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FUZZY COMPLETE LATTICES AND DISTANCE SPACES

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Abstract. In this paper, we introduce the notions of fuzzy join (resp. meet) complete lattices and distance spaces in complete co-residuated lattices. Moreover, we investigate the relations between Alexandrov pretopologies (resp. precotopologies) and fuzzy join (resp. meet) complete lattices, respectively. We give their examples.

1. Introduction

As an algebraic structure for many valued logic, a complete residuated lattice is an important mathematical tool [1-4, 6-11, 15, 16]. For an extension of classical rough sets introduced by Pawlak [12, 13], many researchers [1, 6-11] developed L-lower and L-upper approximation operators in complete residuated lattices. By using the concepts of lower and upper approximation operators, fuzzy concepts, information systems and decision rules are investigated in complete residuated lattices [1-4, 6-11, 15, 16].

Zhang et al. [17, 18] introduced the notion of fuzzy complete lattices using fuzzy partially order on a frame as generalizations of usual complete lattices. Based on residuated lattices as an extension of frame, Zhang [19] introduced the notions of partially orders,join, meet and fuzzy completeness.

Kim et al. [7-10] studied the properties of fuzzy join and meet completeness, L-fuzzy upper and lower approximation spaces and Alexandrov L-topologies with fuzzy partially ordered spaces in complete residuated lattices. Zheng and Wang [20] introduced complete co-residuated lattices. By using this concepts, lower and upper

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approximation operators, fuzzy rough sets and information systems are investigated [6].

In this paper, we introduce the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces [19] in complete co-residuated lattices. We show that fuzzy join (resp. meet) complete lattices and Alexandrov pretopologies (resp. precotopologies) are equivalent, respectively. Moreover, their properties and examples are investigated.

2. Preliminaries

Definition 2.1 ([6, 20])**.** An algebra $(L, \wedge, \vee, \oplus, 0, 1)$ is called a *complete co-residuated lattice* if it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is the top element.

(Q2) $a = a \oplus 0$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$.

(Q3) $(\bigwedge_{i\in\Gamma} a_i) \oplus b = \bigwedge_{i\in\Gamma}(a_i \oplus b)$.

Remark 2.2. (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice [4,15].

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0, 1], \leq, \oplus)$, is a complete co-residuated lattice [1,4,15].

(3) Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge\{z \in L \mid x \oplus z \geq y\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (y \ominus z)$.

(4) $([0, \infty], \leq, \vee, \oplus = +, \wedge, \infty, 0)$ is a commutative unital co-quantale where

$$\begin{aligned} x \ominus y &= \bigwedge\{z \in [0, \infty] \mid x + z \geq y\} \\ &= \bigwedge\{z \in [0, \infty] \mid z \geq -x + y\} = (y - x) \vee 0, \\ &\infty + a = a + \infty = \infty, \forall a \in [0, \infty], \infty \to \infty = 0. \end{aligned}$$

In this paper, we assume $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ is a complete co-residuated lattice. For $\alpha \in L, A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x)$, $(\alpha \oplus A)(x) = \alpha \oplus A(x)$, $\alpha_X(x) = \alpha$.

Lemma 2.3. *Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If* $y \le z$, $(x \oplus y) \le (x \oplus z)$, *then* $x \ominus y \le x \ominus z$ *and* $z \ominus x \le y \ominus x$.

(2) $x \ominus (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$ *and* $(\bigwedge_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$.

(3) $x \ominus (\bigwedge_{i\in\Gamma} y_i) \le \bigwedge_{i\in\Gamma}(x \ominus y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \ominus y \le \bigwedge_{i\in\Gamma}(x_i \ominus y)$.

(5) $x \oplus (x \ominus y) \ge y$, $(x \ominus y) \ominus y \le x$ *and* $(x \ominus y) \oplus (y \ominus z) \ge x \ominus z$.

(6) $(x \oplus y) \ominus z = x \ominus (y \ominus z) = y \ominus (x \ominus z)$.

(7) $x \ominus y \ge (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \ge (y \ominus z) \ominus (x \ominus z)$ *and* $(x \oplus y) \ominus (z \oplus w) \le (x \ominus z) \oplus (y \ominus w)$.

(8) $x \ominus x = 0$, $0 \ominus x = x$. *Moreover,* $x \ominus y = 0$ *iff* $x \ge y$.

Proof. (1) Since $y = y \wedge z$, $x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$. Then $(x \oplus y) \le (x \oplus z)$. Since $y \le z \le x \oplus (x \ominus z)$, $x \ominus y \le x \ominus z$. Since $x \le y \oplus (y \ominus x) \le z \oplus (y \ominus x)$, $z \ominus x \le y \ominus x$.

(2) By (1), $x \ominus (\bigvee_{i\in\Gamma} y_i) \ge \bigvee_{i\in\Gamma}(x \ominus y_i)$. Since $x \oplus \bigvee_{i\in\Gamma}(x \ominus y_i) \ge \bigvee_{i\in\Gamma}(x \oplus (x \ominus y_i)) \ge \bigvee_{i\in\Gamma} y_i$, $x \ominus (\bigvee_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x \ominus y_i)$.

