $C \in \mathcal{P}_{(d-2,\underline{m})}$. Then $P = P_1,\ Q = P_0 \in C$ satisfy the condition. □

Theorem 2.2. *For P, Q as above, $\Gamma(P,Q)$ is the set of all elements appeared in the following Table 7:*

$(1, d-2)$	$(2, d-3+(d-2))$	$\cdots$	$(d-3, 2+(d-4)(d-2))$	$(d-2, 1+(d-3)(d-2))$
	$(2+(d-1), d-3)$	$\cdots$	$(d-3+(d-1), 1+(d-5)(d-2))$	$(d-2+(d-1), 1+(d-4)(d-2))$
		.	$\vdots$	$\vdots$
			.	$\vdots$
				$(d-2+(d-3)(d-1), 1)$

Table 7. $\Gamma(P,Q)$ when $T_P C \centerdot C = dP$ and $T_Q C \centerdot C = (d-2)Q + P + R$

Proof. To use Theorem 2.1 (ii), we arrange the elements of $G(P)$ and $G(Q)$ with $d-2$ columns and rows as in Table 1 and 6.

Note that the lengths of columns in the array in each of Table 1 and 6 are all different. Also note that the sequence in each column of $G(P)$ is increasing by $d-1$ and the sequence in each column of $G(Q)$ is increasing by $d-2$.

By Theorem 2.1 (ii), $(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P,Q)$ if and only if $(\alpha, \beta) \in \Gamma(P,Q)$. It means $\{\alpha, \alpha + (d-1), \cdots, \alpha + k(d-1)\} \subset G(P)$ if and only if $\{\beta, \beta -$

$(d-2), \cdots, \beta - k(d-1)\} \subset G(Q)$. Thus if $(\alpha, \beta) \in \Gamma(P, Q)$ then α and β should belong to the columns of same length in Table 1 and 6. Hence $\Gamma(P, Q)$ is determined as Table 7. □

CASE 2-2. $T_Q C \centerdot C = (d-2)Q + R_1 + R_2$ with $R_1 + R_2 \not\geq P$

Theorem 2.3. (i) *For* $\alpha \geq 0$ *and* $\beta \geq d-2$,

$$\dim(\alpha + d, \beta - (d-2)) = \dim(\alpha, \beta) + 2.$$

(ii) *For* $\alpha \geq 1$ *and* $\beta \geq d-1$,

$$(\alpha + d, \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) *Such a curve and points exist.*

Proof. Note that R_1 and R_2 need not be distinct. When $R_1 \neq R_2$ then let L_1 be a line passing through R_1 but not containing R_2 so $L_1 \neq T_Q C$. When $R_1 = R_2$ then let L_1 be a line passing through R_1 such that $L_1 \neq T_Q C$. In both cases, we have $L_1 \centerdot C = R_1 + S_2 + \cdots + S_d$ for points $S_2, \ldots, S_d \in C$ with $R_2 \neq S_j$ for all j. Since $dP \sim (d-2)Q + R_1 + R_2 \sim L_1 \centerdot C$, we have

$$\begin{aligned} & (\alpha + d)P + (\beta - (d-2))Q \\ \sim\ & \alpha P + \beta Q + R_1 + R_2 \\ \sim\ & \alpha P + (\beta - (d-2))Q + L_1 \centerdot C. \end{aligned}$$

Thus R_1 is not a base point of the linear series $|\alpha P + \beta Q + R_1 + R_2|$ and R_2 is not a base point of the linear series $|\alpha P + \beta Q + R_2| = |\alpha P + (\beta - (d-2))Q + S_2 + \cdots + S_d|$. Hence

$$\begin{aligned} & \dim(\alpha + d, \beta - (d-2)) \\ =\ & \dim|\alpha P + \beta Q + R_1 + R_2| \\ =\ & \dim|\alpha P + \beta Q + R_2| + 1 \\ =\ & \dim(\alpha, \beta) + 2. \end{aligned}$$

Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = 0$, $\underline{m} = (d, d-2, \cdots, d-2)$. Then $\mathcal{P}_{(0,\underline{m})}$ is not empty and $C \in \mathcal{P}_{(0,\underline{m})}$ contains $P_1, P_2, \cdots, P_d$. Then $P = P_1$, $Q = P_2 \in C$ satisfy the condition. Therefore we get the result (iii). □

