J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2022.29.4.401 ISSN(Online) 2287-6081
Volume 29, Number 4 (November 2022), Pages 401–419

PARTIAL S-METRIC SPACES AND FIXED POINT RESULTS

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Abstract. In this paper, we introduce the notion of partial S-metric space and prove a common fixed point theorem in the respective setting. An example is presented to show the effectiveness of this approach.

1. Introduction and Preliminaries

Metrical fixed point theory became one of the most interesting area of research in the last fifty years. A lot of fixed and common fixed point results have been obtained by several authors in various types of spaces, such as metric spaces, fuzzy metric spaces, uniform spaces and others see [1, 2, 3, 6, 7, 11, 13, 15, 16, 17]). One of the most interesting spaces is a partial metric space, which was defined by Matthews in the following way.

Definition 1.1 ([8]). A partial metric on a nonempty set X is a function $p : X\times X \to [0, +\infty)$ such that, for all $x, y, z \in X$,

(p$_1$) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
(p$_2$) $p(x, x) \leq p(x, y)$,
(p$_3$) $p(x, y) = p(y, x)$,
(p$_4$) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is called a *partial metric space.*

On the other hand, S-metric spaces were initiated by Sedghi, Shobe and Aliouche in [14] (see also [4] and references cited therein).

Received by the editors October 06, 2022. Accepted November 07, 2022.
2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, partial metric space, S-metric space, partially weakly increasing mapping.
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Definition 1.2 ([14]). An S-metric on a nonempty set X is a function $S : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

(s_1) $S(x, y, z) = 0 \iff x = y = z$,

(s_2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

In this case, the pair (X, S) is called an *S-metric space.*

It is easy to see that in S-metric space (X, S) we always have $S(x, x, y) = S(y, y, x)$, $x, y \in X$.

In this paper, combining these two concepts, we introduce the notion of partial S-metric space and prove a common fixed point theorem for weakly increasing mappings in ordered spaces of this kind.

We recall some notions and properties in S-metric spaces.

Definition 1.3 ([12]). Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X.

(a) The sequence $\{x_n\}$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) $\{x_n\}$ is said to be a *Cauchy sequence* if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(c) The space (X, S) is said to be *complete* if every Cauchy sequence in it converges.

Lemma 1.4 ([12]). *Let (X, S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

2. Partial S-metric Spaces

In this section, we introduce partial S-metric spaces and investigate some of their simple properties.

Definition 2.1. A partial S-metric or modified partial S-metric on a nonempty set X is a function $S^* : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$,

(s_{p1}) $x = y = z \iff S^*(x, y, z) = S^*(x, x, x) = S^*(y, y, y) = S^*(z, z, z)$,

(s_{p2}) $S^*(x, x, x) \leq S^*(x, y, z)$,

(s_{p3}) $S^*(x, y, z) \leq S^*(x, x, a) + S^*(y, y, a) + S^*(z, z, a) - 2S^*(a, a, a)$.

The pair (X, S^*) is then called a *partial* or *modified partial S-metric space.*

Each S-metric space is also a modified partial S-metric space. The converse is not true, as shown by the following example.

Example 2.2. Let $X = [0, +\infty)$ and let $S^* : X \times X \times X \to [0, +\infty)$ be defined by $S^*(x, y, z) = \max\{x, y, z\}$. Then, it is easy to check that (X, S^*) is a modified partial S-metric space. Obviously, (X, S^*) is not an S-metric space.

Partial S-metric spaces was first introduced by N. Mlaiki in 2014 see [9]. However, we introduced a generalization of partial S-metric spaces. That is due to the fact that:

$$\begin{aligned} S^*(x,y,z) &\leq S^*(x,x,a) + S^*(y,y,a) + S^*(z,z,a) - 2S^*(a,a,a) \\ &\leq S^*(x,x,a) + S^*(y,y,a) + S^*(z,z,a) - S^*(a,a,a). \end{aligned}$$

Now, we present some example of such spaces, i. e. a partial S-metric space that is not a modified partial S-metric space.

Example 2.3. Let $X = [0, +\infty)$ and let $S^* : X \times X \times X \to [0, +\infty)$ be defined by

$$S^*(x,y,z) = \begin{cases} 5, & x \neq y \neq z, \\ 2, & x = y \neq z, \\ 1, & x = y = z. \end{cases}$$

Then, it is easy to check that (X, S^*) is a partial S-metric space. Obviously, (X, S^*) is not a modified partial S-metric space.

Lemma 2.4. *For a modified partial S-metric S^* on X, we have for all $x, y \in X$,*

(a) $S^*(x, x, y) = S^*(y, y, x)$,
(b) *if* $S^*(x, x, y) = 0$ *then* $x = y$.

