J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
https://doi.org/10.7468/jksmeb.2022.29.4.335
Volume 29, Number 4 (November 2022), Pages 335–352

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

BEST PROXIMITY POINT THEOREMS FOR CYCLIC θ-ϕ-CONTRACTION ON METRIC SPACES

Mohamed Rossafi [a], Abdelkarim Kari [b] and Jung Rye Lee [c,*]

Abstract. In this paper, we give an extended version of fixed point results for θ-contraction and θ-ϕ-contraction and define a new type of contraction, namely, cyclic θ-contraction and cyclic θ-ϕ-contraction in a complete metric space. Moreover, we prove the existence of best proximity point for cyclic θ-contraction and cyclic θ-ϕ-contraction. Also, we establish best proximity result in the setting of uniformly convex Banach space.

1. Introduction

In 1922, Polish mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. It is one of the fundamental results in fixed point theory. Due to its importance, many mathematicians have studied many interesting extensions and generalizations in [2, 7, 8, 9, 10].

Best proximity point theorems are those results that present sufficient conditions for the existence of a best proximity point and algorithms for finding best proximity points. It is interesting to see that best proximity point theorems generalize fixed point theorems in a natural way. Some interesting best proximity point theorems in the setting of metric spaces or uniformly convex Banach space can be found in [3, 4, 5].

Our main aim is to resolve a more general problem on the existence of fixed point for θ-contraction and θ-ϕ-contraction. Also, we establish best proximity point in a complete metric space and in a uniformly convex Banach space using this new type mapping.

Received by the editors September 07, 2022. Accepted September 19, 2022.
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, best proximity point, uniformly convex Banach space, θ-contraction, θ-ϕ-contraction.

*Corresponding author.

2. Preliminaries

In this section, we give basic definitions and notations which will be essential throughout the paper.

The following definition was given by Jleli *et al.* in [6].

Definition 2.1 ([6])**.** Let Θ be the family of all functions $\theta :\ ]0,+\infty[\ \to\ ]1,+\infty[$ such that

(θ_1) θ is increasing;

(θ_2) for each sequence $(x_n) \subset\ ]0,+\infty[$,

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta\left(x_n\right) = 1;$$

(θ_3) θ is continuous.

A mapping T : $X \to X$ is said to be *θ-contraction* if there exists $k \in$ such that for all $x, y \in X$ $d(Tx, Ty) > 0$,

$$\theta(d(Tx, Ty)) \leq \theta(d(x, y)). \tag{2.1}$$

Remark 2.2 ([7])**.** From (θ_1) and (2.1), it is clear that every θ-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \forall x, y \in X, Tx \neq Ty.$$

Thus every θ-contraction is a continuous mapping.

In [11], Zheng *et al.* presented the concept of θ-ϕ-contraction in metric spaces.

Definition 2.3 ([11])**.** Let Φ be the family of all functions ϕ: $[1,+\infty[\ \to [1,+\infty[$ such that

(ϕ_1) ϕ is nondecreasing;

(ϕ_2) for each $t \in\ ]1,+\infty[$, $lim_{n\to\infty}\phi^n(t) = 1$;

(ϕ_3) ϕ is continuous.

Lemma 2.4 ([11])**.** *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for all $t \in\ ]1,\infty[$.*

Definition 2.5 ([11])**.** Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a *θ-ϕ-contraction* if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d\left(Tx, Ty\right) > 0$,

$$\theta\left[d\left(Tx, Ty\right)\right] \leq \phi\left[\theta\left(d\left(x, y\right)\right)\right]. \tag{2.2}$$

Remark 2.6. From (θ_1), Lemma 2.4 and (2.2), it is clear that every θ-ϕ-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \forall x, y \in X, Tx \neq Ty.$$

Thus every θ-ϕ-contraction is a continuous mapping.

Theorem 2.7 ([11]). *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.*

Definition 2.8 ([5]). Let A and B be two nonempty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subset B$ and $T(B) \subset A$. We say that T is a *cyclic contraction* if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B)$$

for some $\alpha \in]0, 1[$ and all $x \in A$ and $y \in B$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.
A point $x \in A$ is said to be a *best proximity point* for T if $d(x, Tx) = d(A, B)$.

Definition 2.9 ([5]). A Banach space X is uniformly convex if there exists a strictly increasing function $\delta :]0, 2] \to [0, 1]$ such that the following condition holds for all $x, y, z \in X$, $R > 0$ and $r \in [0, 2R]$ with $\|x - z\| \leq R$, $\|y - z\| \leq R$ and $\|x - y\| \geq r$,

$$\|\frac{x + y}{2} - z\| \leq 1 - \delta\left(\frac{r}{R}\right) R.$$

Definition 2.10 ([5]). A subset K of a metric space X is boundedly compact if every bounded sequence in K has a subsequence converging to a point in K.