By (1), $(\bigwedge_{i\in\Gamma} x_i) \ominus y \ge \bigvee_{i\in\Gamma}(x_i \ominus y)$. Since $(\bigwedge_{i\in\Gamma} x_i) \oplus \bigvee_{i\in\Gamma}(x_i \ominus y) \ge \bigwedge_{i\in\Gamma}(x_i \oplus (x_i \ominus y)) \ge y$, $(\bigwedge_{i\in\Gamma} x_i) \ominus y \le \bigvee_{i\in\Gamma}(x_i \ominus y)$.

(3) and (4) are easily proved from (1).

(5) Since $x \ominus y \ge x \ominus y$, $x \oplus (x \ominus y) \ge y$. Moreover, $x \ge (x \ominus y) \ominus y$. Since $x \oplus (x \ominus y) \oplus (y \ominus z) \ge y \oplus (y \ominus z) \ge z$, $(x \ominus y) \oplus (y \ominus z) \ge x \ominus z$.

(6) We have $x \oplus y \oplus ((x \oplus y) \ominus z) \ge z$ if and only f $x \oplus ((x \oplus y) \ominus z) \ge y \ominus z$. Thus $(x \oplus y) \ominus z \ge x \ominus (y \ominus z)$.

Since $x \oplus y \oplus (x \ominus (y \ominus z)) \ge y \oplus (y \ominus z) \ge z$, $x \ominus (y \ominus z) \ge (x \oplus y) \ominus z$.

Similarly, $(x \oplus y) \ominus z = y \ominus (x \ominus z)$.

(7) Since $(x \oplus z) \oplus (x \ominus y) \ge y \oplus z$, $x \ominus y \ge (x \oplus z) \ominus (y \oplus z)$. Since $x \oplus (x \ominus y) \oplus (y \ominus z) \ge z$, $x \ominus y \ge (y \ominus z) \ominus (x \ominus z)$.

Since $z \oplus w \le x \oplus (x \ominus z) \oplus y \oplus (y \ominus w)$, $(x \oplus y) \ominus (z \oplus w) \le (x \ominus z) \oplus (y \ominus w)$.

(8) For $x \in L$, $x \ominus x = \bigwedge\{z \in L \mid x \oplus z \ge x\} = 0$ and $0 \ominus x = \bigwedge\{z \in L \mid 0 \oplus z \ge x\} = x$. □

Definition 2.4. Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x, x) = 0$ for all $x \in X$,

(M2) $d_X(x, y) \oplus d_X(y, z) \ge d_X(x, z)$, for all $x, y, z \in X$,

(M3) if $d_X(x, y) = d_X(y, x) = 0$, then $x = y$.

The pair (X, d_X) is called a *distance space.*

Remark 2.5. (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a non-symmetric pseudo-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (8), (L, d_L) is a distance space. Moreover, we define a function $d_{L^X} : L^X \times L^X \to L$ as $d_{L^X}(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x))$. Then (L^X, d_{L^X}) is a distance space.

(3) We define a function $d_{[0,\infty]^X} : [0, \infty]^X \times [0, \infty]^X \to [0, \infty]$ as $d_{[0,\infty]^X}(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x)) = \bigvee_{x \in X}((B(x) - A(x)) \vee 0)$. Then $([0, \infty]^X, d_{[0,\infty]^X})$ is a non-symmetric pseudo-metric space.

(4) If (X, d_X) is a distance space and we define a function $d_X^{-1}(x, y) = d_X(y, x)$, then (X, d_X^{-1}) is a distance space.

(5) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Let (X, d_X) be a distance space and define $(d_X \uplus d_X)(x, z) = \bigwedge_{y \in X}(d_X(x, y) \oplus d_X(y, z))$ for each $x, z \in X$. By (M2), $(d_X \uplus d_X)(x, z) \geq d_X(x, z)$ and $(d_X \uplus d_X)(x, z) \leq d_X(x, x) \oplus d_X(x, z) = d(x, z)$. Hence $(d_X \uplus d_X) = d_X$.

(6) If d_X is a distance function and $d_X^{-1}(x, y) = d_X(y, x)$ for each $x, y \in X$, then d_X^{-1} is a distance function.

3. Fuzzy Complete Lattices and Distance Spaces

Definition 3.1. Let (X, d_X) be a distance space and $A \in L^X$.

(1) A point x_0 is called a *fuzzy join* of A, denoted by $x_0 = \sqcup_X A$, if it satisfies

(J1) $A(x) \geq d_X(x_0, x)$,

(J2) $\bigvee_{x \in X}(A(x) \ominus d_X(y, x)) \geq d_X(y, x_0)$.

The pair (X, d_X) is called *fuzzy join complete* if $\sqcup_X A$ exists for each $A \in L^X$.

A point x_1 is called a *fuzzy meet* of A, denoted by $x_1 = \sqcap_X A$, if it satisfies

(M1) $A(x) \geq d_X(x, x_1)$,

(M2) $\bigvee_{x \in X}(A(x) \ominus d_X(x, y)) \geq d_X(x_1, y)$.

The pair (X, d_X) is called *fuzzy meet complete* if $\sqcap_X A$ exists for each $A \in L^X$.

