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APPLICATIONS OF THE JACK'S LEMMA FOR ANALYTIC FUNCTIONS CONCERNED WITH ROGOSINSKI'S LEMMA

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Abstract. In this study, a Schwarz lemma at the boundary for analytic functions at the unit disc, which generalizes classical Schwarz lemma for bounded analytic functions, is considered.The results of Rogosinskis lemma and Jacks lemma have been utilized to derive novel inequalities. Also, these inequalities have been strengthened by considering the critical points which are different from zero.

1. Introduction

Denote by $U = \{z : |z| < 1\}$ the unit disc in the complex plane $\mathbb{C}$ and let $\omega : U \to U$ be an analytic function with $\omega(0) = 0$. The Schwarz lemma tells us that $|\omega(z)| \leq |z|$ for all $z \in U$ and $|\omega'(0)| \leq 1$. In addition, if the equality $|\omega(z)| = |z|$ holds for any $z \neq 0$, or $|\omega'(0)| = 1$, then f is a rotation; that is $\omega(z) = ze^{i\theta}$, θ real ([5], p.329). A sharpened version of this is Rogosinski's Lemma [10], which say that for all $z \in U$

$$|\omega(z) - b_1| \leq r_1,$$

where

$$b_1 = \frac{z\omega'(0)\left(1 - |z|^2\right)}{1 - |z|^2 |\omega'(0)|^2} \quad and \quad r_1 = \frac{|z|^2\left(1 - |\omega'(0)|^2\right)}{1 - |z|^2 |\omega'(0)|^2}.$$

Schwarz lemma has several applications in the field of electrical and electronics engineering. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for the analysis of transfer functions in control engineering and nulti-notch filter design in signal processing [13, 14].

The following lemma, known as Jack's Lemma, is needed in the sequel [6].

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Lemma 1.1. *Let $\omega(z)$ be a non-constant analytic function in U with $\omega(0)=0$. If*

$$|\omega(z_0)| = \max\{|\omega(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + ...$ that are analytic in U. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\left|zf''(z)\right|^2 < \left|f'(z)-1\right|, \; z \in U.$$

The certain analytic functions which is in the class of $\mathcal{M}$ on the unit disc U are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{M}$ by applying Jack's Lemma and Rogosinski's Lemma.

Suppose that $f(z) \in \mathcal{M}$ and consider the following function

$$\vartheta(z) = f'(z) - 1.$$

It is an analytic function in U and $\vartheta(0) = 0$. Now, let us show that $|\vartheta(z)| < 1$ in U. We suppose that there exists a $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |\vartheta(z)| = |\vartheta(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\vartheta(z_0) = e^{i\theta} \text{ and } \frac{z_0\vartheta'(z_0)}{\vartheta(z_0)} = k.$$

Therefore, we have that

$$\frac{|z_0f''(z_0)|^2}{|f'(z_0)-1|} = \frac{|z_0\vartheta'(z_0)|^2}{|\vartheta(z_0)|} = \frac{|k\vartheta(z_0)|^2}{|\vartheta(z_0)|} = k^2\left|e^{i\theta}\right| \geq 1.$$

This contradicts the $f(z) \in \mathcal{M}$. This means that there is no point $z_0 \in U$ such that $\max_{|z|\leq|z_0|} |\vartheta(z)| = |\vartheta(z_0)| = 1$. Hence, we take $|\vartheta(z)| < 1$ in U. By the Schwarz lemma, we obtain

$$\begin{aligned}
\vartheta(z) &= f'(z) - 1 \\
&= 1 + 2a_2z + 3a_3z^2 + ... - 1 \\
&= 2a_2z + 3a_3z^2 + ..., \\
\frac{\vartheta(z)}{z} &= 2a_2 + 3a_3z + ...
\end{aligned}$$

and

$$|a_2| \leq \frac{1}{2}.$$

We thus obtain the following lemma.

