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ON THE SUPERSTABILITY OF THE p-RADICAL SINE TYPE FUNCTIONAL EQUATIONS

Gwang Hui Kim

Abstract. In this paper, we will find solutions and investigate the superstability bounded by constant for the p-radical functional equations as follows:

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = \begin{cases} (i)\ f(x)f(y), \\ (ii)\ g(x)f(y), \\ (iii)\ f(x)g(y), \\ (iv)\ g(x)g(y). \end{cases}$$

with respect to the sine functional equation, where p is an odd positive integer and f is a complex valued function. Furthermore, the results are extended to Banach algebra.

1. Introduction

In 1940, the stability problem is raised by S. M. Ulam [24]. It was solved the case of the additive mapping by Hyers [11] in the next year. In 1979, J. Baker, J. Lawrence and F. Zorzitto in [5] announced the *Superstability*, which is following : if f satisfies the inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. Baker [4] showed the superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{A}$$

The cosine (d'Alembert) functional equation (A) was generalized to the following:

$$f(x+y) + f(x-y) = 2f(x)g(y), \tag{W}$$

$$f(x+y) + f(x-y) = 2g(x)f(y), \tag{K}$$

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in which (W) is called the Wilson equation, and (K) raised by Kim was appeared in Kannappan and Kim ([12]).

In 1983, Cholewa [7] investigated the superstability of the sine functional equation

$$(S) \qquad f(x)f(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2$$

under the condition bounded by constant. Which is improved to the condition bounded by function in R. Badora and R. Ger [3].

It is improved by Kim([14], [15], [20]) which are the superstability of the generalized sine functional equations

$$(S_{fg}) \qquad f(x)g(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2,$$

$$(S_{gf}) \qquad g(x)f(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2,$$

$$(S_{gg}) \qquad g(x)g(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2.$$

The superstability of the trigonometric (cosine(A), sine(S), Wilson(W), Kim(K)) functional equations were founded in Badora[2], Ger[3], Kannappan[12], and Kim (see [12, 14, 15, 16, 17, 18, 21, 22]) and in papers ([8], [9], [12], [17], [21], [23]).

In 2009, Eshaghi Gordji and Parviz [10] introduced the radical functional equation related to the quadratic functional equation

$$(R) \qquad f\big(\sqrt{x^2+y^2}\big) = f(x) + f(y).$$

Recently, Almahalebi*et al.*[1], Kim[13] obtained the superstability of p-radical functional equations related to Wilson and Kim equation.

In this paper, we find solutions and investigate the superstability for the p-radical sine type functional equations

$$(S^r) \qquad f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y),$$

$$(S^r_{gf}) \qquad f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = g(x)f(y),$$

$$(S^r_{fg}) \qquad f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)g(y),$$

$$(S^r_{gg}) \qquad f\left(\sqrt[p]{\frac{x^2+y^2}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^2-y^2}{2}}\right)^2 = g(x)g(y).$$

with respect to the sine functional equation (S), which are under the stability inequality bounded by constant or function. Furthermore, the obtained results can be extended to the Banach algebra.

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f is a nonzero function, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function and p is an odd nonnegative integer.

2. Solution and Stability of the Equations

In this section, we find a solution and investigate the superstability of the p-radical functional equations (S^r),(S^r_{gf}), (S^r_{fg}), (S^r_{gg}) related to the sine functional equation (S). Although the stability of the p-radical sine function equation (S^r) is important, we will obtain it as a corollaries of (S^r_{gf}) and (S^r_{fg}) results to avoid repeat.

We can find a solution of the functional equations (S^r),(S^r_{gf}), (S^r_{fg}), (S^r_{gg}).

(i) A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (S^r) if and only if $f(x) = F(x^p)$ for all $x \in \mathbb{R}$, where F is a solution of (S). Namely, a function $f : \mathbb{R} \to \mathbb{C}$ satisfies $f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y)$ if and only if $f(x) = \sin(x^p) = F(x^p)$ for all $x \in \mathbb{R}$.

(ii) A function $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfies the functional equation (S^r_{gf}) if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of (S^r_{gf}).