The pair (X, d_X) is called *fuzzy complete* if $\sqcap_X A$ and $\sqcup_X A$ exists for each $A \in L^X$.

Theorem 3.2. *Let (X, d_X) be a distance space and $\Phi \in L^X$.*

(1) A point x_0 is a fuzzy join of Φ iff $\bigvee_{x \in X}(\Phi(x) \ominus d_X(y, x)) = d_X(y, x_0)$.

(2) A point x_1 is a fuzzy meet of Φ iff $\bigvee_{x \in X}(\Phi(x) \ominus d_X(x, y)) = d_X(x_1, y)$.

(3) If $\sqcup_X \Phi$ is a fuzzy join of $\Phi \in L^X$, then it is unique. Moreover, if $\sqcap_X \Phi$ is a fuzzy meet of $\Phi \in L^X$, then it is unique.

Proof (1) Let $\sqcup_X \Phi$ be a fuzzy join of $\Phi \in L^X$. By (J1), since $\Phi(x) \geq d_\tau(\sqcup_X \Phi, x)$, we have $\Phi(x) \oplus d_X(y, \sqcup_X \Phi) \geq d_X(\sqcup_X \Phi, x) \oplus d_X(y, \sqcup_X \Phi) \geq d_X(y, x)$. Hence $d_X(y, \sqcup_X \Phi) \geq \bigvee_{x \in X}(\Phi(x) \ominus d_X(y, x))$. By (J2), $d_X(y, \sqcup_X \Phi) = \bigvee_{x \in X}(\Phi(x) \ominus d_X(y, x))$.

Conversely, $d_X(y, \sqcup_X \Phi) \geq (\Phi(x) \ominus d_X(y, x))$ if and only if $\Phi(x) \geq d_X(y, \sqcup_X \Phi) \oplus d_X(y, x)$. Put $y = \sqcup_X \Phi$. Then $\Phi(x) \geq d_X(\sqcup_X \Phi, x)$.

(2) It is similarly proved as (1).

(3) Let x_1, x_2 be fuzzy joins of $\Phi \in L^X$. For all $y \in X$, we have

$$\bigvee_{x \in X} (\Phi(x) \ominus d_X(y, x)) = d_X(y, x_1) = d_X(y, x_2).$$

Put $y = x_1$. Then $0 = d_X(x_1, x_1) = d_X(x_1, x_2)$. Put $y = x_2$. Then □

Theorem 3.3. *Let (X, d_X) be a distance space and $A, B \in L^X$.*

(1) If $\sqcup_X A, \sqcup_X B$ exist, $d_{L^X}(A, B) \geq d_X(\sqcup_X B, \sqcup_X A)$,

(2) If $\sqcap_X A, \sqcap_X B$ exist, $d_{L^X}(A, B) \geq d_X(\sqcap_X A, \sqcap_X B)$.

Proof. (1) For each $A, B \in L^X$, $d_{L^X}(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x)) \geq \bigvee_{x \in X}(A(x) \ominus d_X(\sqcup_X B, x)) \geq d_X(\sqcup_X B, \sqcup_X A)$.

(2) For each $A, B \in L^X$, $d_{L^X}(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x)) \geq \bigvee_{x \in X}(A(x) \ominus d_X(x, \sqcap_X B)) \geq d_X(\sqcap_X A, \sqcap_X B)$. □

Lemma 3.4. *Let (X, d_X) be a distance space. Then the followings hold.*

(1) For each $z \in X$, $\sqcup_X d_X(z, -) = z$ and $\sqcap_X d_X(-, z) = z$.

(2) For $\Phi \in L^X$, $\sqcup_X \Phi = \sqcup_X \bigwedge(\Phi(z) \oplus d_X(z, -))$ and $\sqcap_X \Phi = \sqcap_X \bigwedge(\Phi(z) \oplus d_X(-, z))$.

Proof. (1) Since $d_X(z, x) \oplus d_X(y, z) \geq d_X(y, x)$,

$$d_X(y, z) \geq \bigvee_{x \in X} (d_X(z, x) \ominus d_X(y, x)).$$

From the definition of $\sqcup_X d_X(z, -) = z$,

$$\begin{aligned} d_X(x, \sqcup_X d_X(z, -)) &= \textstyle\bigvee_{x \in X}(d_X(z, x) \ominus d_X(y, x)) \\ &\geq d_X(z, z) \ominus d_X(y, z) = d_X(y, z). \end{aligned}$$

Hence $d_X(x, \sqcup_X d_X(z, -)) = \bigvee_{x \in X}(d_X(z, x) \ominus d_X(y, x)) = d_X(y, z)$. Thus $\sqcup_X d_X(z, -) = z$. Similarly, $d_X(\sqcap_X d_X(-, z), y) = \bigvee_{x \in X}(d_X(x, z) \ominus d_X(x, y)) = d_X(z, y)$. Thus $\sqcap_X d_X(-, z) = z$.