Lemma 1.2. *If $f(z) \in \mathcal{M}$, then we have the inequality*

$$|a_2| \leq \frac{1}{2}. \tag{1.1}$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows:

If ω extends continuously to some boundary point c with $|c| = 1$, and if $|\omega(c)| = 1$ and $\omega'(c)$ exists, then $|\omega'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. In addition to conditions of the boundary Schwarz Lemma, if f fixes the point zero, that is $\omega(z) = a_1 z + a_2 z^2 + ...$, then the inequality

$$\left|\omega'(c)\right| \geq \frac{2}{1 + |\omega'(0)|}. \tag{1.2}$$

is obtained [12]. Inequality (1.2) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 7, 8, 10, 11, 12, 13, 14, 15]. Mercer [9] prove a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [10]. In addition, he obtain a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [11].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [16])

Lemma 1.3 (Julia-Wolff lemma)**.** *Let ω be an analytic function in U, $\omega(0) = 0$ and $\omega(U) \subset U$. If, in addition, the function ω has an angular limit $\omega(c)$ at $c \in \partial U$, $|\omega(c)| = 1$, then the angular derivative $\omega'(c)$ exists and $1 \leq |\omega'(c)| \leq \infty$.*

Corollary 1.4. *The analytic function ω has a finite angular derivative $\omega'(c)$ if and only if ω' has the finite angular limit $\omega'(c)$ at $c \in \partial U$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{M}$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. Also, these inequalities have been strengthened by considering the critical points which are different from zero.

Theorem 2.1. *Let $f(z) \in \mathcal{M}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f'(c) = 2$. Then we have the inequality*

$$\left|f''(c)\right| \geq 1. \tag{2.1}$$

Proof. Let

$$\vartheta(z)=f'(z)-1.$$

$\vartheta(z)$ is an analytic function in U, $\vartheta(0)=0$ and $|\vartheta(z)|<1$ for $z\in U$. Also, we take $|\vartheta(c)|=1$ for $c\in\partial U$ and $f'(c)=2$. Therefore, from Schwarz lemma, we obtain $|\vartheta(z)|\leq|z|$ for $z\in U$ and

$$\left|\frac{\vartheta(z)-1}{|z|-1}\right|\geq\frac{1-|\vartheta(z)|}{1-|z|}\geq\frac{1-|z|}{1-|z|}=1.$$

Without loss of generality, we will assume that $c=1$. Passing to angular limit in the last equality yields

$$\left|\vartheta'(1)\right|\geq 1$$

and

$$\left|f''(1)\right|\geq 1.$$

□

The inequality (2.1) can be strengthened as below by taking into account a_2 which is second coefficient in the expansion of the function $f(z)=z+a_2z^2+a_3z^3+....$

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f''(c)\right|\geq\frac{2}{1+|f''(0)|}. \tag{2.2}$$

Proof. Let $\vartheta(z)$ be the same as in the proof of Theorem 2.1. So, from Rogosinski's lemma, we obtain

$$|\vartheta(z)-b_1|\leq r_1,$$

where

$$b_1=\frac{z\vartheta'(0)\left(1-|z|^2\right)}{1-|z|^2|\vartheta'(0)|^2},\ r_1=\frac{|z|^2\left(1-|\vartheta'(0)|^2\right)}{1-|z|^2|\vartheta'(0)|^2}.$$

Without loss of generality, we will assume that $c=1$. Thus, we obtain

$$\begin{aligned}\left|\frac{\vartheta(z)-1}{z-1}\right| &\geq \frac{1-|b_1|-r_1}{1-|z|}=\frac{1-\frac{|z||\vartheta'(0)|\left(1-|z|^2\right)}{1-|z|^2|\vartheta'(0)|^2}-\frac{|z|^2\left(1-|\vartheta'(0)|^2\right)}{1-|z|^2|\vartheta'(0)|^2}}{1-|z|}\\ &= \frac{1-|z|^2|\vartheta'(0)|^2-|z||\vartheta'(0)|\left(1-|z|^2\right)-|z|^2\left(1-|\vartheta'(0)|^2\right)}{(1-|z|)\left(1-|z|^2|\vartheta'(0)|^2\right)}\\ &= \frac{\left(1-|z|^2\right)\left(^2\left(1-|z||\vartheta'(0)|\right)\right)}{(1-|z|)\left(1-|z|^2|\vartheta'(0)|^2\right)}\\ &= \frac{1+|z|}{1+|z||\vartheta'(0)|}.\end{aligned}$$

Passing to the angular limit in the last inequality yields

$$\left|\vartheta'(1)\right| \geq \frac{2}{1+|\vartheta'(0)|}.$$

Since

$$\left|\vartheta'(1)\right| = \left|f''(1)\right|$$

and

$$\left|\vartheta'(0)\right| = \left|f''(0)\right|$$

we get

$$\left|f''(1)\right| \geq \frac{2}{1+|f''(0)|}.$$

In the following theorem, inequality (2.2) has been strengthened by adding the consecutive terms a_2 and a_3 of $f(z)$ function. □