Proof. (a) By the condition (s_{p3}), we get

(i) $S^*(x,x,y) \leq S^*(x,x,x) + S^*(x,x,x) + S^*(y,y,x) - 2S^*(x,x,x) = S^*(y,y,x)$, and similarly

(ii) $S^*(y,y,x) \leq S^*(y,y,y) + S^*(y,y,y) + S^*(x,x,y) - 2S^*(y,y,y) = S^*(x,x,y)$.
By (i) and (ii), we obtain $S^*(x,x,y) = S^*(y,y,x)$.

(b) By the condition (s_{p2}), we obtain

(iii) $S^*(x,x,x) \leq S^*(x,x,y) = 0$,
and similarly by the relation (a), we also have:

(iv) $S^*(y,y,y) \leq S^*(y,y,x) = S^*(x,x,y) = 0$.

By (iii) and (iv), we get $S^*(x,x,y) = S^*(x,x,x) = S^*(y,y,y) = 0$, which by the condition (s_{p1}) implies that $x = y$. □

Remark 2.5. Dung, Hieu and Radojević [5, Examples 2.1 and 2.2] noted that the class of S-metric spaces is incomparable with the class of G-metric spaces in the sense of Mustafa and Sims [10]. The same examples show that the class of partial S-metric spaces is incomparable with the class of GP-metric spaces in the sense of Zand and Nezhad [18].

Definition 2.6. Let (X, S^*) be a partial S-metric space and $\{x_n\}$ be a sequence in X.

(a) The sequence $\{x_n\}$ converges to $x \in X$ (denoted as $x_n \to x$ as $n \to \infty$) if

$$\lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = S^*(x,x,x).$$

(b) The sequence $\{x_n\}$ is said to be a *Cauchy sequence* if there exists

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

(c) The space (X, S^*) is complete if each Cauchy sequence in X converges.

Note that if $x_n \to x$ as $n \to \infty$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x) - S^*(x,x,x)| < \epsilon, \quad \forall n \geq n_0, \tag{2.1}$$

and

$$|S^*(x_n, x_n, x_n) - S^*(x,x,x)| < \epsilon, \quad \forall n \geq n_0. \tag{2.2}$$

Hence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \epsilon, \quad \forall n \geq n_0. \tag{2.3}$$

Lemma 2.7. *Let (X, S^*) be a partial S-metric space. If a sequence $\{x_n\}$ in X converges to $x \in X$, then x is unique.*

Proof. Let $\{x_n\}$ converge to x and y. Then we have

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, x) = S^*(x,x,x), \tag{2.4}$$

and

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, y) = S^*(y,y,y).$$

By the condition (s_{p3}), the relation (2.4) and Lemma 2.4, we obtain

$$\begin{aligned} S^*(x,x,y) &\le 2S^*(x,x,x_n) + S^*(y,y,x_n) - 2S^*(x_n,x_n,x_n) \\ &= 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + S^*(x_n,x_n,y) - S^*(y,y,y) + S^*(y,y,y). \end{aligned}$$

By taking the limit as $n \to \infty$, we get $S^*(x,x,y) \le S^*(y,y,y)$.

Also, by the condition (s_{p2}), we have $S^*(y,y,y) \le S^*(y,y,x) = S^*(x,x,y)$. Hence we obtain $S^*(x,x,y) = S^*(y,y,y)$. Similarly, we have $S^*(x,x,y) = S^*(x,x,x)$. Therefore, from the condition (s_{p1}), it follows that $x = y$. □

Lemma 2.8. *Let (X, S^*) be a partial S-metric space. Then each convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Proof. Let $\{x_n\}$ converge to x, that is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that the inequalities (2.1), (2.2) and (2.3) hold for all $n \ge n_0$. By the condition (s_{p3}) and these inequalities, we have for $m, n \ge n_0$,

(2.5)
$$\begin{aligned} S^*(x_n,x_n,x_m) &\le S^*(x_n,x_n,x) + S^*(x_n,x_n,x) + S^*(x_m,x_m,x) - 2S^*(x,x,x) \\ &\le 2(S^*(x_n,x_n,x) - S^*(x,x,x)) + S^*(x_m,x_m,x) - S^*(x,x,x) + S^*(x,x,x) \\ &< 2\epsilon + \epsilon + S^*(x,x,x). \end{aligned}$$

Similarly, by the condition (s_{p3}) and Lemma 2.7, we obtain

$$\begin{aligned} S^*(x,x,x) &\le S^*(x,x,x_n) + S^*(x,x,x_n) + S^*(x,x,x_n) - 2S^*(x_n,x_n,x_n) \\ &= 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + S^*(x,x,x_n) \\ &\le 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + 2S^*(x,x,x_m) \\ &\qquad + S^*(x_n,x_n,x_m) - 2S^*(x_m,x_m,x_m). \\ &< 2\epsilon + 2\epsilon + S^*(x_n,x_n,x_m). \end{aligned} \tag{2.6}$$

Hence, by (2.5) and (2.6), we get

$$|S^*(x_n,x_n,x_m) - S^*(x,x,x)| < 4\epsilon$$

for $m, n \ge n_0$. Thus $\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) = S^*(x,x,x)$, and the sequence $\{x_n\}$ is a Cauchy sequence. □

The notion of S_b-metric spaces was introduced independently in [13] and [16].