(2) From the definitions of $\sqcup_X \bigwedge(\Phi(z) \oplus d_X(z,-))$ and $\sqcap_X \bigwedge(\Phi(z) \oplus d_X(-,z))$,

$d_X(y, \sqcup \bigwedge(\Phi(z) \oplus d_X(z,-))) = \bigvee_{x\in X}(\bigwedge_{z\in X}(\Phi(z) \oplus d_X(z,x)) \ominus d_X(y,x))$
$= \bigvee_{x,z\in X}(\Phi(z) \ominus (d_X(z,x) \ominus d_X(y,x)) = \bigvee_{z\in X}(\Phi(z) \ominus \bigvee_{x\in X}(d_X(z,x) \ominus d_X(y,x))$
$= \bigvee_{z\in X}(\Phi(z) \ominus d_X(y,z)) = d_X(y, \sqcup\Phi)$,

$d_X(\sqcap \bigwedge(\Phi(z) \oplus d_X(-,z)), y) = \bigvee_{x\in X}(\bigwedge_{z\in X}(\Phi(z) \oplus d_X(x,z)) \ominus d_X(x,y))$
$= \bigvee_{x,z\in X}(\Phi(z) \ominus (d_X(x,z) \ominus d_X(x,y)) = \bigvee_{z\in X}(\Phi(z) \ominus \bigvee_{x\in X}(d_X(x,z) \ominus d_X(x,y))$
$= \bigvee_{z\in X}(\Phi(z) \ominus d_X(z,y)) = d_X(\sqcap\Phi, y)$.

□

Theorem 3.5. *Let* (X, d_X) *be a distance space. Then the following are equivalent:*

(1) $\sqcup_X \Phi$ *exists for every* $\Phi \in L^X$.

(2) $\sqcap_X \Phi$ *exists for every* $\Phi \in L^X$.

Proof. (1) ⇒ (2). For every $\Phi \in L^X$ and $\bigvee(\Phi(y) \ominus d_X(y,-)) \in L^X$, there exists $z = \sqcup_X(\bigvee(\Phi(y) \ominus d_X(y,-)))$. We will show that $z = \sqcap_X \Phi$.

(M2) By the definition of $\sqcup_X(\bigvee(\Phi(y) \ominus d_X(y,-)))$, by (J1),

$$\bigvee(\Phi(y) \ominus d_X(y,x)) \geq d_X(\sqcup_X(\bigvee(\Phi(y) \ominus d_X(y,-))), x) = d_X(z,x).$$

(M1) Since $(\Phi(y) \ominus d_X(y,x)) \oplus \Phi(y) \geq d_X(y,x)$ iff $\Phi(y) \geq (\Phi(y) \ominus d_X(y,x)) \ominus d_X(y,x)$,

$$\begin{aligned} \Phi(y) &\geq \bigvee_{x\in X}((\Phi(y) \ominus d_X(y,x)) \ominus d_X(y,x)) \\ &\geq \bigvee_{x\in X}(\bigvee_{y\in X}(\Phi(y) \ominus d_X(y,x)) \ominus d_X(y,x)) \\ &= d_X(y, \sqcup_X(\bigvee(\Phi(y) \ominus d_X(y,-)))) = d_X(y,z). \end{aligned}$$

(2) ⇒ (1). For every $\Psi \in L^X$ and $\bigvee(\Psi(y) \to d_X(-,y)) \in L^X$, there exists $w = \sqcap_X(\bigvee(\Psi(y) \ominus d_X(-,y)))$. We will show that $z = \sqcup_X \Psi$.

(J2) Since $w = \sqcap_X(\bigvee(\Psi(y) \ominus d_X(-,y)))$,

$$\bigvee(\Psi(y) \ominus d_X(x,y)) \geq d_X(x, \sqcap_X(\bigvee(\Psi(y) \ominus d_X(y,-)))) = d_X(x,w).$$

(J1) Since $(\Psi(y) \ominus d_X(x,y)) \oplus \Phi(y) \geq d_X(x,y)$ iff $\Psi(y) \geq (\Psi(y) \ominus d_X(x,y)) \ominus d_X(x,y)$,

$$\begin{aligned} \Psi(y) &\geq \bigvee_{x\in X}((\Psi(y) \ominus d_X(x,y)) \ominus d_X(x,y)) \\ &\geq \bigvee_{x\in X}(\bigvee_{y\in X}(\Psi(y) \ominus d_X(x,y)) \ominus d_X(x,y)) \\ &= d_X(\sqcap_X(\bigvee(\Psi(y) \ominus d_X(-,y))), y). \end{aligned}$$

Hence $\sqcup_X \Psi = \sqcap_X(\bigvee(\Psi(y) \ominus d_X(-,y))) = w$. □

Definition 3.6. (1) A subset $\tau \subset L^X$ is called an *Alexandrov pretopology* on X iff it satisfies the following conditions:

(O1) if $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i\in I} A_i \in \tau$.

(O2) if $A \in \tau$ and $\alpha \in L$, then $\alpha \ominus A \in \tau$.

(2) A subset $\eta \subset L^X$ is called an *Alexandrov precotopology* on X iff it satisfies the following conditions:

(CO1) if $A_i \in \eta$ for all $i \in I$, then $\bigwedge_{i\in I} A_i \in \eta$.