Theorem 2.3. *Let $f(z) \in \mathcal{M}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f'(c) = 2$. Then we have the inequality*

$$\left|f''(c)\right| \geq 1 + \frac{4\left(1-|f''(0)|\right)^2}{2\left(1-|f''(0)|^2\right)+|f'''(0)|}. \tag{2.3}$$

Proof. Let $\vartheta(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$k(z) = \frac{\vartheta(z)}{z}$$

and

$$s(z) = \frac{k(z)-k(0)}{1-\overline{k(0)}k(z)}$$

The function $s(z)$ is analytic in U, $s(0) = 0$, $|s(z)| < 1$ for $|z| < 1$ and

$$s'(0) = \frac{k'(0)}{\left(1-|k(0)|^2\right)} = \frac{\vartheta''(0)}{2\left(1-|\vartheta'(0)|^2\right)}.$$

From Rogosinski's Lemma and [9, 10], we have

$$|\vartheta(z) - b_2| \leq r_2, \tag{2.4}$$

where

$$b_2 = \frac{z\,|\vartheta'(0)|\left(1-\rho^2\right)}{1-\rho^2\,|\vartheta'(0)|^2},\quad r_2 = \frac{\rho\,|z|\left(1-|\vartheta'(0)|^2\right)}{1-\rho^2\,|\vartheta'(0)|^2},\quad \rho = |z|\,\frac{|z|+|s'(0)|}{1+|z|\,|s'(0)|}.$$

Without loss of generality, we will assume that $c = 1$. So, from (2.4), we obtain

$$\begin{aligned}
\left|\frac{\vartheta(z)-1}{z-1}\right| &\geq \frac{1-|b_2|-r_2}{1-|z|} = \frac{1-\frac{|z||\vartheta'(0)|\left(1-\rho^2\right)}{1-\rho^2|\vartheta'(0)|^2}-\frac{\rho|z|\left(1-|\vartheta'(0)|^2\right)}{1-\rho^2|\vartheta'(0)|^2}}{1-|z|} \\
&= \frac{1-\rho^2\left|\vartheta'(0)\right|^2-|z|\left|\vartheta'(0)\right|\left(1-\rho^2\right)-\rho\left|z\right|\left(1-\left|\vartheta'(0)\right|^2\right)}{(1-|z|)\left(1-\rho^2\left|\vartheta'(0)\right|^2\right)} \\
&= \frac{\left(1-\rho\left|\vartheta'(0)\right|\right)\left(1+\rho\left|\vartheta'(0)\right|-|z|\left|\vartheta'(0)\right|-\rho\left|z\right|\right)}{(1-|z|)\left(1-\rho^2\left|\vartheta'(0)\right|^2\right)} \\
&= \frac{1+\rho\left|\vartheta'(0)\right|-|z|\left|\vartheta'(0)\right|-\rho\left|z\right|}{(1-|z|)\left(1+\rho\left|\vartheta'(0)\right|\right)}.
\end{aligned}$$

Since $\rho = |z|\,\frac{|z|+|s'(0)|}{1+|z||s'(0)|}$, we take

$$\begin{aligned}
&\left|\frac{\vartheta(z)-1}{z-1}\right| \geq \frac{1+|z|\frac{|z|+|s'(0)|}{1+|z||s'(0)|}|\vartheta'(0)|-|z||\vartheta'(0)|-|z|\frac{|z|+|s'(0)|}{1+|z||s'(0)|}|z|}{(1-|z|)\left(1+|z|\frac{|z|+|s'(0)|}{1+|z||s'(0)|}|\vartheta'(0)|\right)} \\
&= \frac{1-|z|^3+|z||s'(0)|(1-|z|)-|z||\vartheta'(0)|(1-|z|)+|z||\vartheta'(0)||s'(0)|(1-|z|)}{(1-|z|)\left(1+|z||s'(0)|+|z|^2|\vartheta'(0)|+|z||\vartheta'(0)||s'(0)|\right)} \\
&= \frac{\left(1+|z|+|z|^2\right)+|z||s'(0)|-|z||\vartheta'(0)|+|z||s'(0)||\vartheta'(0)|}{1+|z||s'(0)|+|z|^2|\vartheta'(0)|+|z||\vartheta'(0)||s'(0)|}.
\end{aligned}$$