Theorem 2.4. *For P, Q as above, $\Gamma(P,Q)$ is the set of all elements appeared in the following Table 8 :*

$(1, 1+(d-3)(d-2))$	$(2, 2+(d-4)(d-2))$	$\cdots$	$(d-3, d-3+(d-2))$	$(d-2, d-2)$
$(1+d, 1+(d-4)(d-2))$	$(2+d, 2+(d-5)(d-2))$	$\cdots$	$(d-3+d, d-3)$	
$\vdots$	$\vdots$	$\cdot$		
$(1+(d-4)d, 1+(d-2))$	$(2+(d-4)d, 2)$			
$(1+(d-3)d, 1)$				

Table8. $\Gamma(P,Q)$ when $T_PC \centerdot C = dP$ and $T_QC \centerdot C = (d-2)Q+R_1+R_2$ with $R_1+R_2 \not\geq P$

Proof. To use Theorem 2.3 (ii), we rearrange the elements of $G(P)$ and $G(Q)$ with $d-2$ columns and rows such that the sequence in each column of $G(P)$ is increasing by d and the sequence in each column of $G(Q)$ is increasing by $d-2$. Then $G(P)$ and $G(Q)$ can be represented as Table 2 and 6.

Note that the lengths of columns in the array in each of Table 2 and 6 are all different. So in view of Theorem 2.3 (ii), if $(\alpha, \beta) \in \Gamma(P,Q)$ then α and β should belong to the columns of same length in Table 2 and 6. The proof is similar to that of Thoerem 2.2 and $\Gamma(P,Q)$ is determined as Table 8. □

3. At a Pair (P, Q) with $i_PC = d-1$ and $i_QC = d-2$

In this case, there are points $R_1, R_2, R_3 \in C$ such that $T_PC \centerdot C = (d-1)P + R_1$ with $R_1 \neq P$ and $T_QC \centerdot C = (d-2)Q + R_2 + R_3$ with $R_2 + R_3 \not\geq Q$. There are 4 possible cases for points P, Q, and $R_i{}'s$.

Case 3-1. $R_1 = Q$ (Then $R_2 + R_3 \not\geq P$ since $T_PC \neq T_QC$.)

Case 3-2. $R_1 \neq Q, R_3 = P$

Case 3-3. $R_1 \neq Q, R_1 = R_3 \neq P$

Case 3-4. $R_1 \neq Q, R_2 + R_3 \not\geq P, R_2 + R_3 \not\geq R_1$

We find $\Gamma(P,Q)$ for each cases through this section.

Case 3-1. $T_PC \centerdot C = (d-1)P + Q$ and $T_QC \centerdot C = (d-2)Q + R_2 + R_3$ with $R_2 + R_3 \not\geq P$

Theorem 3.1. (i) *For $\alpha \geq 0, \beta \geq d-2$,*

$$\dim(\alpha + (d-1), \beta - (d-3)) = \dim(\alpha, \beta) + 2.$$

(ii) *For* $\alpha \geq 1, \beta \geq d-1$,

$$(\alpha + (d-1), \beta - (d-3)) \in \Gamma(P,Q) \iff (\alpha, \beta) \in \Gamma(P,Q).$$

(iii) *Such a curve and points exist.*

Proof. Let L_1 be general line passing through R_2 but not containing Q and $L_1 . C = R_2 + S_2 + \cdots + S_d$ with $R_2 \neq S_j$ and $R_3 \neq S_j$ for all j. Since $(d-1)P \sim (d-3)Q + R_2 + R_3$,

$$\begin{aligned} & (\alpha + (d-1))P + (\beta - (d-3))Q \\ \sim\ & \alpha P + \beta Q + R_2 + R_3 \\ =\ & \alpha P + (\beta - (d-2))Q + ((d-2)Q + R_2 + R_3) \\ \sim\ & \alpha P + (\beta - (d-2))Q + R_2 + S_2 + \cdots + S_d. \end{aligned}$$