(iii) A function $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfies (S^r_{fg}) if and only if $f(x) = F(x^p)$ and $g(x) = G(x^p)$, where F and G are solutions of (S^r_{fg}).

2.1. Stability of the Equation (S^r_{gf})

Theorem 1. *Assume that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right| \leq \varepsilon \tag{2.1}$$

for all $x, y \in \mathbb{R}$.

Then, either g is bounded or f and g satisfy (S^r).

Proof. Inequality (2.1) may equivalently be written as

$$|g(\sqrt[p]{2}x)f(\sqrt[p]{2}y) - f\left(\sqrt[p]{x^p + y^p}\right)^2 + f\left(\sqrt[p]{x^p - y^p}\right)^2| \leq \varepsilon \quad \forall\, x, y \in \mathbb{R} \tag{2.2}$$

Let g be unbounded. Then we can choose a sequence $\{x_n\}$ in $\mathbb{R}$ such that

$$0 \neq |g(\sqrt[p]{2}x_n)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{2.3}$$

Taking $x = x_n$ in (2.2), we obtain

$$\left| f(\sqrt[p]{2}y) - \frac{f\left(\sqrt[p]{x_n^p + y^p}\right)^2 - f\left(\sqrt[p]{x_n^p - y^p}\right)^2}{g(\sqrt[p]{2}x_n)} \right| \leq \frac{\varepsilon}{|g(\sqrt[p]{2}x_n)|},$$

that is, using (2.3)

$$f(\sqrt[p]{2}y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right)^2 - f\left(\sqrt[p]{x_n^p - y^p}\right)^2}{g(\sqrt[p]{2}x_n)} \tag{2.4}$$

Using (2.1), we have

$$\begin{aligned}
2\varepsilon \geq & \left| g(\sqrt[p]{2x_n^p + x^p})f(y) - f\left(\sqrt[p]{\frac{2x_n^p + x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{2x_n^p + x^p - y^p}{2}}\right)^2 \right| \\
& + \left| g(\sqrt[p]{2x_n^p - x^p})f(y) - f\left(\sqrt[p]{\frac{2x_n^p - x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{2x_n^p - x^p - y^p}{2}}\right)^2 \right| \\
\geq & \; | \left(g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})\right) f(y) \\
& - \left(f\left(\sqrt[p]{x_n^p + \frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p + y^p}{2}}\right)^2 \right) \\
& + \left(f\left(\sqrt[p]{x_n^p + \frac{x^p - y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p - y^p}{2}}\right)^2 \right) \Bigg|
\end{aligned}$$

for all $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}
\frac{2\varepsilon}{|g(\sqrt[p]{2}x_n)|} \geq & \left| \frac{g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})}{g(\sqrt[p]{2}x_n)} f(y) \right. \\
& - \frac{f\left(\sqrt[p]{x_n^p + \frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p+y^p}{2}}\right)^2}{g(\sqrt[p]{2}x_n)} \\
& \left. + \frac{f\left(\sqrt[p]{x_n^p + \frac{x^p-y^p}{2}}\right)^2 - f\left(\sqrt[p]{x_n^p - \frac{x^p-y^p}{2}}\right)^2}{g(\sqrt[p]{2}x_n)} \right|
\end{aligned}$$

for all $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ with the use of (2.3) and (2.4), we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{g(\sqrt[p]{2x_n^p + x^p}) + g(\sqrt[p]{2x_n^p - x^p})}{g(\sqrt[p]{2}x_n)}, \tag{2.5}$$

where the obtained function $h : G \to \mathbb{C}$ satisfies the equation as even

$$f(\sqrt[p]{x^p + y^p}) - f(\sqrt[p]{x^p - y^p}) = h(x)f(y) \quad \forall x, y \in \mathbb{R}. \tag{2.6}$$