Passing to the angular limit in the last inequality yields

$$\begin{aligned}
\left|\varphi'(1)\right| &\geq \frac{3+|s'(0)|-|\vartheta'(0)|+|s'(0)|\,|\vartheta'(0)|}{1+|s'(0)|+|\vartheta'(0)|+|s'(0)|\,|\vartheta'(0)|} \\
&= \frac{3+|s'(0)|-|\vartheta'(0)|+|s'(0)|\,|\vartheta'(0)|}{(1+|s'(0)|)\,(1+|\vartheta'(0)|)}.
\end{aligned}$$

A little manipulation gives

$$\begin{aligned}
\left|\varphi'(1)\right| &\geq 1+\frac{2\left(1-|\vartheta'(0)|\right)^2}{(1+|s'(0)|)\left(1-|\vartheta'(0)|^2\right)} \\
&= 1+\frac{4\left(1-|\vartheta'(0)|\right)^2}{2\left(1-|\vartheta'(0)|^2\right)+|\vartheta''(0)|}.
\end{aligned}$$

Since

$$\left|\vartheta'(1)\right| = \left|f''(1)\right|,$$

$$\left|\vartheta'(0)\right| = \left|f''(0)\right|$$

and

$$\left|\vartheta''(0)\right| = \left|f'''(0)\right|,$$

we obtain

$$\left|f''(1)\right| \geq 1 + \frac{4\left(1-|f''(0)|\right)^2}{2\left(1-|f''(0)|^2\right)+|f'''(0)|}$$

□

If $f(z)-z$ a have zeros different from $z=0$, taking into account these critical points, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{M}$. Assume that, for some $c\in\partial U$, f has an angular limit $f(c)$ at c, $f'(c)=2$. Let $z_1,z_2,...,z_n$ be critical points of the function $f(z)-z$ in E that are different from zero. Then we have the inequality*

$$\left|f''(c)\right| \geq \left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right.$$

$$\left. + \frac{2\left(\prod_{i=1}^{n}|z_i|-2|a_2|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4|a_2|^2+\prod_{i=1}^{n}|z_i|\left|3a_3+2a_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right). \tag{2.5}$$

Proof. Let $\vartheta(z)$ be as in the proof of Theorem 2.1 and $z_1,z_2,...,z_n$ be critical points of the function $f(z)-z$ in U that are different from zero. Let

$$B(z)=z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}.$$

$B(z)$ is an analytic function in U and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in U$, we have $|\vartheta(z)|\leq|B(z)|$. Consider the function

$$\begin{aligned} u(z) &= \frac{\vartheta(z)}{B(z)} \\ &= \frac{f'(z)-1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}} \\ &= \frac{2a_2z+3a_3z^2+...}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}, \end{aligned}$$

$$= \frac{2a_2 + 3a_3 z + ...}{\prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}}.$$

$u(z)$ is analytic in U and $|u(z)| < 1$ for $z \in U$. In particular, we have

$$|u(0)| = \frac{2\,|a_2|}{\prod_{i=1}^{n} |z_i|}$$

and

$$|u'(0)| = \frac{\left| 3a_3 + 2a_2 \sum_{i=1}^{n} \frac{1 - |z_i|^2}{z_i} \right|}{\prod_{i=1}^{n} |z_i|}.$$

Moreover, with the simple calculations, we get

$$\frac{c\vartheta'(c)}{\vartheta(c)} = |\vartheta'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)}$$

and

$$|B'(c)| = 1 + \sum_{i=1}^{n} \frac{1 - |z_i|^2}{|c - z_i|^2}.$$

The auxiliary function

$$g(z) = \frac{u(z) - u(0)}{1 - \overline{u(0)}u(z)}$$

is analytic in the unit disc U, $g(0) = 0$, $|g(z)| < 1$ for $z \in U$ and $|g(c)| = 1$ for $c \in \partial U$. From (1.2), we obtain

$$\begin{aligned} \frac{2}{1 + |g'(0)|} &\leq |g'(c)| = \frac{1 + |u(0)|^2}{\left|1 - \overline{u(0)}u(c)\right|^2} |u'(c)| \\ &\leq \frac{1 + |u(0)|}{1 - |u(0)|} \left\{ |\vartheta'(c)| - |B'(c)| \right\}. \end{aligned}$$