3. Main Results

In the following, using the idea introduced by Kanta and Mondal in [5], we obtain fixed point results for cyclic type θ-contraction and θ-ϕ-contraction.

Theorem 3.1. *Suppose that there exist two nonempty closed subsets A and B of a complete metric space (X, d) and that a mapping $T : A \cup B \to A \cup B$ satisfies*

(1) *$T(A) \subset B$ and $T(B) \subset A$,*
(2) *for all $x \in A$ and $y \in B$, there exists $k \in]0, 1[$ such that*

$$d(T(x), T(y)) > 0 \Rightarrow \theta(d(T(x), T(y))) \leq [\theta(d(x, y))]^k.$$

Then T has a unique fixed point in $A \cap B$.

Proof. First, note that if x and y are two different fixed points of T, then $d(x,y) > 0$. Now by the definition of θ-contraction we have

$$\theta(d(x,y)) = \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k < \theta(d(x,y)).$$

Since $k \in]0,1[$, we have

$$\theta(d(x,y)) < \theta(d(x,y)).$$

So $x = y$. Thus the uniqueness part of the theorem is done.

For the existence part, suppose $x_0 \in A \cap B$. Define a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Set $\lambda_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Note that if $\lambda_n = 0$ for some $n \geq 0$, then $x_n = x_{n+1} = Tx_n$. Therefore, $x_n \in A \cap B$ is a fixed point of T and the proof follows.

We now assume that $\lambda_n \neq 0$ for all $n \geq 0$.
Since T is θ-contraction, from (2.1),

$$\theta(\lambda_n) \leq [\theta(\lambda_{n-1})]^k \leq [\theta(\lambda_{n-2})]^{k^2} \leq \cdots \leq [\theta(\lambda_0)]^{k^n}.$$

Using (θ_1), we get

$$\lambda_n < \lambda_{n-1}.$$

Therefore, $\{\lambda_n\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \lambda_n = \alpha.$$

Now, we claim that $\alpha = 0$. Arguing by contraction, we assume that $\alpha > 0$. Since $\{\lambda_n\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$\lambda_n \geq \alpha \quad \forall n \in \mathbb{N}.$$

By the property of θ, we get

$$1 < \theta(\alpha) \leq \theta(\lambda_n)^{k^n}. \tag{3.1}$$

Letting $n \to \infty$ in (3.1), we obtain

$$1 < \theta(\alpha) \leq 1,$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} \lambda_n = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Suppose to the contrary. Then there is an $\varepsilon > 0$ such that for an integer k

there exists two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that $d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \geq \varepsilon$ and $d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon$. Using the triangular inequality, we have

(3.2)
$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right).$$

Taking the limit as $k \to \infty$ in (3.2), we have

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon.$$

On the other hand,

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)}}\right), \tag{3.3}$$

(3.4)
$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right).$$

Taking the limit as $k \to \infty$ in (3.3) and (3.4), we have

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon.$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (2.1), we obtain

$$\theta\left[d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \leq \left[\theta\left(d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{k}.$$

Letting $k \to \infty$ the above inequality and applying the continuity of θ, we obtain

$$\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \left[\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]^{k}.$$

Therefore,

$$\theta(\varepsilon) \leq [\theta(\varepsilon)]^{k} < \theta(\varepsilon).$$

Since θ is increasing, we get

$$\varepsilon < \varepsilon,$$

which is a contradiction. Thus

$$\lim_{n,m\to\infty} d\left(x_m, x_n\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Let $\lim_{n\to\infty} x_n = z$. Then there are infinitely many number of terms of x_n in A as well as in B. Therefore $z \in A \cap B$ and so $\in A \cap B \neq \phi$.

Now (1) implies $T : A \cap B \to A \cap B$ while (2) implies that T restricted to $A \cap B$ is a θ-contraction. By Theorem 3.1, applied to T on $A \cap B$, we have a unique fixed point in $A \cap B$. □

Theorem 3.2. *Suppose that there exist two nonempty closed subsets A and B of a complete metric space (X,d) and that a mapping $T: A\cup B \to A\cup B$ satisfies*

(1) *$T(A)\subset B$ and $T(B)\subset A$;*
(2) *for all $x\in A$ and $ly\in B$, there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that*

$$d(T(x),T(y))>0 \Rightarrow \theta(d(T(x),T(y)))\leq \phi\left[\theta(d(x,y))\right].$$

Then T has a unique fixed point in $A\cap B$.

Proof. First, note that if x and y are two different fixed points of T, then $d(x,y)>0$. Now by the definition of θ-ϕ-contraction, we have

$$\theta(d(x,y))=\theta(d(Tx,Ty))\leq \phi\left[\theta(d(x,y))\right]<\theta(d(x,y)).$$

By Lemma 2.4, we have

$$\theta(d(x,y))<\theta(d(x,y)).$$

So $x=y$. Thus the uniqueness part of the theorem is done.