From the definition of h, we get the equality $h(0) = 2$, which jointly with (2.6) implies that f is an odd. Keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned} f(\sqrt[p]{x^p + y^p})^2 - f(\sqrt[p]{x^p - y^p})^2 &= [f(\sqrt[p]{x^p + y^p}) + f(\sqrt[p]{x^p - y^p})]h(x)f(y) \\ &= \left[f(\sqrt[p]{2x^p + y^p}) + f(\sqrt[p]{2x^p - y^p})\right]f(y) \\ &= \left[f(\sqrt[p]{y^p + 2x^p}) - f(\sqrt[p]{y^p - 2x^p})\right]f(y) \\ &= h(y)f(\sqrt[p]{2}x)f(y). \end{aligned} \tag{2.7}$$

The oddness of f forces it vanish at 0. Putting $x = y$ in (2.6) we conclude with the above result that

$$f(\sqrt[p]{2}y) = f(y)h(y) \text{ for all } x, y \in \mathbb{R}.$$

This, in return, leads to the equation

$$f(\sqrt[p]{x^p + y^p})^2 - f(\sqrt[p]{x^p - y^p})^2 = f(\sqrt[p]{2}x)f(\sqrt[p]{2}y), \tag{2.8}$$

valid for all $x, y \in \mathbb{R}$, which f, divided by $\sqrt[p]{2}$, states nothing else but (S^r).

Next, by showing $g = f$, we will prove that g also is a solution of (S^r).

If f is bounded, choose $y_0 \in G$ such that $f(2y_0) \neq 0$, and then by (2.2) we obtain

$$\begin{aligned} |g(\sqrt[p]{2}x)| - \Big|\frac{f\left(\sqrt[p]{x^p + y^p}\right)^2 - f\left(\sqrt[p]{x^p - y^p}\right)^2}{f(\sqrt[p]{2}y_0)}\Big| \\ \leq \Big|\frac{f\left(\sqrt[p]{x^p + y^p}\right)^2 - f\left(\sqrt[p]{x^p - y^p}\right)^2}{f(\sqrt[p]{2}y_0)} - g(\sqrt[p]{2}x)\Big| \\ \leq \frac{\varepsilon}{|f(\sqrt[p]{2}y_0)|} \end{aligned} \tag{2.9}$$

and it follows that g is also bounded on $\mathbb{R}$.

Since the unbounded assumption of g implies that f also is unbounded, we can choose a sequence $\{y_n\}$ such that $0 \neq |f(\sqrt[p]{2}y_n)| \to \infty$ as $n \to \infty$.

A slight change applied after (2.3) gives us

$$g(\sqrt[p]{2}x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right)^2 - f\left(\sqrt[p]{x^p - y_n^p}\right)^2}{f(\sqrt[p]{2}y_n)} \tag{2.10}$$

Since we have shown that f satisfies (2.8) whenever g is unbounded, the above limit equation (2.10) is represented as

$$g(\sqrt[p]{2}x) = f(\sqrt[p]{2}x).$$

By the $\sqrt[p]{2}x$-divisibility of $\mathbb{R}$, we obtain $f = g$. Therefore it is completed that g also satisfies (S^r). □

Theorem 2. *Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| g(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right| \le \varepsilon, \quad \forall x,\, y \in \mathbb{R} \tag{2.11}$$

which satisfies one of the cases $g(0) = 0$, $f(x)^2 = f(-x)^2$.

Then either f is bounded or g satisfies (S^r).

Proof. In Theorem 1, the inequality (2.11) be written equivalently as (2.2).

Let f be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that $f(\sqrt[p]{2}y_n)| \to \infty$ as $n \to \infty$. An obvious slight change in the proof steps applied in the start of Theorem 1 in (2.11) gives us

$$g(\sqrt[p]{2}x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right)^2 - f\left(\sqrt[p]{x^p - y_n^p}\right)^2}{f(\sqrt[p]{2}y_n)} \tag{2.12}$$

and allows, with an applying of (2.12), one to state the existence of a limit function

$$k(y) := \lim_{n\to\infty} \frac{f(\sqrt[p]{y^p + 2y_n^p}) + f(\sqrt[p]{-y^p + 2y_n^p})}{f(\sqrt[p]{2}y_n)}, \tag{2.13}$$

where the obtained function $k : G \to \mathbb{C}$ satisfies the equation

$$g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = g(x)k(y) \quad \forall x, y \in \mathbb{R}. \tag{2.14}$$

From the definition of k, we get the equality $k(y) = k(-y)$.