Thus $\dim(\alpha + (d-1), \beta - (d-3)) = \dim(\alpha, \beta) + 2$ and (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = d-1$, $\underline{m} = (d-2)$. Then $\mathcal{P}_{(d-1,\underline{m})}$ is not empty and $C \in \mathcal{P}_{(d-1,\underline{m})}$ contains $P = P_0$, $Q = P_1$ which satisfy the condition. Therefore we get the result (iii). □

Theorem 3.2. *For* P, Q *as above,* $\Gamma(P,Q)$ *is the set of all elements appeared in the following Table 9 :*

$(1, d-2+(d-3)(d-3))$	$(2, d-3+(d-4)(d-3))$	$\cdots$	$(d-3, 2+(d-3))$	$(d-2, 1)$
$(1+(d-1), d-2+(d-4)(d-3))$	$(2+(d-1), d-3+(d-5)(d-3))$	$\cdots$	$(d-3+(d-1), 2)$	
$\vdots$	$\vdots$	$.$		
$(1+(d-4)(d-1), d-2+(d-3))$	$(2+(d-4)(d-1), d-3)$			
$(1+(d-3)(d-1), d-2)$				

Table 9. $\Gamma(P,Q)$ when $T_PC . C = (d-1)P + Q$ and $T_QC . C = (d-2)Q + R_2 + R_3$ with $R_2 + R_3 \not\geq P$

Proof. We rearrange the elements of $G(P)$ and $G(Q)$ with $d-2$ columns and rows such that the sequence in each column of $G(P)$ is increasing by $d-1$ and the sequence in each column of $G(Q)$ is increasing by $d-3$. Then $G(P)$ and $G(Q)$ can be represented as Table 4 and 5.

Note that the lengths of columns in the array in each of Table 4 and 5 are all different. In view of Theorem 3.1 (ii), if $(\alpha, \beta) \in \Gamma(P,Q)$ then α and β should belong to the columns of same length in Table 4 and 5. Hence $\Gamma(P,Q)$ is determined as Table 9. □

CASE 3-2. $T_PC \cdot C = (d-1)P + R_1$ and $T_QC \cdot C = (d-2)Q + R_2 + P$ with $R_1 \neq R_2$

Theorem 3.3. (i) *For $\alpha \geq 0, \beta \geq d-2$,*

$$\dim(\alpha + (d-2), \beta - (d-2)) = \dim(\alpha, \beta).$$

(ii) *For $\alpha \geq 1, \beta \geq d-1$,*

$$(\alpha + (d-2), \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) *Such a curve and points exist.*

Proof. Since $(d-2)P + R_1 \sim (d-2)Q + R_2$,

$$(\alpha + (d-2))P + (\beta - (d-2))Q + R_1 \sim \alpha P + \beta Q + R_2.$$

Thus neither R_1 nor R_2 is a base point of the linear series

$$|(\alpha + (d-2))P + (\beta - (d-2))Q + R_1| = |\alpha P + \beta Q + R_2|.$$

Hence $\dim(\alpha + (d-2), \beta - (d-2)) = \dim(\alpha, \beta) = \dim |\alpha P + \beta Q + R_2| - 1$.

Thus (i) is proved and by Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = d-2$, $\underline{m} = (d-1, d-1)$. Then $\mathcal{P}_{(d-2,\underline{m})}$ is not empty and $C \in \mathcal{P}_{(d-2,\underline{m})}$ contains $Q = P_0, P = P_1$ which satisfy the condition. Therefore we get the result (iii). □

Theorem 3.4. *For P, Q as above, $\Gamma(P, Q)$ is the set of all elements appeared in the following Table 10:*

$$\begin{array}{llllll}
(1,d-2) & (2,d-3+(d-2)) & \cdots & \cdots & (d-3,2+(d-4)(d-2)) & (d-2,1+(d-3)(d-2)) \\
 & (2+(d-2),d-3) & \cdots & \cdots & (d-3+(d-2),1+(d-5)(d-2)) & (d-2+(d-2),1+(d-4)(d-2)) \\
 & & . & \vdots & \vdots & \vdots \\
 & & & . & \vdots & \vdots \\
 & & & & (d-3+(d-4)(d-2),2) & (d-2+(d-4)(d-2),1+(d-2)) \\
 & & & & & (d-2+(d-3)(d-2),1)
\end{array}$$