For the existence part, suppose $x_0\in A\cap B$. Define a sequence $\{x_n\}$ such that $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. Set $\lambda_n=d(x_n,x_{n+1})$ for all $n\in\mathbb{N}$. Note that if $\lambda_n=0$ for some $n\geq 0$, then $x_n=x_{n+1}=Tx_n$. Therefore, $x_n\in A\cap B$ is a fixed point of T and the proof follows.

We now assume that $\lambda_n\neq 0$ for all $n\geq 0$. Since T is θ-ϕ-contraction, from (2.2),

$$\theta(\beta_n)\leq\phi\left[\theta(\beta_{n-1})\right]\leq\phi^2\left[\theta(\beta_{n-2})\right]\leq\cdots\leq\phi^n\left[\theta(\beta_0)\right].$$

Using (θ_1) and Lemma 2.4, we get

$$\beta_n<\beta_{n-1}.$$

Therefore, $\{\beta_n\}_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\mu\geq 0$ such that

$$\lim_{n\to\infty}\beta_n=\mu.$$

Now, we claim that $\mu=0$. Arguing by contraction, we assume that $\mu>0$. Since $\{\beta_n\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$\beta_n\geq\mu\quad\forall n\in\mathbb{N}.$$

By the properties of θ and ϕ, we get

$$1<\theta\left(\mu\right)\leq\phi^n\left[\theta\left(\beta_0\right)\right]. \tag{3.5}$$

Letting $n\to\infty$ in (3.5), we obtain

$$1<\theta\left(\mu\right)\leq 1$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} \mu_n = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Suppose to the contrary. Then there is an $\varepsilon > 0$ such that for any integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that $d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \geq \varepsilon$ and $d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon$.

As in the proof of Theorem 3.1, we have

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon$$

and

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon.$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (2.2), we obtain

$$\theta\left[d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \leq \phi\left[\theta\left(d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

Letting $k \to \infty$ the above inequality and applying the continuity of θ and ϕ and using Lemma 2.4, we obtain

$$\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

Therefore,

$$\theta(\varepsilon) \leq \phi\left[\theta(\varepsilon)\right] < \theta(\varepsilon).$$

Since θ is increasing, we get

$$\varepsilon < \varepsilon,$$

which is a contradiction. Thus

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Let $\lim_{n\to\infty} x_n = z$. Then there are infinitely many number of terms of x_n in A as well as in B. Therefore $z \in A \cap B$ and so $\in A \cap B \neq \phi$.

Now (1) implies $T : A \cap B \to A \cap B$ while (2) implies that T, restricted to $A \cap B$, is a θ-contraction. By Theorem 3.1, applied to T on $A \cap B$, we have a unique fixed point in $A \cap B$. □

To study the convergence and existence results for best proximity points, we first introduce new concepts of cyclic θ-contraction and θ-ϕ-contraction. We also prove the existence results for best proximity points of cyclic θ-contraction and θ-ϕ-contraction in the setting of uniformly convex Banach space.

Definition 3.3. Let A and B be two nonempty closed subsets of a complete metric space (X,d) and $T: A\cup B \to A\cup B$ be a mapping. Then we say that T is a *θ-cyclic contraction* if the following conditions hold:

(i) $T(A) \subset B$ and $T(B) \subset A$;

(ii) for all $x \in A$ and $y \in B$, there exist $k \in]0,1[$ and $\theta \in \Theta$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta\left(d(Tx,Ty)\right) \leq \alpha\left[\theta\left(d(x,y)\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right)$$

for some $\alpha \in]0,1[$ provided $d(A,B) > 0$.

Note that (ii) implies that T satisfies $\theta\left(d(Tx,Ty)\right) \leq \alpha\left[\theta\left(d(x,y)\right)\right]^k$ for all $x \in A$ and $y \in B$. On the other hand, (ii) can be written as:

$$\begin{aligned}\theta\left(d(Tx,Ty)\right) - \theta\left(d(A,B)\right) &\leq \alpha\left[\left(\theta\left(d(x,y)\right)\right)^k - \theta\left(d(A,B)\right)\right]\\ &\leq \left(\theta\left(d(x,y)\right)\right)^k - \theta\left(d(A,B)\right).\end{aligned}$$

Example 3.4. Consider the complete metric space $X = \mathbb{R}$ with the usual metric d. If we consider $\theta(t) = e^{\sqrt{t}}, t > 0$, then clearly θ satisfies all the conditions from $(\theta_1) - (\theta_3)$. In this case, a mapping $T: A\cup B \to A\cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ satisfying the condition

$$d(Tx,Ty) \leq \left[\alpha + (1-\alpha)\sqrt{d(A,B)}\left(\sqrt{d(x,y)}\right)^{-k}\right]\left(\sqrt{d(x,y)}\right)^k$$

for all $x \in A$ and $y \in B$ with $d(Tx,Ty) > 0$, $\alpha \in]0,1[$ and $k \in]0,1[$, provided $d(A,B) > 0$, is a cyclic θ-contraction.