(CO2) if $A \in \eta$ and $\alpha \in L$, then $\alpha \oplus A \in \eta$.

A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X iff it is both Alexandrov pretopology and Alexandrov precotopology on X.

Theorem 3.7. *Let $\tau \subset L^X$. Define $d_\tau : \tau \times \tau \to L$ as $d_\tau(A, B) = \bigvee_{x\in X}(A(x) \ominus B(x))$. Then the following statements hold.*

(1) *(τ, d_τ) is a distance space.*

(2) *$\sqcup_\tau \Phi$ is a fuzzy join of $\Phi \in L^\tau$ iff $\bigvee_{A\in\tau}(\Phi(A) \ominus d_\tau(B, A)) = d_\tau(B, \sqcup_\tau \Phi)$.*

(3) *$\sqcap_\tau \Phi$ is a fuzzy meet of $\Phi \in L^\tau$ iff $\bigvee_{A\in\tau}(\Phi(A) \ominus d_\tau(A, B)) = d_\tau(\sqcap_\tau \Phi, B)$.*

(4) *If $\sqcup_\tau \Phi$ is a fuzzy join of $\Phi \in L^\tau$, then it is unique. Moreover, if $\sqcap_\tau \Phi$ is a fuzzy meet of $\Phi \in L^\tau$, then it is unique.*

Proof. (1) (M1) For each $A \in \tau$, $d_\tau(A, A) = \bigvee_{x\in X}(A(x) \ominus A(x)) = 0$.

(M2) By Lemma 2.3(5), $d_\tau(A, B) \oplus d_\tau(B, C) = \bigvee_{x\in X}(A(x) \ominus B(x)) \oplus \bigvee_{x\in X}(B(x) \ominus C(x)) \geq \bigvee_{x\in X}((A(x) \ominus B(x)) \oplus (B(x) \ominus C(x))) \geq d_\tau(A, C)$, for all $A, B, C \in \tau$.

(M3) If $d_\tau(A, B) = d_\tau(B, A) = 0$, by Lemma 2.3(8), $A = B$. Hence (τ, d_τ) is a distance space.

(2), (3) and (4) follow from Theorem 3.2 □

Theorem 3.8. *Let (X, d_X) be a distance space. Then (L^X, d_{L^X}) is a complete lattice.*

Proof. For every $\Phi \in L^{L^X}$ and $A \in L^X$, we obtain that $\sqcap_{L^X}\Phi(x) = \bigwedge_{A\in L^X}(\Phi(A) \oplus A(x))$ and $\sqcup_{L^X}\Phi(x) = \bigvee_{A\in L^X}(\Phi(A) \ominus A(x))$, since

$$\begin{aligned}
& d_{L^X}(\textstyle\bigwedge_{A\in L^X}(\Phi(A) \oplus A(-)), B) = \bigvee_{x\in X}(\bigwedge_{A\in L^X}(\Phi(A) \oplus A(x)) \ominus B(x)) \\
& = \textstyle\bigvee_{A\in L^X}(\Phi(A) \ominus \bigvee_{x\in X}(A(x) \ominus B(x))) \text{ (by Lemma 2.3(6))} \\
& = \textstyle\bigvee_{A\in L^X}(\Phi(A) \ominus d_{L^X}(A, B)) = d_{L^X}(\sqcap_{L^X}\Phi, B),
\end{aligned}$$

$$\begin{aligned}
& d_{L^X}(B, \textstyle\bigvee_{A\in L^X}(\Phi(A) \ominus A(-))) = \bigvee_{x\in X}(B(x) \ominus \bigvee_{A\in L^X}(\Phi(A) \ominus A(x))) \\
& = \textstyle\bigvee_{A\in L^X}(\Phi(A) \ominus \bigvee_{x\in X}(B(x) \ominus A(x))) \text{ (by Lemma 2.3(6))} \\
& = \textstyle\bigvee_{A\in L^X}(\Phi(A) \ominus d_{L^X}(B, A)) = d_{L^X}(B, \sqcup_{L^X}\Phi).
\end{aligned}$$

□

Theorem 3.9. *Let $\tau \subset L^X$. Then the following statements are equivalent:*

(1) *(τ, d_τ) is fuzzy join complete.*

(2) *τ is an Alexandrov pretopology on X.*

Proof. (1) ⇒ (2) Since (τ, d_τ) is fuzzy join complete, for each $\Phi \in L^\tau$, we have

$$d_\tau(B, \sqcup_\tau \Phi) = \bigvee_{C\in\tau}(\Phi(C) \ominus d_\tau(B,C))$$
$$= \bigvee_{C\in\tau} d_\tau(B, \Phi(C) \ominus C) = d_\tau(B, \bigvee_{C\in\tau} \Phi(C) \ominus C) \text{ (by Lemma 2.3(6))}.$$

By Lemma 3.2(4), $\sqcup_\tau \Phi = \bigvee_{C\in\tau}(\Phi(C) \ominus C) \in \tau$.

(O1) Define $\Phi : \tau \to L$ as $\Phi(A) = \alpha$ for $A \in \tau$ and $\Phi(B) = 1$, otherwise. Then

$$\sqcup_\tau \Phi(x) = \bigvee_{A\in\tau}(\Phi(A) \ominus A(x)) = \alpha \ominus A(x).$$

So, $\sqcup_\tau \Phi = \alpha \ominus A \in \tau$.