Table 10. $\Gamma(P, Q)$ when $T_PC \cdot C = (d-1)P + R_1$ and $T_QC \cdot C = (d-2)Q + R_2 + P$, $R_1 \neq R_2$

Proof. We use the array in Table 3 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 3 [resp. Table 6] is increasing by $(d-2)$ [resp. $(d-2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.3 (ii), we obtain $\Gamma(P, Q)$. □

CASE 3-3. $T_PC \centerdot C = (d-1)P + R_1$ and $T_QC \centerdot C = (d-2)Q + R_1 + R_2$

Theorem 3.5. (i) *For* $\alpha \geq 0, \beta \geq d-2$,

$$\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1.$$

(ii) *For* $\alpha \geq 1, \beta \geq d-1$,

$$(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P,Q) \iff (\alpha, \beta) \in \Gamma(P,Q).$$

(iii) *Such a curve and points exist.*

Proof. Since $(d-1)P \sim (d-2)Q + R_2$,

$$(\alpha + (d-1))P + (\beta - (d-2))Q \sim \alpha P + \beta Q + R_2.$$

Since R_2 is not a base point of $|\alpha P + \beta Q + R_2|$, $\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1$ holds. Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

Modifying the idea in [1], we construct a desired polynomial of degree d. Consider a linear system $\{ay^{d-2}(y+x)z + b\prod_{n=0}^{d-1}(x-nz) \mid (a,b) \in \mathbb{P}^1\}$. By Bertini's theorem, a general element in this system is smooth. In fact, easy calculation shows that $C := ay^{d-2}(y+x)z + b\prod_{n=0}^{d-1}(x-nz)$ is smooth and for $P = (0,0,1)$ and $Q = (1,0,1)$, $T_PC = \{x = 0\}$ and $T_PQ = \{x = z\}$ satisfy the conditions. Note that $R_1 = (0,1,0)$ is contained in all of C, T_PC and T_QC. Therefore we get the result (iii). □

Theorem 3.6. *For* P, Q *as above,* $\Gamma(P,Q)$ *is the set of all elements appeared in the following Table 11.*

$(1, 1 + (d-3)(d-2))$	$(2, 2 + (d-4)(d-2))$	$\cdots$	$(d-3, d-3+(d-2))$	$(d-2, d-2)$
$(1+(d-1), 1+(d-4)(d-2))$	$(2+(d-1), 2+(d-5)(d-2))$	$\cdots$	$(d-3+(d-1), d-3)$	
$\vdots$	$\vdots$	.		
$(1 + (d-4)(d-1), 1 + (d-2))$	$(2 + (d-4)(d-1), 2)$			
$(1 + (d-3)(d-1), 1)$				

Table 11. $\Gamma(P,Q)$ when $T_PC \centerdot C = (d-1)P + R_1$ and $T_QC \centerdot C = (d-2)Q + R_1 + R_2$

Proof. We use the array in Table 4 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 4 [resp. Table 6] is increasing by $(d-1)$ [resp. $(d-2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.5 (ii), we obtain $\Gamma(P,Q)$. □

CASE 3-4. $T_PC \cdot C = (d-1)P + R_1, R_1 \neq Q$ and $T_QC \cdot C = (d-2)Q + R_2 + R_3$ with $R_2 + R_3 \not\geq R_1, P$

Theorem 3.7. (i) *For* $\alpha \geq 0, \beta \geq d-2$,

$$\dim(\alpha + (d-1), \beta - (d-2)) = \dim(\alpha, \beta) + 1.$$

(ii) *For* $\alpha \geq 1, \beta \geq d-1$,

$$(\alpha + (d-1), \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) *Such a curve and points exist.*

Proof. Let L_1 be a line passing through R_2 differnet from T_QC and $L_1 \cdot C \sim R_2 + S_2 + \cdots + S_d$ with $R_2 \neq S_j$ for all j. Then

$$\begin{aligned} & (\alpha + (d-1))P + (\beta - (d-2))Q + R_1 \\ \sim \ & \alpha P + \beta Q + R_2 + R_3 \\ \sim \ & \alpha P + (\beta - (d-2))Q + L_1 \cdot C \\ \sim \ & \alpha P + (\beta - (d-2))Q + R_2 + S_2 + \cdots + S_d. \end{aligned}$$