Definition 3.5. Let A and B be two nonempty closed subsets of a complete metric space (X,d) and $T: A\cup B \to A\cup B$ be a mapping. Then we say that T is a *cyclic θ-ϕ-contraction* if the following conditions hold:

(i) $T(A) \subset B$ and $T(B) \subset A$;

(ii) for all $x \in A$ and $y \in B$, there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta\left(d(Tx,Ty)\right) \leq \alpha\phi\left[\theta\left(d(x,y)\right)\right] + (1-\alpha)\theta\left(d(A,B)\right)$$

for some $\alpha \in]0,1[$ provided $d(A,B) > 0$.

Note that (ii) implies that T satisfies $\theta\left(d(Tx,Ty)\right) \leq \alpha\phi\left[\theta\left(d(x,y)\right)\right]$ for all $x \in A$ and $y \in B$. On the other hand, (ii) can be written as

$$\begin{aligned}\theta\left(d(Tx,Ty)\right) - \theta\left(d(A,B)\right) &\leq \alpha\left[\phi\left(\theta\left(d(x,y)\right)\right) - \theta\left(d(A,B)\right)\right]\\ &\leq \phi\left(\theta\left(d(x,y)\right)\right) - \theta\left(d(A,B)\right).\end{aligned}$$

Example 3.6. Consider the complete metric space $X = \mathbb{R}$ with the usual metric d. If we consider $\theta(t) = \sqrt{t} + 1,\ t > 0$ and $\phi(t) = \frac{t+1}{2}$, then clearly θ satisfies all the conditions from $(\theta_1) - (\theta_3)$ and ϕ satisfies all the conditions from $(\phi_1) - (\phi_3)$. In this case, a mapping $T : A \cup B \to A \cup B$ such that $T(A) \subseteq\ B$ and $T(B) \subseteq A$ satisfying the condition

$$\sqrt{d(Tx,Ty)}+1 \leq \left[\alpha + (1-\alpha)\left(\left(\sqrt{d(A,B)}+1\right)\left(\frac{2}{\sqrt{d(x,y)}+2}\right)\right)\right]\left(\frac{\sqrt{d(x,y)}+2}{2}\right)$$

for all $x \in A$ and $y \in B$ with $d(Tx,Ty) > 0$, $\alpha \in \left]0,1\right[$, provided $d(A,B) > 0$, is a cyclic θ-ϕ-contraction.

Now we present our main results regarding the convergence and existence of best proximity point as below.

Proposition 3.7. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that a mapping $T : A \cup B \to A \cup B$ is a cyclic θ-contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = T_n$ for all $n \in \mathbb{N}$. Suppose that $\{x_{2n}\}$ has a convergent subsequence in A. Then we have*

$$\lim_{n\to+\infty} d(x_n, Tx_n) = d(A,B).$$

Proof. Suppose that $x_0 \in A \cup B$. Define a sequence $\{x_n\}$ such that $x_{n+1} = T_n$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned}\theta(d(x_1,x_2)) &= \theta(d(Tx_0,Tx_1))\\ &\leq \alpha\left[\theta\left(d(x_0,x_1)\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right)\\ &< \left[\theta\left(d(x_0,x_1)\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right),\end{aligned}$$

which implies that

$$\theta(d(x_1,x_2) - \theta\left(d(A,B)\right) \leq \alpha\left[\left[\theta\left(d(x_0,x_1)\right)\right]^k + d(A,B)\right].$$

Again

$$\begin{aligned}\theta(d(x_2,x_3)) &= \theta(d(Tx_1,Tx_2))\\ &\leq \alpha\left[\theta\left(d(x_1,x_2)\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right)\\ &< \alpha\theta\left(d(x_1,x_2)\right) + (1-\alpha)\theta\left(d(A,B)\right),\end{aligned}$$

which implies that

$$\begin{aligned}\theta(d(x_2,x_3)) - d(A,B) &< \alpha\left[\theta\left(d(x_1,x_2)\right) - \theta\left(d(A,B)\right)\right]\\ &\leq \alpha^2\left[\theta\left(d(x_0,x_1)\right) - \theta\left(d(A,B)\right)\right].\end{aligned}$$

Recursively we obtain that for $\alpha \in]0,1[$,

$$\theta(d(x_n,x_{n+1})) - d(A,B) < \alpha^n\left[\theta\left(d(x_0,x_1)\right) - \theta\left(d(A,B)\right)\right].$$