Definition 2.9. Let X be a nonempty set and $b \ge 1$ be a given real number. An S_b-metric on X, with a parameter b, is a function $S_b : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

(s_{b1}) $S_b(x,y,z)=0 \iff x=y=z$,

(s_{b2}) $S_b(x,x,y)=S_b(y,y,x)$,

(s_{b3}) $S_b(x,y,z)\leq b(S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a))$.

In this case, the pair (X,S_b) is called an S_b*-metric space.*

A connection between partial S-metric and S_b-metric spaces is given by the following lemma.

Lemma 2.10. *If* (X,S^*) *is a partial* S*-metric space, then* $S^s: X\times X\times X \to [0,+\infty)$, *given by*

$$S^s(x,y,z)=S^*(x,x,y)+S^*(y,y,z)+S^*(z,z,x)-S^*(x,x,x)-S^*(y,y,y)-S^*(z,z,z),$$

is an S_b*-metric on* X*, with a parameter* $b=2$.

Proof. First of all, by the condition (s_{p2}) and the definition of S^s, we have $S^s(x,y,z)\geq 0$. Further, we check that the conditions of Definition 2.9 are fulfilled.

(s_{b1}) If $S^s(x,y,z)=0$ then it follows that $S^*(x,y,z)=S^*(x,x,x)=S^*(y,y,y)=S^*(z,z,z)$. That is, $x=y=z$. Conversely, if $x=y=z$, then we have $S^s(x,y,z)=0$.

(s_{b2}) By the definition of S^s and Lemma 2.4, we get

$$\begin{aligned} S^s(x,x,y) &= S^*(x,x,x)+S^*(x,x,y)+S^*(y,y,x)-S^*(x,x,x) \\ &\quad -S^*(x,x,x)-S^*(y,y,y) \\ &= S^*(x,x,x)+S^*(x,x,y)+S^*(x,x,y)-S^*(x,x,x) \\ &\quad -S^*(x,x,x)-S^*(y,y,y) \\ &= 2S^*(x,x,y)-S^*(x,x,x)-S^*(y,y,y). \end{aligned}$$

Similarly, we can show that

$$S^s(y,y,x)=2S^*(x,x,y)-S^*(x,x,x)-S^*(y,y,y).$$

Therefore, $S^s(x,x,y)=S^s(y,y,x)$. We always have that $S^*(x,x,y)-S^*(x,x,x)\leq S^s(x,x,y)$.

(s_{b3}) By the condition (s_{p3}) and Lemma 2.4, we obtain

$$\begin{aligned} S^s(x,y,z) &= S^*(x,x,y)+S^*(y,y,z)+S^*(z,z,x)-S^*(x,x,x) \\ &\quad -S^*(y,y,y)-S^*(z,z,z) \end{aligned}$$

$$
\begin{aligned}
&\leq 2S^*(x,x,a) - 2S^*(a,a,a) + S^*(y,y,a) \\
&\quad + 2S^*(y,y,a) - 2S^*(a,a,a) + S^*(z,z,a) \\
&\quad + 2S^*(z,z,a) - 2S^*(a,a,a) + S^*(x,x,a) \\
&\quad - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z) \\
&= 3S^*(a,a,x) - 2S^*(a,a,a) - S^*(x,x,x) + S^*(a,a,x) - S^*(x,x,x) \\
&\quad + 3S^*(a,a,y) - 2S^*(a,a,a) - S^*(y,y,y) + S^*(a,a,y) - S^*(y,y,y) \\
&\quad + 3S^*(a,a,z) - 2S^*(a,a,a) - S^*(z,z,z) + S^*(a,a,z) - S^*(z,z,z) \\
&= 2[S^s(x,x,a) + S^s(y,y,a) + S^s(z,z,a)],
\end{aligned}
$$

as desired. □

Lemma 2.11. *Let (X, S^*) be a partial S-metric space and S^s be the respective S_b-metric introduced in Lemma 2.10.*

(a) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X, S^*) if and only if it is a Cauchy sequence in (X, S^s).

(b) The space (X, S^*) is complete if and only if the space (X, S^s) is complete. Furthermore, $\lim_{n\to\infty} S^s(x_n, x_n, x) = 0$ if and only if

$$S^*(x,x,x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, S^*). Then there exists

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(x_n, x_n, x_n).$$

Since

$$S^s(x_n, x_n, x_m) = 2S^*(x_n, x_n, x_m) - S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m),$$

we have

$$
\begin{aligned}
&\lim_{n,m\to\infty} S^s(x_n, x_n, x_m) \\
&\quad = 2\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) - \lim_{n\to\infty} S^*(x_n, x_n, x_n) - \lim_{m\to\infty} S^*(x_m, x_m, x_m) = 0.
\end{aligned}
$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X, S^s).