(O2) Let $\{A_i \in \tau \mid i \in \Gamma\}$ be given. Define $\Phi : \tau \to L$ as $\Phi(A_i) = 0$ for $i \in \Gamma$ and $\Phi(B) = 1$, otherwise. Then

$$\sqcup_\tau \Phi(x) = \bigvee_{A\in\tau}(\Phi(A) \ominus A(x)) = \bigvee_{i\in\Gamma}(0 \ominus A_i(x)) = \bigvee_{i\in\Gamma} A_i(x).$$

So, $\sqcup_\tau \Phi = \bigvee_{i\in\Gamma} A_i \in \tau$.

(2) ⇒ (1) For each $\Phi \in L^\tau$, by (O1) and (O2), $\bigvee_{C\in\tau}(\Phi(C) \ominus C) \in \tau$. Thus,

$$d_\tau(B, \sqcup_\tau \Phi) = \bigvee_{C\in\tau}(\Phi(C) \ominus d_\tau(B,C))$$
$$= \bigvee_{C\in\tau} d_\tau(B, \Phi(C) \ominus C) = d_\tau(B, \bigvee_{C\in\tau} \Phi(C) \ominus C) \text{ (by Lemma 2.3(6))}.$$

By Theorem 3.2 (3), $\sqcup_\tau \Phi$ is a fuzzy join of Φ. □

Theorem 3.10. *Let $\tau \subset L^X$. Then the following statements are equivalent:*

(1) *(τ, d_τ) is fuzzy meet complete.*

(2) *τ is an Alexandrov precotopology on X.*

Proof. (1) ⇒ (2) Since (τ, d_τ) is fuzzy meet complete, for each $\Phi \in L^\tau$, we have

$$d_\tau(\sqcap_\tau \Phi, B) = \bigvee_{C\in\tau}(\Phi(C) \ominus d_\tau(C,B))$$
$$= \bigvee_{C\in\tau} d_\tau(\Phi(C) \oplus C, B) = d_\tau(\bigwedge_{C\in\tau}(\Phi(C) \oplus C), B) \text{ (by Lemma 2.3(6))}.$$

By Theorem 3.2(3), $\sqcap_\tau \Phi = \bigwedge_{C\in\tau}(\Phi(C) \oplus C) \in \tau$.

(CO1) Define $\Phi : \tau \to L$ as $\Phi(A) = \alpha$ for $A \in \tau$ and $\Phi(B) = 1$, otherwise. Then

$$\sqcap_\tau \Phi(x) = \bigwedge_{A\in\tau}(\Phi(A) \oplus A(x)) = \alpha \oplus A(x).$$

So, $\sqcap_\tau \Phi = \alpha \oplus A \in \tau$.

(CO2) Let $\{A_i \in \tau \mid i \in \Gamma\}$ be given. Define $\Phi : \tau \to L$ as $\Phi(A_i) = 0$ for $i \in \Gamma$ and $\Phi(B) = 1$, otherwise. Then

$$\sqcap_\tau \Phi(x) = \bigwedge_{A\in\tau}(\Phi(A) \oplus A(x)) = \bigwedge_{i\in\Gamma}(0 \oplus A_i(x)) = \bigwedge_{i\in\Gamma} A_i(x).$$

So, $\sqcap_\tau \Phi = \bigwedge_{i\in\Gamma} A_i \in \tau$.

(2) ⇒ (1) For each $\Phi \in L^\tau$, by (CO1) and (CO2), $\bigwedge_{C\in\tau} \Phi(C) \oplus C \in \tau$. Thus,

$$d_\tau(\sqcap_\tau \Phi, B) = \bigvee_{C\in\tau}(\Phi(C) \ominus d_\tau(C,B))$$
$$= \bigvee_{C\in\tau} d_\tau(\Phi(C) \oplus C, B) = d_\tau(\bigwedge_{C\in\tau} \Phi(C) \oplus C, B).$$

By Theorem 3.2 (2), $\sqcap_\tau \Phi$ is a fuzzy meet of Φ. $\square$

Theorem 3.11. *Let $\mathcal{D} : L^X \to L^X$ be a map. The following statements are equivalent.*

(1) $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{D}(A), \mathcal{D}(B))$ *for all* $A, B \in L^X$.

(2) $\alpha \oplus \mathcal{D}(A) \geq \mathcal{D}(\alpha \oplus A)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{D}(A) \leq \mathcal{D}(B)$ *for* $A \leq B$.

(3) $\mathcal{D}(\alpha \ominus A) \geq \alpha \ominus \mathcal{D}(A)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{D}(A) \leq \mathcal{D}(B)$ *for* $A \leq B$.

Proof. (1) $\Rightarrow$ (2). If $B \leq A$, by Lemma 2.3(8), $d_{L^X}(A, B) = 0$ and $d_{L^X}(\mathcal{D}(A), \mathcal{D}(B)) = 0$. Thus $\mathcal{D}(B) \leq \mathcal{D}(A)$. Since $\alpha \geq d_{L^X}(A, \alpha \oplus A) \geq d_{L^X}(\mathcal{D}(A), \mathcal{D}(\alpha \oplus A))$, we have $\alpha \oplus \mathcal{D}(A) \geq \mathcal{D}(\alpha \oplus A)$.