This implies that

$$\lim_{n\to\infty}\theta(d(x_n,x_{n+1})) = \theta(d(A,B)).$$

Since θ is increasing,

$$\lim_{n\to+\infty}(d(x_n,x_{n+1})) = d(A,B).$$

That is,

$$\lim_{n\to+\infty}(d(x_n,Tx_n)) = d(A,B).$$

This completes the proof. □

Proposition 3.8. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that a mapping $T : A\cup B \to A\cup B$ is a cyclic θ-ϕ-contraction. Let $x_0 \in A\cup B$ and $x_{n+1} = T_n$ for all $n \in \mathbb{N}$. Suppose that $\{x_{2n}\}$ has a convergent subsequence in A. Then we have*

$$\lim_{n\to+\infty} d(x_n,Tx_n) = d(A,B).$$

Proof. Suppose that $x_0 \in A\cup B$. Define a sequence $\{x_n\}$ such that $x_{n+1} = T_n$ for all $n \in \mathbb{N}$. Now,

$$\begin{aligned}\theta(d(x_1,x_2)) &= \theta(d(Tx_0,Tx_1))\\ &\leq \alpha\phi\left[\theta\left(d(x_0,x_1)\right)\right] + (1-\alpha)\theta\left(d(A,B)\right)\\ &< \alpha\left[\theta\left(d(x_0,x_1)\right)\right] + (1-\alpha)\theta\left(d(A,B)\right),\end{aligned}$$

which implies that

$$\theta(d(x_1,x_2) - \theta\left(d(A,B)\right) \leq \alpha\left[\phi\left[\theta\left(d(x_0,x_1)\right)\right] + d(A,B)\right].$$

Again

$$\begin{aligned}\theta(d(x_2,x_3)) &= \theta(d(Tx_1,Tx_2))\\ &\le \alpha\phi\left[\theta\left(d(x_1,x_2)\right)\right]+(1-\alpha)\theta\left(d(A,B)\right)\\ &< \alpha\theta\left(d(x_1,x_2)\right)+(1-\alpha)\theta\left(d(A,B)\right).\end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned}\theta(d(x_2,x_3))-d(A,B) &< \alpha\left[\theta\left(d(x_1,x_2)\right)-\theta\left(d(A,B)\right)\right]\\ &\le \alpha^2\left[\theta\left(d(x_0,x_1)\right)-\theta\left(d(A,B)\right)\right].\end{aligned}$$

Recursively we obtain that for $\alpha\in\,]0,1[$,

$$\theta(d(x_n,x_{n+1}))-d(A,B)<\alpha^n\left[\theta\left(d(x_0,x_1)\right)-\theta\left(d(A,B)\right)\right].$$

This implies that

$$\lim_{n\to\infty}\theta(d(x_n,x_{n+1}))=\theta(d(A,B)).$$

Since θ is increasing,

$$\lim_{n\to+\infty}(d(x_n,x_{n+1}))=d(A,B).$$

That is,

$$\lim_{n\to+\infty}(d(x_n,Tx_n))=d(A,B).$$

This completes the proof. □

Theorem 3.9. *Let A and B be two nonempty closed subsets of a complete metric space X and $T:A\cup B\to A\cup B$ be a cyclic θ-contraction mapping. Let $x_0\in A\cup B$ and $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x\in A$ such that $d(x,Tx)=d(A,B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ with $x_{2n_k}\to x\in A$. Since

$$d(A,B)\le d(x,x_{2n_k})\le d(x,x_{2n_{k-1}})+d(x_{2n_{k-1}},x_{2n_k}) \tag{3.6}$$

for all $k\ge 1$, by Proposition 3, letting $n\to\infty$ in (3.6), we have

$$d(A,B)\le d(x,x_{2n_k})\le d(x,x_{2n_{k-1}})\le d(A,B)$$

and so $d(x,x_{2n_k})\to d(A,B)$ as $k\to\infty$.
Again

$$d(A,B)\le d(x_{2n_k},Tx)=d(Tx_{2n_{k-1}},Tx)\le d(x_{2n_{k-1}},x)$$

for all $k\ge 1$ and hence $d(x,Tx)=d(A,B)$. □

Theorem 3.10. *Let A and B be two nonempty closed subsets of a complete metric space X and $T : A \cup B \to A \cup B$ be a cyclic θ-ϕ-contraction. Let $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ with $x_{2n_k} \to x \in A$. Since

$$d(A,B) \leq d(x, x_{2n_k}) \leq d(x, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n_k}) \tag{3.7}$$

for all $k \geq 1$, by Proposition 3.8, letting $n \to \infty$ in 3.7, we have

$$d(A,B) \leq \lim_{k \to +\infty} d(x, x_{2n_k}) \leq d(A,B)$$

and so $d(x, x_{2n_k}) \to d(A,B)$ as $k \to \infty$.
Again

$$d(A,B) \leq d(x_{2n_k}, Tx) = d(Tx_{2n_{k-1}}, Tx) \leq d(x_{2n_{k-1}}, x)$$

for all $k \geq 1$ and hence $d(x, Tx) = d(A,B)$. □

Now we present the following lemmas from [5] in order to prove our best proximity result of cyclic θ-contraction and θ-ϕ-contraction in uniformly convex Banach space.