Thus R_1 is not a base point of $|\alpha P + \beta Q + R_2 + R_3|$ and

$$\begin{aligned} & \dim(\alpha + (d-1), \beta - (d-2)) \\ = \ & \dim |(\alpha + (d-1))P + (\beta - (d-2))Q + R_1| - 1 \\ = \ & \dim |\alpha P + \beta Q + R_2 + R_3| - 1 \\ = \ & \dim(\alpha, \beta) + 1 \end{aligned}$$

since R_2 is not a base point of

$$|\alpha P + \beta Q + R_2 + R_3| = |(\alpha + (d-1))P + (\beta - (d-2))Q + R_1|,$$

and R_3 is not a base point of

$$|\alpha P + \beta Q + R_3| = |\alpha P + (\beta - (d-2))Q + S_2 + \cdots + S_d|.$$

Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = 0$, $\underline{m} = (d-1, d-2, \cdots, d-2)$. Choose three lines T_1, T_2, T_3 which are not concurrent. Then $\mathcal{P}_{(0,\underline{m})}$ is not empty and take $C \in \mathcal{P}_{(0,\underline{m})}$ which satisfy $T_1 \cap T_2 \nsubseteq C$ or $T_1 \cap T_3 \nsubseteq C$, since T_1 meet C at only one more point other than P_1. We may assume $T_1 \cap T_2 \nsubseteq C$. Then $P = P_1$ and $Q = P_2 \in C$ satisfy the condition. Therefore we get the result (iii). □

Theorem 3.8. *For P, Q as above, $\Gamma(P,Q)$ is the set of all elements appeared in the Table 11 of Theorem 3.6.*

Proof. We use the array in Table 4 [resp. Table 6] as $G(P)$ [resp. $G(Q)$] since the sequence in each column of Table 4 [resp. Table 6] is increasing by $(d-1)$ [resp. $(d-2)$]. Now the proof is similar to that of Theorem 3.2. By applying Theorem 3.7 (ii), we obtain $\Gamma(P,Q)$. □

4. At a Pair (P,Q) with $i_PC = d-2$ and $i_QC = d-2$

In this case, $T_PC \centerdot C = (d-2)P + R_1 + R_2$ and $T_QC \centerdot C = (d-2)Q + S_1 + S_2$. There are 3 possible cases for points P, Q, $R_i's$ and $S_i's$:

Case 4-1. $R_2 = Q$ (Then $S_1 + S_2 \not\geq P$ since $T_PC \neq T_QC$.)

Case 4-2. $R_1, R_2, S_1, S_2 \notin \{P, Q\}$, $R_2 = S_2$

Case 4-3. $R_1, R_2, S_1, S_2 \notin \{P, Q\}$, $R_1, R_2 \notin \{S_1, S_2\}$ (maybe $R_1 = R_2$ or $S_1 = S_2$)

Case 4-1. $T_PC \centerdot C = (d-2)P + R_1 + Q$ and $T_QC \centerdot C = (d-2)Q + S_1 + S_2$

Theorem 4.1. (i) *For $\alpha \geq 0, \beta \geq d-2$,*

$$\dim(\alpha + (d-2), \beta - (d-3)) = \dim(\alpha, \beta) + 1.$$

(ii) *For $\alpha \geq 1, \beta \geq d-1$,*

$$(\alpha + (d-2), \beta - (d-3)) \in \Gamma(P,Q) \iff (\alpha, \beta) \in \Gamma(P,Q).$$

(iii) *Such a curve and points exist.*

Proof. Let L_1 be a line passing through S_1 different from T_QC and $L_1 \centerdot C = S_1 + U_2 + \cdots + U_d$ with $S_1 \neq U_j$ for all j.