Next we prove that the completeness of (X, S^s) implies the completeness of (X, S^*). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X, S^*) then it is also a Cauchy sequence in (X, S^s). Since the space (X, S^s) is complete, we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} S^s(x_n, x_n, y) = 0$ since $S^s(x_n, x_n, y) = 2S^*(x_n, x_n, y) -$

$S^*(y,y,y) - S^*(x_n,x_n,x_n)$. Also, we know that

$$0 \le S^*(x_n,x_n,y) - S^*(y,y,y) < S^s(x_n,x_n,y),$$

and

$$0 \le S^*(x_n,x_n,y) - S^*(x_n,x_n,x_n) < S^s(x_n,x_n,y).$$

Therefore, we have

$$\lim_{n\to\infty} S^*(x_n,x_n,y) = \lim_{n\to\infty} S^*(x_n,x_n,x_n) = \lim_{n\to\infty} S^*(y,y,y).$$

Hence, we get that $\{x_n\}$ is a convergent sequence in (X,S^*).

Now we prove that every Cauchy sequence $\{x_n\}$ in (X,S^s) is a Cauchy sequence in (X,S^*). Let $\epsilon = \dfrac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $S^s(x_n,x_n,x_m) < \dfrac{1}{2}$ for all $n,m \ge n_0$. Since

$$\begin{aligned}
&S^*(x_n,x_n,x_n)\\
&\quad \le 4S^*(x_{n_0},x_{n_0},x_n) - 3S^*(x_n,x_n,x_n) - S^*(x_{n_0},x_{n_0},x_{n_0}) + S^*(x_n,x_n,x_n)\\
&\quad \le 2S^s(x_n,x_n,x_{n_0}) + S^*(x_{n_0},x_{n_0},x_{n_0}),
\end{aligned}$$

we get

$$\begin{aligned}
S^*(x_n,x_n,x_n) &\le 2S^s(x_n,x_n,x_{n_0}) + S^*(x_{n_0},x_{n_0},x_{n_0})\\
&\le 1 + S^*(x_{n_0},x_{n_0},x_{n_0}).
\end{aligned}$$

Consequently, the sequence $\{S^*(x_n,x_n,x_n)\}$ is bounded in $\mathbb{R}$ and so there exists an $\alpha \in \mathbb{R}$ such that the subsequence $\{S^*(x_{n_k},x_{n_k},x_{n_k})\}$ is convergent to α, i.e., $\lim_{k\to\infty} S^*(x_{n_k},x_{n_k},x_{n_k}) = \alpha$.

It remains to prove that $\{S^*(x_n,x_n,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in (X,S^s), for given $\epsilon > 0$, there exists n_ϵ such that $S^s(x_n,x_n,x_m) < \dfrac{\epsilon}{2}$ for all $n,m \ge n_\epsilon$. Thus, for all $n,m \ge n_\epsilon$,

$$\begin{aligned}
&|S^*(x_n,x_n,x_n) - S^*(x_m,x_m,x_m)|\\
&\qquad \le 4S^*(x_n,x_n,x_m) - 3S^*(x_n,x_n,x_n) - S^*(x_m,x_m,x_m)\\
&\qquad\quad + S^*(x_n,x_n,x_n) - S^*(x_m,x_m,x_m)\\
&\qquad \le 2S^s(x_n,x_n,x_m) < \epsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|S^*(x_n,x_n,x_n) - \alpha| &\le |S^*(x_n,x_n,x_n) - S^*(x_{n_k},x_{n_k},x_{n_k})| + |S^*(x_{n_k},x_{n_k},x_{n_k}) - \alpha|\\
&< \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \alpha$. Now,

$$\begin{aligned}|2S^*(x_n, x_n, x_m) - 2\alpha| &= |S^s(x_n, x_n, x_m) + S^*(x_n, x_n, x_n) - \alpha + S^*(x_m, x_m, x_m) - \alpha| \\ &\leq S^s(x_m, x_m, x_m) + |S^*(x_n, x_n, x_n) - \alpha| + |S^*(x_m, x_m, x_m) - \alpha| \\ &< \frac{\epsilon}{2} + 2\epsilon + 2\epsilon = \frac{9}{2}\epsilon.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, S^*).