Lemma 3.11 ([5]). *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and $\{z_n\}$ be a sequence in B satisfying*

(i) *$\|x_n - y_n\| \to d(A,B)$;*
(ii) *for every $\varepsilon > 0$ there exists N_0 such that for all $m > n > N_0$, $\|x_m - y_n\| \leq d(A,B) + \varepsilon$.*

Then for every $\varepsilon > 0$ there exists N_1 such that for all $m > n > N_1$, $\|x_m - z_n\| \leq \varepsilon$.

Lemma 3.12 ([5]). *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and $\{z_n\}$ be a sequence in B satisfying*

(i) *$\|x_n - y_n\| \to d(A,B)$;*
(ii) *$\|y_n - z_n\| \to d(A,B)$.*

Then $\|x_n - z_n\| \to 0$.

Theorem 3.13. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space. Suppose that $T : A \cup B \to A \cup B$ is a cyclic θ-contraction. Then there exists a unique best proximity point $x \in A$ with $\|x - Tx\| =$*

$d(A,B)$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Proof. Suppose that $d(A,B) \neq 0$. Then by Proposition 3,

$$\|x_{2n} - x_{2n+1}\| = \|x_{2n} - Tx_{2n}\| \to d(A,B)$$

and

$$\|x_{2n+2} - x_{2n+1}\| = \|T^2x_{2n} - Tx_{2n}\| \to d(A,B).$$

Then by Lemma 3.12, $\|x_{2n+2} - x_{2n}\| \to 0$. Similarly, we can show that $\|Tx_{2n+2} - Tx_{2n}\| \to 0$.

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n > N_0$, $\|x_{2m} - Tx_{2n}\| \leq d(A,B) + \varepsilon$. Let $\varepsilon > 0$. If possible, suppose that for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ such that

$$\|x_{2m_k} - Tx_{2n_k}\| \geq d(A,B) + \varepsilon$$

and

$$\|x_{2m_{k-1}} - Tx_{2n_k}\| \leq d(A,B) + \varepsilon.$$

Now,

$$\begin{aligned} d(A,B) + \varepsilon &\leq \|x_{2m_k} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2m_{k-1}}\| + \|x_{2m_{k-1}} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_{2m_{k-1}}\| + d(A,B) + \varepsilon. \end{aligned}$$

Thus

$$\lim_{k\to+\infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A,B) + \varepsilon.$$

Consequently,

$$\begin{aligned} \|x_{2m_k} - Tx_{2n_k}\| \leq{}& \|x_{2m_k} - x_{2(m_{k+1})}\| \\ &+ \|x_{2(m_{k+1})} - Tx_{2(n_{k+1})}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|. \end{aligned}$$

Since θ is continuous and increasing, we get

$$\begin{aligned} &\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right) \\ &\leq \theta\left[\|x_{2m_k} - x_{2(m_{k+1})}\| + \|x_{2(m_{k+1})} - Tx_{2(n_{k+1})}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right] \\ &= \theta\left[\|x_{2m_k} - x_{2(m_{k+1})}\| + \|Tx_{2m_k+1} - Tx_{2n_k+2}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right]. \end{aligned}$$

Thus

$$
\begin{aligned}
&\lim_{k\to+\infty} \theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\\
&= \theta\left(\lim_{k\to+\infty} \|x_{2m_k} - Tx_{2n_k}\|\right)\\
&\leq \theta\left[\lim_{k\to+\infty} \|x_{2m_k} - x_{2(m_{k+1})}\| + \|Tx_{2m_k+1} - Tx_{2n_k+2}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right]\\
&= \theta\left[\lim_{k\to+\infty} \|Tx_{2m_k+1} - Tx_{2n_k+2}\|\right].
\end{aligned}
$$

Since T is a cyclic θ-contraction, we get

$$
\begin{aligned}
&\lim_{k\to+\infty} \theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\\
&\leq \alpha\left[\theta\left(\lim_{k\to+\infty} \|x_{2m_k+1} - x_{2n_k+2}\|\right)\right]^k + (1-\alpha)\theta\left(D(A,B)\right)\\
&= \alpha \lim_{k\to+\infty} \left[\theta\left(\|Tx_{2m_k} - T^2x_{2n_k}\|\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right)\\
&< \alpha \lim_{k\to+\infty} \left[\theta\left(\|Tx_{2m_k} - T^2x_{2n_k}\|\right)\right] + (1-\alpha)\theta\left(d(A,B)\right)\\
&\leq \alpha\left[\alpha \lim_{k\to+\infty} \left[\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\right]^k + (1-\alpha)\theta\left(d(A,B)\right)\right] + (1-\alpha)\theta\left(d(A,B)\right)\\
&= \alpha^2 \lim_{k\to+\infty} \left[\left[\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\right]^k + (1-\alpha^2)\theta\left(d(A,B)\right)\right].
\end{aligned}
$$