(2)$\Rightarrow$ (1). Put $\alpha = d_{L^X}(A, B)$. Then $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{D}(A), \mathcal{D}(B))$, since

$$d_{L^X}(A, B) \oplus \mathcal{D}(A) \geq \mathcal{D}(d_{L^X}(A, B) \oplus A) \geq \mathcal{D}(B).$$

(1) $\Rightarrow$ (3). If $A \leq B$, then $\mathcal{D}(A) \leq \mathcal{D}(B)$. Since $\alpha \geq d_{L^X}(\alpha \ominus A, A) \geq d_{L^X}(\mathcal{D}(\alpha \ominus A), \mathcal{D}(A))$, we have $\mathcal{D}(\alpha \ominus A) \geq \alpha \ominus \mathcal{D}(A)$.

(3)$\Rightarrow$ (1). Put $\alpha = d_{L^X}(A, B)$. Then $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{D}(A), \mathcal{D}(B))$, since

$$\mathcal{D}(A) \geq \mathcal{D}(d_{L^X}(A, B) \ominus B) \geq d_{L^X}(A, B) \ominus \mathcal{D}(B).$$

$\square$

Theorem 3.12. *Let $\mathcal{D} : L^X \to L^X$ be a map. The following statements hold.*

(1) $\sqcup_{L^X} \mathcal{D}^{\to}(\Phi) \leq \mathcal{D}(\sqcup_{L^X} \Phi)$ *for each* $\Phi \in L^{L^X}$ *where* $\mathcal{D}^{\to}(\Phi)(B) = \bigvee_{B=\mathcal{D}(A)} \Phi(A)$ *iff* $\mathcal{D}(\alpha \ominus A) \geq \alpha \ominus \mathcal{D}(A)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{D}(A) \leq \mathcal{D}(B)$ *for* $A \leq B$.

(2) $\mathcal{D}(\sqcap_{L^X} \Phi) \leq \sqcap_{L^X} \mathcal{D}^{\to}(\Phi)$ *for each* $\Phi \in L^{L^X}$ *iff* $\alpha \oplus \mathcal{D}(A) \geq \mathcal{D}(\alpha \oplus A)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{D}(A) \leq \mathcal{D}(B)$ *for* $A \leq B$.

Proof. (1) ($\Rightarrow$) For all $\Phi \in L^{L^X}$,

$$\begin{aligned} d_{L^X}(B, \sqcup_{L^X} \Phi) &= \textstyle\bigvee_{A \in L^X} (\Phi(A) \ominus d_{L^X}(B, A)) \\ &= \textstyle\bigvee_{A \in L^X} d_{L^X}(B, \Phi(A) \ominus A) = d_{L^X}(B, \bigvee_{A \in L^X} \Phi(A) \ominus A), \end{aligned}$$

$$\begin{aligned} d_{L^X}(B, \sqcup_{L^X} \mathcal{D}^{\to}(\Phi)) &= \textstyle\bigvee_{C \in L^X} (\mathcal{D}^{\to}(\Phi)(C) \ominus d_{L^X}(B, C)) \\ &= \textstyle\bigvee_{C \in L^X} d_{L^X}(B, \mathcal{D}^{\to}(\Phi)(C) \ominus C) = d_{L^X}(B, \bigvee_{C \in L^X} \mathcal{D}^{\to}(\Phi)(C) \ominus C)). \end{aligned}$$

By Theorem 3.2(3), $\sqcup_{L^X} \Phi = \bigvee_{A \in L^X} (\Phi(A) \ominus A)$ and $\sqcup_{L^X} \mathcal{D}^{\to}(\Phi) = \bigvee_{C \in L^X} (\mathcal{D}^{\to}(\Phi)(C) \ominus C)$. Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 1$, otherwise. Then

$$(\sqcup_{L^X} \Phi_1)(x) = \bigvee_{D \in L^X} (\Phi_1(D) \ominus D(x)) = \alpha \ominus A(x).$$

Since $\mathcal{D}^{\rightarrow}(\Phi_1)(B) = \bigvee_{B=\mathcal{D}(A)} \Phi_1(A)$ and $\mathcal{D}(\sqcup_{L^X}\Phi_1) \geq \sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi_1)$ for all $\Phi_1 \in L^{L^X}$, we have

$$\begin{aligned}&\sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi_1)(x) = \textstyle\bigvee_{C=\mathcal{D}(A)\in L^X}(\mathcal{D}^{\rightarrow}(\Phi_1)(C)\ominus C(x))\\ &= \Phi_1(A)\ominus\mathcal{D}(A)(x) = \alpha\ominus\mathcal{D}(A)(x) \leq \mathcal{D}(\sqcup_{L^X}\Phi_1)(x) = \mathcal{D}(\alpha\ominus A)(x).\end{aligned}$$

Hence $\alpha\ominus\mathcal{D}(A) \leq \mathcal{D}(\alpha\ominus A)$.