First, let us consider the case $g(0) = 0$, then it forces by (2.14) g is odd. Putting $y = x$ in (2.14), we get

$$g(\sqrt[p]{2}x) = g(x)k(x) \quad \forall x, y \in \mathbb{R}. \tag{2.15}$$

From (2.14), the oddness of g and (2.15), we obtain the equation

$$\begin{aligned} g(\sqrt[p]{x^p+y^p})^2 - g(\sqrt[p]{x^p-y^p})^2 &= g(x)k(y)[g(\sqrt[p]{x^p+y^p}) - g(\sqrt[p]{x^p-y^p})] \\ &= g(x)[g(\sqrt[p]{x^p+2y^p}) - g(\sqrt[p]{x^p-2y^p})] \\ &= g(x)[g(\sqrt[p]{2y^p+x^p}) + g(\sqrt[p]{2y^p-x^p})] \\ &= g(x)g(\sqrt[p]{2}y)k(x) \\ &= g(\sqrt[p]{2}x)g(\sqrt[p]{2}y), \end{aligned}$$

that holds true for all $x, y \in \mathbb{R}$, which states nothing else but (S^r).

In next case $f(x)^2 = f(-x)^2$, it is enough to show that $g(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $g(0) = c$: constant.

Putting $x = 0$ in (2.11), from the above assumpition, we obtain the inequality

$$|f(y)| \le \frac{\varepsilon}{c} \quad \forall\, y \in G.$$

This inequality means that f is globally bounded – a contradiction by unboundedness assumpition. Thus the claimed $g(0) = 0$ holds, so the proof of theorem is completed. □

2.2. Stability of the equation (S_{fg}) We will investigate the stability of the functional equation (S_{fg}) throughout the same proceedings as Subsection 2. The proof processes are the same word by word as it is in subsection 2.1, so we will represent only the main equations.

Theorem 3. *Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 \right| \le \varepsilon \tag{2.16}$$

Then, either f is bounded or g satisfy (S^r).

Proof. Let f be unbounded solution of the inequality (2.16). Then, there exists a sequence $\{x_n\}$ in G such that $0 \neq |f(\sqrt[p]{2}x_n)| \to \infty$ as $n \to \infty$.

Putting $x = \sqrt[p]{2}x, y = \sqrt[p]{2}y$ in inequality (2.16), taking $x = x_n$ in the obtained inequality, dividing both sides by $|f(\sqrt[p]{2}x_n)|$ and passing to the limit as $n \to \infty$, then it arrive that

$$g(\sqrt[p]{2}y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p+y^p}\right)^2 - f\left(\sqrt[p]{x_n^p-y^p}\right)^2}{f(\sqrt[p]{2}x_n)} \tag{2.17}$$

An obvious slight change in the proof steps applied in Theorem 1 allows, with an applying of (2.17), us to state the existence of a limit function p such that

$$p(x) := \lim_{n\to\infty} \frac{f(\sqrt[p]{2x_n^p + x^p}) + f(\sqrt[p]{2x_n^p - x^p})}{f(\sqrt[p]{2}x_n)}, \tag{2.18}$$

where $p : G \to \mathbb{C}$ satisfies as even

$$g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = p(x)g(y) \quad \forall x, y \in \mathbb{R}. \tag{2.19}$$

From the definition of p, we get $p(0) = 2$ as even, which jointly with (2.19) implies that g is odd.

The oddness of g forces it vanish at 0. Putting $x = y$ in (2.19), it implies by letting $\tilde{k} := \frac{1}{\sqrt[p]{2}}k$

$$g(\sqrt[p]{2}y) = p(y)g(y) = \sqrt[p]{2}\tilde{p}(x)g(y) \text{ for all } x, y \in \mathbb{R}. \tag{2.20}$$

Since some calculation of the oddness of g, (2.19), and (2.20) lead that g satisfies (2.8), so g satisfies (S^r). □

Theorem 4. *Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right| \le \varepsilon$$

which satisfies one of the cases $f(0) = 0$, $f(x)^2 = f(-x)^2$.

Then, either g is bounded or f and g satisfy (S^r).