Since $\lim\limits_{k\to+\infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A,B) + \varepsilon$, we have

$$
\begin{aligned}
\theta\left(d(A,B)+\varepsilon\right) &\leq \alpha^2\left[\theta\left(d(A,B)+\varepsilon\right)\right]^k + (1-\alpha^2)\theta\left(d(A,B)\right)\\
&< \alpha^2\theta\left(d(A,B)+\varepsilon\right) + (1-\alpha^2)\theta\left(d(A,B)\right).
\end{aligned}
$$

Hence

$$\theta\left(d(A,B)+\varepsilon\right) < \theta\left(d(A,B)\right),$$

which is a contradiction. Therefore $\{2x_n\}$ is a Cauchy sequence by Lemma 3.11 and hence it converges to some $x \in A$. From Proposition 3, it follows that $\|x - Tx\| = d(A,B)$. Suppose that $x, y \in A$ and $x \neq y$ such that $\|x - Tx\| = d(A,B)$ and $\|y - Ty\| = d(A,B)$, where, necessarily, $T^2x = x$ and $T^2y = y$. By Remark 2.2, we have

$$\|Tx - y\| = \|Tx - T^2y\| < \|x - Ty\|,$$

$$\|Ty - x\| = \|Ty - T^2x\| < \|y - Tx\|,$$

which imply $\|Ty - x\| < \|y - Tx\| < \|Ty - x\|$, which is a contradiction. Thus $x = y$. $\square$

Theorem 3.14. *Let A and B be two nonempty closed and convex subsets of a uniformly convex Banach space. Suppose that $T : A \cup B \to A \cup B$ is a cyclic θ-ϕ-contraction. Then there exists a unique best proximity point $x \in A$ with $\|x - Tx\| = d(A, B)$. Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Suppose that $d(A, B) \neq 0$. Then by Proposition 3.8,

$$\|x_{2n} - x_{2n+1}\| = \|x_{2n} - Tx_{2n}\| \to d(A, B)$$

and

$$\|x_{2n+2} - x_{2n+1}\| = \|T^2 x_{2n} - Tx_{2n}\| \to d(A, B).$$

Then by Lemma 3.12, $\|x_{2n+2} - x_{2n}\| \to 0$. Similarly, we can show that $\|Tx_{2n+2} - Tx_{2n}\| \to 0$.

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n > N_0$, $\|x_{2m} - Tx_{2n}\| \leq d(A, B) + \varepsilon$. Let $\varepsilon > 0$. If possible, suppose for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ such that,

$$\|x_{2m_k} - Tx_{2n_k}\| \geq d(A, B) + \varepsilon$$

and

$$\|x_{2m_{k-1}} - Tx_{2n_k}\| \leq d(A, B) + \varepsilon.$$

As in the proof of Theorem 3.13, we conclude that

$$\lim_{k \to +\infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A, B) + \varepsilon.$$

Consequently,

$$\begin{aligned}\|x_{2m_k} - Tx_{2n_k}\| &\leq \|x_{2m_k} - x_{2(m_{k+1})}\| \\ &\quad + \|x_{2(m_{k+1})} - Tx_{2(n_{k+1})}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\end{aligned}$$

Since θ is continuous and increasing, we get

$$\begin{aligned}&\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right) \\ &\leq \theta\left[\|x_{2m_k} - x_{2(m_{k+1})}\| + \|x_{2(m_{k+1})} - Tx_{2(n_{k+1})}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right] \\ &= \theta\left[\|x_{2m_k} - x_{2(m_{k+1})}\| + \|Tx_{2m_k+1} - Tx_{2n_k+2}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right].\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{k\to+\infty} \theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right) \\
&= \theta\left(\lim_{k\to+\infty} \|x_{2m_k} - Tx_{2n_k}\|\right) \\
&\leq \theta\left[\lim_{k\to+\infty} \|x_{2m_k} - x_{2(m_{k+1})}\| + \|Tx_{2m_k+1} - Tx_{2n_k+2}\| + \|x_{2(n_{k+1})} - Tx_{2n_k}\|\right] \\
&= \theta\left[\lim_{k\to+\infty} \|Tx_{2m_k+1} - Tx_{2n_k+2}\|\right].
\end{aligned}$$