In order to complete the proof, we have to prove that (X, S^s) is complete if (X, S^*) is complete. Let $\{x_n\}$ be a Cauchy sequence in (X, S^s). Then $\{x_n\}$ is a Cauchy sequence in (X, S^*) and so it is convergent to a point $y \in X$ with

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(y, y, x_n) = S^*(y, y, y).$$

Thus, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$|S^*(y, y, x_n) - S^*(y, y, y)| < \frac{\epsilon}{2} \quad \text{and} \quad |S^*(y, y, y) - S^*(x_n, x_n, x_n)| < \frac{\epsilon}{2},$$

whenever $n \geq n_\epsilon$. Hence we have

$$\begin{aligned}S^s(y, y, x_n) &= 2S^*(y, y, x_n) - S^*(x_n, x_n, x_n) - S^*(y, y, y) \\ &\leq |S^*(y, y, x_n) - S^*(y, y, y)| + |S^*(y, y, x_n) - S^*(x_n, x_n, x_n)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore (X, S^s) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty} S^s(a, a, x_n) = 0$ if and only if

$$S^*(a, a, a) = \lim_{n\to\infty} S^*(a, a, x_n) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

This completes the proof. □

Lemma 2.12. *Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences to $x \in X$ and $y \in X$, respectively, in a partial S-metric space (X, S^*). Then*

$$\lim_{n\to\infty} S^*(x_n, y_n, y_n) = S^*(x, y, y).$$

In particular, $\lim_{n\to\infty} S^*(x_n, y_n, z) = S^*(x, y, z)$ *for every $z \in X$.*

Proof. By the assumptions, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y) - S^*(y, y, y)| < \frac{\epsilon}{4},$$

$$|S^*(x_n, x_n, x_n) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y, y, y)| < \frac{\epsilon}{4},$$

$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y_n, y_n, y)| < \frac{\epsilon}{4},$$

hold for all $n \geq n_0$. By the condition (s_{p3}), for $n \geq n_0$, we have

$$\begin{aligned} S^*(x_n, x_n, y_n) &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, x) - 2S^*(x, x, x) \\ &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, y) + S^*(y_n, y_n, y) \\ &\quad + S^*(x, x, y) - 2S^*(y, y, y) - 2S^*(x, x, x) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x, x, y), \end{aligned}$$

and so we obtain

$$S^*(x_n, x_n, y_n) - S^*(x, x, y) < \epsilon.$$

Also,

$$\begin{aligned} S^*(x, x, y) &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, x_n) - 2S^*(x_n, x_n, x_n) \\ &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, y_n) + S^*(y, y, y_n) \\ &\quad + S^*(x_n, x_n, y_n) - 2S^*(y_n, y_n, y_n) - 2S^*(x_n, x_n, x_n) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x_n, x_n, y_n). \end{aligned}$$

Thus

$$S^*(x, x, y) - S^*(x_n, x_n, y_n) < \epsilon.$$

Hence for all $n \geq n_0$, we get $|S^*(x_n, x_n, y_n) - S^*(x, x, y)| < \epsilon$ and the result follows. □

Lemma 2.13. *If (X, S^*) is a partial S-metric space, then the S_b-metrics S^s (defined in Lemma 2.10) and $S^m : X \times X \times X \to \mathbb{R}^+$ given by*

$$S^m(x, y, z) = \max \left\{ \begin{array}{l} 2S^*(x, x, y) - S^*(x, x, x) - S^*(y, y, y), \\ 2S^*(y, y, z) - S^*(y, y, y) - S^*(z, z, z), \\ 2S^*(z, z, x) - S^*(z, z, z) - S^*(x, x, x) \end{array} \right\}$$

for all $x, y, z \in X$, are equivalent.

Proof. It is easy to see that S^m is an S_b-metric on X. Let $x, y, z \in X$. It is obvious that

$$S^m(x, y, z) \leq 2S^s(x, y, z).$$

On the other hand, since $a + b + c \leq 3 \max\{a, b, c\}$, it follows that

$$\begin{aligned} S^s(x, y, z) &= S^*(x, x, y) + S^*(y, y, z) + S^*(z, z, x) - S^*(x, x, x) \\ &\quad - S^*(y, y, y) - S^*(z, z, z) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y)] \\
&\quad + \frac{1}{2}[2S^*(y,y,z) - S^*(y,y,y) - S^*(z,z,z)] \\
&\quad + \frac{1}{2}[2S^*(z,z,x) - S^*(z,z,z) - S^*(x,x,x)] \\
&\leq \frac{3}{2}\max\left\{\begin{array}{c} 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y), \\ 2S^*(y,y,z) - S^*(y,y,y) - S^*(z,z,z), \\ 2S^*(z,z,x) - S^*(z,z,z) - S^*(x,x,x) \\ \end{array}\right\} \\
&= \frac{3}{2}S^m(x,y,z).
\end{aligned}
$$

Thus we have

$$\frac{1}{2}S^m(x,y,z) \leq S^s(x,y,z) \leq \frac{3}{2}S^m(x,y,z).$$

These inequalities imply that S^s and S^m are equivalent. □

Lemma 2.14. *Let (X, S^*) be a partial S-metric space. Define a relation $\sim$ on X as follows:*

$$x \sim y \iff S^*(x,x,y) = S^*(x,x,x).$$

Then $\sim$ is an equivalence relation on X.