Proof. As like Theorem 3, the same process as theorem 2 is performed line by line, f is satisfied (S^r), g also is satisfied through a process similar to (2.7), (2.9) and (2.10) in Theorem 1. □

2.3. Stability of the equation (S_{gg}) We will investigate the stability of the generalized functional equation (S_{gg}) of (S^r).

Theorem 5. *Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right| \le \varepsilon$$

Then either g is bounded or g satisfies (S^r).

Proof. The term $f(y)$ and $f(x)$ in Theorem 1 and Theorem 2 converts to $g(y)$ and $g(x)$, respectively. □

Finally, by converting g in Theorems 1, 2, 3, 4, and 5, to f, we can be obtained the stability of the p-radical sine functional equation (S^r) related to the sine functional equation (S) as corollary.

Corollary 1. *Assume that $f : \mathbb{R} \longrightarrow \mathbb{C}$ satisfy the inequality*

$$\left| f(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right| \le \varepsilon$$

for all $x, y \in \mathbb{R}$.

Then, either f is bounded or f satisfies (S^r).

3. Extention of the Stability Results on the Banach Algebra

All results in the Section 2 also can be extended to the stability on the Banach algebra. The following threorem due to Theorem 1 and Theorem 2.

Theorem 6. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| g(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \le \varepsilon \quad \forall x, y \in \mathbb{R}. \tag{3.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *if the superposition $x^* \circ g$ fails to be bounded, then f and g satisfy (S^r),*

(ii) *if the superposition $x^* \circ f$ under the cases $g(0) = 0$ or $f(x)^2 = f(-x)^2$ fails to be bounded, then g satisfies (S^r).*

Proof. Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}
\varepsilon &\ge \left\| g(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \\
&= \sup_{\|y^*\|=1} \left| y^*\left(g(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right) \right| \\
&\ge \left| x^*(g(x)) \cdot x^*(f(y)) - x^*\left(f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)\right) + x^*\left(f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2\right) \right|,
\end{aligned}$$

which states that the superposition $x^* \circ g$ and $x^* \circ f$ yields a solution of stability inequality (2.1) of Theorem 1. Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 1 shows that the function $x^* \circ f$ solves the generalized sine equation (S^r). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in \mathbb{R}$, the difference $\mathcal{D}S^r_{gf} : \mathbb{R} \times \mathbb{R} \to E$ defined by

$$\mathcal{D}S^r_{gf}(x,y) := f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 - g(x)f(y)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}S^r_{gf}(x,y) \in \bigcap\{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}S(x,y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. The case(ii) also are the same. □

The following results are also formed by the same logic as Theorem 6.

Theorem 7. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| f(x)g(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \le \varepsilon \quad \forall x, y \in \mathbb{R}. \tag{3.2}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(i) *if the superposition $x^* \circ f$ fails to be bounded, then g satisfy (S^r),*

(ii) *if the superposition $x^* \circ g$ under the cases $f(0) = 0$ or $f(x)^2 = f(-x)^2$ fails to be bounded, then g satisfies (S^r).*

Theorem 8. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| g(x)g(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \le \varepsilon \quad \forall x, y \in \mathbb{R}. \tag{3.3}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ g$ is bounded or g satisfies (S^r).

The above theorems also imply the following corollary by putting $g = f$, immediately.

Theorem 9. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : \mathbb{R} \longrightarrow E$ satisfy the inequality*

$$\left\| f(x)f(y) - f\left(\sqrt[p]{\frac{x^p + y^p}{2}}\right)^2 + f\left(\sqrt[p]{\frac{x^p - y^p}{2}}\right)^2 \right\| \leq \varepsilon \quad \forall x, y \in \mathbb{R}. \tag{3.4}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ f$ is bounded or f satisfies (S^r).

Remark 1. Applying $p = 1$ in all of the p-radical sine and the p-radical sine type functional equations $((S^r),(S^r_{gf}),(S^r_{fg}),(S^r_{gg}))$, then it implies the sine and the sine type functional equations $((S),(S_{gf}),(S_{fg}),(S_{gg}))$.