Since T is a cyclic θ-ϕ-contraction, θ is continuous and increasing and ϕ is continuous and increasing, we get

$$\begin{aligned}
&\lim_{k\to+\infty} \theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right) \\
&\leq \alpha\phi\left[\theta\left(\lim_{k\to+\infty} \|x_{2m_k+1} - x_{2n_k+2}\|\right)\right] + (1-\alpha)\theta\left(D(A,B)\right) \\
&= \alpha \lim_{k\to+\infty} \phi\left[\theta\left(\|Tx_{2m_k} - T^2x_{2n_k}\|\right)\right] + (1-\alpha)\theta\left(d(A,B)\right) \\
&< \alpha \lim_{k\to+\infty} \left[\theta\left(\|Tx_{2m_k} - T^2x_{2n_k}\|\right)\right] + (1-\alpha)\theta\left(d(A,B)\right) \\
&\leq \alpha\left[\alpha \lim_{k\to+\infty} \phi\left[\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\right] + (1-\alpha)\theta\left(d(A,B)\right)\right] + (1-\alpha)\theta\left(d(A,B)\right) \\
&= \alpha^2 \lim_{k\to+\infty} \left[\phi\left[\theta\left(\|x_{2m_k} - Tx_{2n_k}\|\right)\right] + (1-\alpha^2)\theta\left(d(A,B)\right)\right].
\end{aligned}$$

Since $\lim_{k\to+\infty} \|x_{2m_k} - Tx_{2n_k}\| = d(A,B) + \varepsilon$, by Lemma 2.4, we have

$$\begin{aligned}
\theta\left(d(A,B)+\varepsilon\right) &\leq \alpha^2\phi\left[\theta\left(d(A,B)+\varepsilon\right)\right] + (1-\alpha^2)\theta\left(d(A,B)\right) \\
&< \alpha^2\theta\left(d(A,B)+\varepsilon\right) + (1-\alpha^2)\theta\left(d(A,B)\right).
\end{aligned}$$

Hence

$$\theta\left(d(A,B)+\varepsilon\right) < \theta\left(d(A,B)\right),$$

which is a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence by Lemma 3.11 and hence it converges to some $x \in A$. From Proposition 3.8, it follows that $\|x - Tx\| = d(A,B)$. Suppose that $x, y \in A$ and $x \neq y$ such that $\|x - Tx\| = d(A,B)$ and $\|y - Ty\| = d(A,B)$, where, necessarily, $T^2x = x$ and $T^2y = y$. By Remark 2.6, we conclude that

$$\|Tx - y\| = \|Tx - T^2y\| < \|x - Ty\|$$

and

$$\|Ty - x\| = \|Ty - T^2x\| < \|y - Tx\|,$$

which imply $\|Ty - x\| = \|Tx - y\|$. But, since $\|y - Tx\| > dist(A, B)$, it follows that $\|Tx - y\| < \|Tx - y\|$, which is a contradiction. Thus $x = y$. □

Declarations

Availablity of data and materials

Not applicable.

Conflict of interest

The authors declare that they have no competing interests.

Fundings

Not applicable.

References

1. S. Banach: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. **3** (1922), 133-181.
2. F.E. Browder: On the convergence of successive approximations for nonlinear functional equations. Indag. Math. **30** (1968), 27-35.
3. L.K. Dey & S. Mondal: Best proximity point of F-contraction in complete metric space. Bull. Allahabad Math. Soc. **30** (2015), no. 2, 173-189.
4. L.K. Dey & S. Mondal: Best proximity point theorems for cyclic Wardowski type contraction. Thai J. Math. **18** (2020), 1857-1864.
5. A.A. Eldred & P. Veeramani: Existence and convergence of best proximity points. J. Math. Anal. Appl. **323** (2006), 1001-1006.
6. M. Jleli, E. Karapinar & B. Samet: Further generalizations of the Banach contraction principle. J. Inequal. Appl. **2014** (2014), Paper No. 439.
7. M. Jleli & B. Samet: A new generalization of the Banach contraction principle. J. Inequal. Appl. **2014** (2014), Paper No. 38.
8. R. Kannan: Some results on fixed points-II. Am. Math. Monthly **76** (1969), 405-408.
9. A. Kari, M. Rossafi, E. Marhrani & M. Aamri: Fixed-point theorem for nonlinear F-contraction via w-distance. Adv. Math. Phys. **2020** (2020), Article ID 6617517.
10. S. Reich: Some remarks concerning contraction mappings. Canad. Math. Bull. **14** (1971), no. 2, 121-124.
11. D. Zheng , Z. Cai & P. Wang: New fixed point theorems for θ-ϕ-contraction in complete metric spaces. J. Nonlinear Sci. Appl. **10** (2017), 2662-2670.

[a]Professor: LASMA Laboratory Department of Mathematics, Faculty of Sciences, Dhar El Mahraz University, Sidi Mohamed Ben Abdellah, Fes, Morocco
Email address: `rossafimohamed@gmail.com; mohamed.rossafi@usmba.ac.ma`

[b]PROFESSOR: LABORATORY OF ALGEBRA, ANALYSIS AND APPLICATIONS, FACULTY OF SCIENCES BEN M'SIK, HASSAN II UNIVERSITY, CASABLANCA, MOROCCO
Email address: abdkrimkariprofes@gmail.com

[c]PROFESSOR: DEPARTMENT OF DATA SCIENCE, DAEJIN UNIVERSITY, KYUNGGI 11159, KOREA
Email address: jrlee@daejin.ac.kr