Since $(d-2)P + R_1 \sim (d-3)Q + S_1 + S_2$, we have

$$\begin{aligned} & (\alpha + (d-2))P + (\beta - (d-3))Q + R_1 \\ \sim\ & \alpha P + \beta Q + S_1 + S_2 \\ \sim\ & \alpha P + (\beta - (d-2))Q + S_1 + U_2 + \cdots + U_d. \end{aligned}$$

Hence

$$\begin{aligned} & \dim |(\alpha + (d-2))P + (\beta - (d-3))Q| + 1 \\ =\ & \dim |(\alpha + (d-2))P + (\beta - (d-3))Q + R_1| \\ =\ & \dim |\alpha P + \beta Q + S_1 + S_2| \\ =\ & \dim(\alpha, \beta) + 2. \end{aligned}$$

Thus (i) is proved.

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = d-2$, $\underline{m} = (d-2, d-2)$. Then $\mathcal{P}_{(d-2,\underline{m})}$ is not empty and $C \in \mathcal{P}_{(d-2,\underline{m})}$ contains $P = P_0$, $Q = P_1$ which satisfy the condition. Therefore we get the result (iii). □

Theorem 4.2. *For P, Q as above, $\Gamma(P,Q)$ is the set of all elements appeared in the following Table 12 :*

$(1, d-2+(d-3)(d-3))$	$(2, d-3+(d-4)(d-3))$	$\cdots$	$(d-3, 2+(d-3))$	$(d-2, 1)$
$(1+(d-2), d-2+(d-4)(d-3))$	$(2+(d-2), d-3+(d-5)(d-3))$	$\cdots$	$(d-3+(d-2), 2)$	
$\vdots$	$\vdots$	$.$		
$(1+(d-4)(d-2), d-2+(d-3))$	$(2+(d-4)(d-2), d-3)$			
$(1+(d-3)(d-2), d-2)$				

Table 12. $\Gamma(P,Q)$ when $T_PC \cdot C = (d-2)P + R_1 + Q$ and $T_QC \cdot C = (d-2)Q + S_1 + S_2$

Proof. The proof is similar to the proof of Theorem 3.2. In this proof, we use Table 6 for $G(P)$ and Table 5 for $G(Q)$. Then we obtain $\Gamma(P,Q)$. □

CASE 4-2. $T_PC \cdot C = (d-2)P + R_1 + R_2$ and $T_QC \cdot C = (d-2)Q + S_1 + R_2$

Theorem 4.3. (i) *For $\alpha \geq 0$, $\beta \geq d-2$,*

$$\dim(\alpha + (d-2), \beta - (d-2)) = \dim(\alpha, \beta).$$

(ii) *For $\alpha \geq 1, \beta \geq d-1$,*

$$(\alpha + (d-2), \beta - (d-2)) \in \Gamma(P,Q) \iff (\alpha,\beta) \in \Gamma(P,Q).$$

(iii) *Such a curve and points exist.*

Proof. Since $(d-2)P + R_1 \sim (d-2)Q + S_1$,

$$(\alpha + (d-2))P + (\beta - (d-2))Q + R_1 \sim \alpha P + \beta Q + S_1.$$

Thus $\dim(\alpha + (d-2), \beta - (d-2)) = \dim(\alpha, \beta)$.

By Theorem 1.2, (ii) is proved.

Consider a generic smooth curve C given in the proof of Theorem 3.5. Then $P = (1,0,1), Q = (2,0,1), R_1 = (1,-1,1), S_1 = (2,-2,1), R_2 = (0,1,0)$ on C satisfy the condition and (iii) is proved. □

Theorem 4.4. *For P, Q as above, $\Gamma(P,Q)$ is the set of all elements appeared in the following Table 13 :*

$(1, 1 + (d-3)(d-2))$	$(2, 2 + (d-4)(d-2))$	$\cdots$	$(d-3, d-3+(d-2))$	$(d-2, d-2)$
$(1+(d-2), 1+(d-4)(d-2))$	$(2+(d-2), 2+(d-5)(d-2))$	$\cdots$	$(d-3+(d-2), d-3)$	
$\vdots$	$\vdots$			
$(1 + (d-4)(d-2), 1 + (d-2))$	$(2 + (d-4)(d-2), 2)$			
$(1 + (d-3)(d-2), 1)$				

Table 13. $\Gamma(P, Q)$ when $T_PC \centerdot C = (d-2)P + R_1 + R_2$ and $T_QC \centerdot C = (d-2)Q + S_1 + R_2$