Thus, all results of the p-radical sine and the p-radical sine type functional equations mean no other than the stability of the sine and the sine-type function equation, which are founded Cholewa [7], Badora and Ger [3], and Kim ([14], [15], [16],[20]).

References

1. M. Almahalebi, R. El Ghali, S. Kabbaj & C. Park: Superstability of p-radical functional equations related to WilsonKannappanKim functional equations. Results Math. **76** (2021), Paper No. 97.
2. R. Badora: On the stability of cosine functional equation. Rocznik Naukowo-Dydak., Prace Mat. **15** (1998), 1-14.
3. R. Badora & R. Ger: On some trigonometric functional inequalities. Functional Equations- Results and Advances, 3-15 (2002).
4. J.A. Baker: The stability of the cosine equation. Proc. Amer. Math. Soc. **80** (1980), 411-416.
5. J. Baker, J. Lawrence & F. Zorzitto: The stability of the equation $f(x+y) = f(x)f(y)$. Proc. Amer. Math. Soc. **74** (1979), 242-246.
6. B. Bouikhalene, E. Elquorachi & J.M. Rassias: The superstability of d'Alembert's functional equation on the Heisenberg group. Appl. Math. Lett. **23** (2010), 105-109.
7. P.W. Cholewa: The stability of the sine equation. Proc. Amer. Math. Soc. **88** (1983), 631-634.
8. E. Elqorachi & M. Akkouchi: On Hyers-Ulam stability of the generalized Cauchy and Wilson equations. Publ. Math. Debrecen **66** (2005), 283-301.

9. P. de P. Friis: d'Alembert's and Wilson's equations on Lie groups. Aequationes Math. **67** (2004), 12-25.
10. M. Eshaghi Gordji & M. Parviz: On the Hyers-Ulam-Rassias stability of the functional equation $f(\sqrt{x^2+y^2}) = f(x) + f(y)$. Nonlinear Funct. Anal. Appl. **14** (2009), 413-420.
11. D.H. Hyers: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222-224.
12. Pl. Kannappan & G.H. Kim: On the stability of the generalized cosine functional equations. Annales Acadedmiae Paedagogicae Cracoviensis - Studia Mathematica **1** (2001), 49-58.
13. G.H. Kim: Superstability of the p-radical functional equations related to Wilson's and Kim's equation. Int. J. Nonlinear. Anal. Appl. **12** (2021), 571-582.
14. G.H. Kim: On the stability of the generalized sine functional equations. Acta Math. Sin., Engl. Ser. **25** (2009), 965-972
15. G.H. Kim: A stability of the generalized sine functional equations. Jour. Math. Anal & Appl. **331** (2007), 886-894.
16. G.H. Kim: On the superstability of the pexider type trigonometric functional equation. Journal of Inequalities Applications **2010** (2010), Doi:10.1155/2010/897123.
17. G.H. Kim: The stability of the d'Alembert and Jensen type functional equations. J. Math. Anal & Appl. **325** (2007), 237-248.
18. G.H. Kim: The stability of pexiderized cosine functional equations. Korean J. Math. **16** (2008), no. 1, 103-114.
19. G.H. Kim: A stability of the generalized sine functional equations. J. Math. Anal & Appl. preprint (2007).
20. G.H. Kim: On the stability of mixed trigonometric functional equations. Banach J. Math. Anal. **2** (2007), 227-236.
21. G.H. Kim & Sever S. Dragomir: On the the Stability of generalized d'Alembert and Jensen functional equation. Intern. Jour. Math. & Math. Sci. preprint (2006).
22. G.H. Kim & S.H. Lee: Stability of the d'Alembert type functional equations. Nonlinear Funct. Anal. Appl. **9** (2004), 593-604.
23. P. Sinopoulos: Generalized sine equations. III. Aequationes Math. **51** (1996), 311-327.
24. S.M. Ulam: 'Problems in Modern Mathematics' Chap. VI, Science editions. Wiley, New York, (1964)

DEPARTMENT OF APPLIED MATHEMATICS, KANGNAM UNIVERSITY, GYEONGGI-DO 16979, REPUBLIC OF KOREA
Email address: ghkim@kangnam.ac.kr