Since

$$\left|g'(0)\right| = \frac{|u'(0)|}{1-|u(0)|^2} = \frac{\dfrac{\left|3a_3+2a_2\sum\limits_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\prod\limits_{i=1}^{n}|z_i|}}{1-\left(\dfrac{2|a_2|}{\prod\limits_{i=1}^{n}|z_i|}\right)^2}$$

$$= \prod_{i=1}^{n}|z_i| \frac{\left|3a_3+2a_2\sum\limits_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-4|a_2|^2},$$

we get

$$\frac{2}{1+\prod\limits_{i=1}^{n}|z_i|\dfrac{\left|3a_3+2a_2\sum\limits_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-4|a_2|^2}} \leq$$

$$\frac{1+\dfrac{2|a_2|}{\prod\limits_{i=1}^{n}|z_i|}}{1-\dfrac{2|a_2|}{\prod\limits_{i=1}^{n}|z_i|}}\left\{|f''(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

$$\frac{2\left(\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-4|a_2|^2\right)}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-4|a_2|^2+\prod\limits_{i=1}^{n}|z_i|\left|3a_3+2a_2\sum\limits_{i=1}^{n}\dfrac{1-|z_i|^2}{z_i}\right|}$$

$$\leq \frac{\prod\limits_{i=1}^{n}|z_i|+2|a_2|}{\prod\limits_{i=1}^{n}|z_i|-2|a_2|}\left\{|f''(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

$$\frac{2\left(\prod_{i=1}^{n}|z_i|-2|a_2|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4|a_2|^2+\prod_{i=1}^{n}|z_i|\left|3a_3+2a_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq |f''(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}$$

and

$$|f''(c)| \geq \left(1+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right.$$

$$\left.+\frac{2\left(\prod_{i=1}^{n}|z_i|-2|a_2|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2-4|a_2|^2+\prod_{i=1}^{n}|z_i|\left|3a_3+2a_2\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right).$$

□

Theorem 2.5. *Let $f(z)\in\mathcal{M}$. Assume that, for $1\in\partial U$, f has an angular limit $f(1)$ at 1, $f'(1)=2$. Then we have the inequality*

$$f''(1) \geq 1+\frac{|1-f''(0)|^2}{1-|f''(0)|^2}\frac{2}{1+\Re\left(\frac{1-\overline{f''(0)}}{1-f''(0)}\frac{f'''(0)}{1-|f''(0)|^2}\right)}. \tag{2.6}$$

Proof. Let $\vartheta(z)$ be as in the proof of Theorem 2.1. So, from hypothesis, we have

$$\vartheta(1)=f'(1)-1=1$$

and

$$\vartheta(1)=1,$$

where 1 is a boundary fixed point of $\vartheta(z)$. Also, we have

$$\begin{aligned}\vartheta(z) &= f'(z)-1=2a_2z+3a_3z^2+\ldots\\ &= c_1z+c_2z^2+\ldots\end{aligned}$$

Let

$$\Delta(z)=\frac{1-\overline{c_1}}{c_1-1}\frac{c_1z-\vartheta(z)}{z-\overline{c_1}\vartheta(z)}.$$

$\Delta(z)$ is analytic in U, $|\Delta(z)|<1$ for $|z|<1$ and 1 is a boundary fixed point of $\Delta(z)$. That is, $\Delta(1)=1$. Also, with the simple calculations, we obtain

$$\Delta'(1)=\frac{1-|c_1|^2}{|1-c_1|^2}\left(\vartheta'(1)-1\right)$$

and

$$\Delta'(0) = \frac{1-\overline{c_1}}{1-c_1}\frac{c_2}{1-|c_1|^2}.$$

In particular, from (1.2), we have

$$\Delta'(1) \geq \frac{2}{1+\Re\Delta'(0)}. \tag{2.7}$$

Let us substitute the values of $\Delta'(1)$ and $\Delta'(0)$ into (2.7). Therefore, we take

$$\frac{1-|c_1|^2}{|1-c_1|^2}\left(\vartheta'(1)-1\right) \geq \frac{2}{1+\Re\left(\frac{1-\overline{c_1}}{1-c_1}\frac{c_2}{1-|c_1|^2}\right)}$$

and

$$\vartheta'(1) \geq 1+\frac{|1-c_1|^2}{1-|c_1|^2}\frac{2}{1+\Re\left(\frac{1-\overline{c_1}}{1-c_1}\frac{c_2}{1-|c_1|^2}\right)}.$$

Since

$$\vartheta'(1) = f''(1),\ c_1 = f''(0),\ c_2 = f'''(0),$$

we obtain the inequality (2.6). □

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