Proof. (i) It is easy to see that $x \sim x$.

(ii) Let $x \sim y$ and $y \sim x$. Then $S^*(x,x,y) = S^*(x,x,x)$ and $S^*(y,y,x) = S^*(y,y,y)$. Since $S^*(x,x,y) = S^*(y,y,x)$,

$$S^*(x,x,x) = S^*(x,x,y) = S^*(y,y,y),$$

and by the condition (s_{p1}) we have $x = y$.

(iii) Let $x \sim y$ and $y \sim z$, and so $S^*(x,x,y) = S^*(x,x,x)$ and $S^*(y,y,z) = S^*(y,y,y)$. So

$$
\begin{aligned}
S^*(x,x,z) = S^*(z,z,x) &\leq 2S^*(z,z,y) + S^*(x,x,y) - 2S^*(y,y,y) \\
&= 2S^*(y,y,z) + S^*(x,x,x) - 2S^*(y,y,y) \\
&= S^*(x,x,x).
\end{aligned}
$$

Hence, by (s_{p2}), we get $S^*(x,x,x) \leq S^*(x,x,z)$. So $S^*(x,x,z) = S^*(x,x,x)$, that is, $x \sim z$. □

3. A Common Fixed Point Result in Ordered Partial S-metric Spaces

In what follows, we will consider the following special type of partial S-metric spaces.

Definition 3.1. A partial S-metric space (X, S^*) is said to be of the *first type* if for all $x, y, z \in X$

$$S^*(x, x, y) \leq \min\{S^*(x, y, z), S^*(z, x, y)\}$$

holds.

It is easy to check that the partial S-metric space of Example 2.2 is of the first type.

Definition 3.2 ([1])**.** Let $(X, \preceq)$ be a partially ordered set and $S, T : X \to X$ be two mappings. The pair (S, T) is said to be *partially weakly increasing* if $Sx \preceq TSx$ holds for each $x \in X$.

In the sequel, we use the following notations.

(i) $\mathcal{F}$ denotes the set of all functions $F : [0, \infty) \to [0, \infty)$ such that F is nondecreasing, continuous, $F(0) = 0 < F(t)$ for each $t > 0$ and $F(x + y) \leq F(x) + F(y)$ for all $x, y \in [0, +\infty)$;

(ii) Ψ denotes the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that ψ is continuous, nondecreasing and $\sum_{n=0}^{\infty} \psi^n(t)$ is convergent for each $t > 0$. From the conditions on ψ, it is clear that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for each $t > 0$.

The following theorem is the main result of this section.

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a first type partial S-metric S^* on X such that (X, S^*) is a complete partial S-metric space. Let $S, T, R : X \to X$ be mappings such that the pairs (S, T), (T, R) and (R, S) are partially weakly increasing and for some $F \in \mathcal{F}$, $\psi \in \Psi$ and all $x, y, z \in X$, comparable in pairs with respect to $\preceq$, the inequality*

$$F(S^*(Sx, Ty, Rz)) \leq \psi(F(\varphi(x, y, z))) \tag{3.1}$$

holds, where

$$\varphi(x, y, z) = \max\{S^*(x, y, z), S^*(x, x, Sx), S^*(y, y, Ty), S^*(z, z, Rz)\}. \tag{3.2}$$

Further, assume that for every nondecreasing sequence $\{x_n\}$ in X, converging to $x \in X$, we have $x_n \preceq x$.

Then S, T and R have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X as follows:

$$x_{3n+1} = Sx_{3n},\ x_{3n+2} = Tx_{3n+1} \text{ and } x_{3n+3} = Rx_{3n+2} \text{ for } n = 0, 1, \dots .$$

Since the pairs (S, T), (T, R) and (R, S) are partially weakly increasing, we have

$$x_1 = Sx_0 \preceq TSx_0 = x_2 = Tx_1 \preceq RTx_1 = x_3 = Rx_2 \preceq SRx_2 = x_4,$$

and continuing this process we have

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Consider, first of all, the case when there exists $n_0 \in \mathbb{N}$ such that $S^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}) = 0$. We will show also that $S^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Indeed, otherwise, from (3.1), we get

$$\begin{aligned} F(S^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})) &\leq F(S^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3})) \\ &= F(S^*(Sx_{3n_0}, Tx_{3n_0+1}, Rx_{3n_0+2})) \\ &\leq \psi(F(\varphi(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}))) \\ &= \psi(F(S^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3}))) \\ &< F(S^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})), \end{aligned}$$

which is a contradiction. Hence $S^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Therefore, $x_{3n_0} = x_{3n_0+1} = x_{3n_0+2} = x_{3n_0+3}$ and thus $Sx_{3n_0} = Tx_{3n_0} = Rx_{3n_0} = x_{3n_0}$. That is, x_{3n_0} is a common fixed point of S, T and R.