Proof. The proof is similar to the proof of Theorem 3.2. In this proof, we use Table 6 for both $G(P)$ and $G(Q)$. Then we obtain $\Gamma(P, Q)$. □

Case 4-3. $T_PC \centerdot C = (d-2)P + R_1 + R_2$ and $T_QC \centerdot C = (d-2)Q + S_1 + S_2$

Theorem 4.5. (i) *For* $\alpha \geq 0, \beta \geq d-2$,

$$\dim(\alpha + (d-2), \beta - (d-2)) = \dim(\alpha, \beta).$$

(ii) *For* $\alpha \geq 1, \beta \geq d-1$,

$$(\alpha + (d-2), \beta - (d-2)) \in \Gamma(P, Q) \iff (\alpha, \beta) \in \Gamma(P, Q).$$

(iii) *Such a curve and points exist.*

Proof. Let L_1 be a line passing through R_1 different from T_PC and $L_1 \centerdot C \sim R_1 + R_2{}' + \cdots + R_d{}'$ with $R_1 \neq R_j{}'$ for all j. Let L_2 be a line passing through S_1 different from T_QC and $S_1 \centerdot C \sim S_1 + S_2{}' + \cdots + S_d{}'$ with $S_1 \neq S_j{}'$ for all j.

Since $(d-2)P + R_1 + R_2 \sim (d-2)Q + S_1 + S_2$, we have

$$\begin{aligned} & \alpha P + (\beta - (d-2))Q + R_1 + R_2{}' + \cdots + R_d{}' \\ \sim\ & (\alpha + (d-2))P + (\beta - (d-2))Q + R_1 + R_2 \\ \sim\ & \alpha P + \beta Q + S_1 + S_2 \\ \sim\ & \alpha P + (\beta - (d-2))Q + S_1 + S_2{}' + \cdots + S_d{}'. \end{aligned}$$

Hence

$$\begin{aligned} & \dim((\alpha + (d-2), \beta - (d-2)) \\ =\ & \dim |(\alpha + (d-2))P + (\beta - (d-2))Q + R_1 + R_2| - 2 \\ =\ & \dim |\alpha P + \beta Q + S_1 + S_2| - 2 \\ =\ & \dim(\alpha, \beta). \end{aligned}$$

By Theorem 1.2, (ii) is proved.

In Theorem 1.4, let $e = 0$, $\underline{m} = (d-2, d-2, \cdots, d-2)$. Then $\mathcal{P}_{(0,\underline{m})}$ is not empty and take $C \in \mathcal{P}_{(0,\underline{m})}$. Then $P = P_1, Q = P_2 \in C$ satisfy the condition. Therefore we get the result (iii). □

Theorem 4.6. *For P, Q as above, $\Gamma(P, Q)$ is the same table as that in Theorem 4.4*

Proof. The proof is same as that of Theorem 4.4. □

Acknowledgment

The first author was partially supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (2022R1A2C1012291).

References

1. M. Coppens and T. Kato: The Weierstrass gap sequence at an inflection point on a nodal plane curve, aligned inflection points on plane curves. Bollettino U.M.I. **7** (1997), no. 11-B, 1-33.
2. M. Homma: The Weierstrass semigroup of a pair of points on a curve. Arch. Math. **67** (1996), 337-348.
3. E. Kang & S.J. Kim: Special pairs in the generating subset of the Weierstrass semigroup at a pair. Geom. Dedicata **99** (2003), no. 1, 167-177.
4. E. Kang & S.J. Kim: A Weierstrass semigroup at a pair of inflection points on a smooth plane curve. Bull. Korean Math. Soc. **44** (2007), no. 2, 369-378.
5. S.J. Kim: On the index of the Weierstrass semigroup of a pair of points on a curve. Arch. Math. **62** (1994), 73-82.
6. S.J. Kim & J. Komeda: Weierstrass semigroups of pairs of points whose first non-gaps are three. Geom. Dedicata **93** (2002), no. 2, 113-119.

[a]Professor: Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Republic of Korea
Email address: skim@gnu.ac.kr

[b]Professor: Department of Information and Communication Engineering, Honam University, Gwangju 62399, Republic of Korea
Email address: ejkang@honam.ac.kr