Assume now that $S^*(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for all $n \in \mathbb{N}$. We will prove that

$$F(S^*(x_{n-1}, x_n, x_{n+1})) \leq \psi(F(S^*(x_{n-2}, x_{n-1}, x_n))). \tag{3.3}$$

Setting $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ in (3.2), we obtain

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) = \max \left\{ \begin{array}{l} S^*(x_{3n}, x_{3n+1}, x_{3n+2}), S^*(x_{3n}, x_{3n}, x_{3n+1}), \\ S^*(x_{3n+1}, x_{3n+1}, x_{3n+2}), S^*(x_{3n+2}, x_{3n+2}, x_{3n+3}) \end{array} \right\}.$$

Since S^* is of the first type, we get

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) \leq \max\{S^*(x_{3n}, x_{3n+1}, x_{3n+2}), S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\}.$$

If $S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})$ is the maximum in the above inequality, we have from (3.1) that

$$\begin{aligned}F(S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) &= F(S^*(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}))\\ &< \psi(F(\varphi(x_{3n}, x_{3n+1}, x_{3n+2})))\\ &\leq \psi\big(F(\max\{S^*(x_{3n}, x_{3n+1}, x_{3n+2}), S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\})\big)\\ &= \psi\big(F(S^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))\big)\\ &< F(S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})),\end{aligned}$$

which is a contradiction. Thus

$$F(S^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) \leq \psi(F(S^*(x_{3n}, x_{3n+1}, x_{3n+2}))).$$

Similarly, we obtain

$$F(S^*(x_{3n+2}, x_{3n+3}, x_{3n+4})) \leq \psi(F(S^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))),$$

and

$$F(S^*(x_{3n}, x_{3n+1}, x_{3n+2})) \leq \psi(F(S^*(x_{3n-1}, x_{3n}, x_{3n+1}))).$$

Therefore, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}F(S^*(x_n, x_{n+1}, x_{n+2})) &\leq \psi(F(S^*(x_{n-1}, x_n, x_{n+1})))\\ &\leq \dots\\ &\leq \psi^n(F(S^*(x_0, x_1, x_2))).\end{aligned}$$

Hence $\lim_{n\to\infty} F(S^*(x_n, x_{n+1}, x_{n+2})) = 0$ and so

$$\lim_{n\to\infty} S^*(x_n, x_{n+1}, x_{n+2}) = 0. \tag{3.4}$$

Since S^* is of the first type and F is nondecreasing, we obtain

$$\begin{aligned}F(S^*(x_n, x_n, x_{n+1})) &\leq F(S^*(x_n, x_{n+1}, x_{n+2}))\\ &\leq \psi^n F(S^*(x_0, x_1, x_2))).\end{aligned}$$

Since $F(x + y) \leq F(x) + F(y)$ and $S^s(x_n, x_n, x_{n+1}) \leq 2S^*(x_n, x_n, x_{n+1})$, we have

$$F(S^s(x_n, x_n, x_{n+1})) \leq 2F(S^*(x_n, x_n, x_{n+1})) \leq 2\psi^n(F(S^*(x_0, x_1, x_2))).$$

Now from $S^s(x_{n+k}, x_{n+k}, x_n) \leq 2S^s(x_{n+k}, x_{n+k}, x_{n+k-1}) + \cdots + 2S^s(x_{n+1}, x_{n+1}, x_n)$, we get

$$\begin{aligned} F(S^s(x_{n+k}, x_{n+k}, x_n)) &\leq F(2S^s(x_{n+k}, x_{n+k}, x_{n+k-1})) + \cdots + F(2S^s(x_{n+1}, x_{n+1}, x_n)) \\ &\leq 2\psi^{n+k}(S^*(x_0, x_1, x_2)) + \cdots + 2\psi^{n+1}(S^*(x_0, x_1, x_2)) \\ &\leq 2\sum_{i=n}^{\infty} \psi^i(S^*(x_0, x_1, x_2)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for each $t > 0$, it follows that $\{x_n\}$ is a Cauchy sequence in the S_b-metric space (X, S^s). Since (X, S^*) is complete, it follows from Lemma 2.11 that the sequence $\{x_n\}$ converges to some x in the S_b-metric space (X, S^s). Hence

$$\lim_{n\to\infty} S^s(x_n, x_n, x) = 0.$$

Again, from Lemma 2.11, we have

$$S^*(x, x, x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m). \tag{3.5}$$

Since $\{x_n\}$ is a Cauchy sequence in the S_b-metric space (X, S^s) and

$$S^s(x_n, x_n, x_m) = 2S^*(x_n, x_n, x_m) - S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m),$$

we obtain

$$\lim_{n,m\to\infty} S^s(x_n, x_n, x_m) = 0,$$

and from (3.4), it follows that

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = 0.$$

Thus by the definition of S^s we have

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = 0.$$

Therefore by (3.5), we obtain

$$S^*(x, x, x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = 0.$$

Now, from the inequality (3.1) for $x = x$, $y = x_{3n+1}$ and $z = x_{3n+2}$, we get

$$F(S^*(Sx, x_{3n+2}, x_{3n+3})) \leq \psi(F(\varphi(x, x_{3n+1}, x_{3n+2})).$$

Letting $n \to \infty$ and using Lemma 2.12, we obtain

$$F(S^*(Sx, x, x)) \leq \psi(F(S^*(Sx, x, x)) < F(S^*(Sx, x, x)),$$

which is a contradiction. Hence $S^*(Sx, x, x) = 0$. Thus $Sx = x$.

Similarly, by using the inequality (3) for $y = x, x = x_{3n}$ and $z = x_{3n+2}$, we get

$$F(S^*(x_{3n}, Tx, x_{3n+3})) \leq \psi(F(\varphi(x_{3n}, x, x_{3n+2})),$$

and letting $n \to \infty$ and using Lemma 2.12, we obtain

$$F(S^*(x, Tx, x)) \leq \psi(F(S^*(x, Tx, x)) < F(S^*(x, Tx, x)),$$

which is a contradiction. Hence $S^*(x, Tx, x) = 0$. Thus $Tx = x$.

In a similar manner, applying the inequality (3.1) for $z = x$, $x = x_{3n}$ and $y = x_{3n+1}$, we can show that $Rx = x$. □

Taking $S = T = R$ in Theorem 3.3, one obtains the following

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a first type partial S-metric S^* on X such that (X, S^*) is a complete partial S-metric space. Let $T : X \to X$ be a mapping such that $Tx \preceq T^2x$ for $x \in X$ and*

$$F(S^*(Tx, Ty, Tz)) \leq \psi(F(\varphi(x, y, z))) \tag{3.6}$$

for all $x, y, z \in X$ with x, y, z comparable in pairs with respect to the partial order $\preceq$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$\varphi(x, y, z) = \max\{S^*(x, y, z), S^*(x, x, Tx), S^*(y, y, Ty), S^*(z, z, Tz)\}.$$

Further, assume that for every increasing sequence $\{x_n\}$ in X, converging to $x \in X$, we have $x_n \preceq x$.

Then T has a fixed point.

Now, we present an example which supports Corollary 3.4 and shows that the obtained fixed point results can be applied in the situations when some other known results fail.

Example 3.5. Let $X = [0, \infty)$ be equipped with the first type S-metric $S^*(x, y, z) = \max\{x, y, z\}$ and the partial order $\preceq$ be defined by

$$x \preceq y \iff x = y \vee 0 \leq y \leq x \leq 1.$$

Consider the mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \dfrac{x^2}{2(1+x)}, & 0 \leq x \leq 1, \\ x, & x > 1. \end{cases}$$

Choose $F \in \mathcal{F}$ and $\psi \in \Psi$ by $F(t) = t$ and

$$\psi(t) = \begin{cases} \dfrac{t^2}{1+t}, & 0 \le t \le 1, \\ \dfrac{t}{2}, & t > 1. \end{cases}$$

We will check that the conditions of Corollary 3.4 are fulfilled.

First of all, $Tx \preceq T^2x$ holds for $x \in X$ (indeed, for $0 \le x \le 1$ this reduces to

$$\frac{x^4}{4(1+x)(x^2+2x+2)} \le \frac{x^2}{2(1+x)},$$

and is trivial for $x > 1$). Let, further, x, y, z be comparable in pairs w.r.t. $\preceq$, e.g., $x \preceq y \preceq z$. The only nontrivial case to consider is when $0 \le z \le y \le x \le 1$. Then

$$S^*(Tx, Ty, Tz) = \max\left\{\frac{x^2}{2(1+x)}, \frac{y^2}{2(1+y)}, \frac{z^2}{2(1+z)}\right\} = \frac{x^2}{2(1+x)},$$

and

$$\varphi(x,y,z) = \max\{\max\{x,y,z\}, \max\{x,Tx\}, \max\{y,Ty\}, \max\{z,Tz\}\} = x.$$

Hence, the condition (3.6) reduces to

$$\frac{x^2}{2(1+x)} \le \frac{x^2}{1+x},$$

which is satisfied for $0 \le x \le 1$. By Corollary 3.4, the mapping T has a fixed point (which is 0). Note that, if the same problem is considered in one of usual S-metrics on X:

$$S(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\},$$

then the conclusion cannot be obtained in a similar way. Namely, if one takes $x = 1$ and $y = z = \dfrac{3}{4}$ (and the same functions F and ψ as before), then the inequality (3.6) (with S instead of S^*) would reduce to $\dfrac{5}{56} < \dfrac{1}{20}$, which is false. Also, if the problem is considered in the space without partial order, the condition would also not be satisfied.

Competing Interests

The author declares that he has no competing interests.

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