Let $A \leq B$ be given. Define $\Phi_2 : L^X \to L$ as $\Phi_2(A) = \Phi_2(B) = 0$ and $\Phi_2(C) = 1$, otherwise. Then

$$(\sqcup_{L^X}\Phi_2)(x) = \bigvee_{D\in L^X}(\Phi_2(D)\ominus D(x)) = A(x)\vee B(x) = B(x).$$

Since $\mathcal{D}^{\rightarrow}(\Phi_2)(B) = \bigvee_{B=\mathcal{D}(A)} \Phi_2(A)$ and $\mathcal{D}(\sqcup_{L^X}\Phi_2) \geq \sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi_2)$ for $\Phi_2 \in L^{L^X}$,

$$\begin{aligned}&\sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi_2)(x) = \textstyle\bigvee_{C=\mathcal{D}(A)\in L^X}(\mathcal{D}^{\rightarrow}(\Phi_2)(C)\ominus C(x))\\ &= (\Phi_2(A)\ominus\mathcal{D}(A)(x))\vee(\Phi_2(B)\ominus\mathcal{D}(B)(x)) = \mathcal{D}(A)(x)\vee\mathcal{D}(B)(x)\\ &\leq \mathcal{D}(\sqcup_{L^X}\Phi_1)(x) = \mathcal{D}(A\vee B)(x) = \mathcal{D}(B)(x).\end{aligned}$$

Hence $\mathcal{D}(A) \leq \mathcal{D}(B)$.

($\Leftarrow$) $\sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi) \leq \mathcal{D}(\sqcup_{L^X}\Phi)$, since

$$\begin{aligned}&\sqcup_{L^X}\mathcal{D}^{\rightarrow}(\Phi)(y) = \textstyle\bigvee_{A\in L^X}\Phi(A)\ominus\mathcal{D}(A)(y)\\ &\leq \mathcal{D}(\textstyle\bigvee_{A\in L^X}(\Phi(A)\ominus A))(y) = \mathcal{D}(\sqcup_{L^X}\Phi)(y).\end{aligned}$$

(2) ($\Rightarrow$) For all $\Phi \in L^{L^X}$,

$$\begin{aligned}&d_{L^X}(\sqcap_{L^X}\Phi, B) = \textstyle\bigvee_{A\in L^X}(\Phi(A)\ominus d_{L^X}(A,B))\\ &= \textstyle\bigvee_{A\in L^X} d_{L^X}(\Phi(A)\oplus A, B) = d_{L^X}(\textstyle\bigwedge_{A\in L^X}(\Phi(A)\oplus A), B),\end{aligned}$$

$$\begin{aligned}&d_{L^X}(\sqcap_{L^X}\mathcal{D}^{\rightarrow}(\Phi), B) = \textstyle\bigvee_{C\in L^X}(\mathcal{D}^{\rightarrow}(\Phi)(C)\ominus d_{L^X}(C,B))\\ &= \textstyle\bigvee_{C\in L^X}((\textstyle\bigvee_{\mathcal{D}(A)=C}\Phi(A)\ominus d_{L^X}(C,B)))\\ &= \textstyle\bigvee_{A\in L^X}(\Phi(A)\ominus d_{L^X}(\mathcal{D}(A),B) = \textstyle\bigvee_{A\in L^X} d_{L^X}(\Phi(A)\oplus\mathcal{D}(A),B)\\ &= d_{L^X}(\textstyle\bigwedge_{A\in L^X}\Phi(A)\oplus\mathcal{D}(A),B).\end{aligned}$$

By Theorem 3.2(3), $\sqcap_{L^X}\Phi = \bigwedge_{A\in L^X}(\Phi(A)\oplus A)$ and $\sqcap_{L^X}\mathcal{D}^{\rightarrow}(\Phi) = \bigwedge_{A\in L^X}(\Phi(A)\oplus\mathcal{D}(A)) \in L^X$. Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 1$, otherwise. Then

$$(\sqcap_{L^X}\Phi_1) = \bigwedge_{A\in L^X}(\Phi_1(A)\oplus A) = \alpha\oplus A.$$

Since $\mathcal{D}^{\rightarrow}(\Phi_1)(B) = \bigvee_{B=\mathcal{D}(A)}\Phi_1(A)$ and $\mathcal{D}(\sqcap_{L^X}\Phi_1) \leq \sqcap_{L^X}\mathcal{D}^{\rightarrow}(\Phi_1)$ for $\Phi_1 \in L^{L^X}$,

$$\begin{aligned}&\sqcap_{L^X}\mathcal{D}^{\rightarrow}(\Phi_1)(y) = \textstyle\bigwedge_{B\in L^X}(\Phi_1(A)\oplus\mathcal{D}(A)(y))\\ &= \alpha\oplus\mathcal{D}(A)(y) \geq \mathcal{D}(\sqcap_{L^X}\Phi_1)(y) = \mathcal{D}(\alpha\oplus A)(y).\end{aligned}$$

Hence $\mathcal{D}(\alpha\oplus A) \leq \alpha\oplus\mathcal{D}(A) \in L^X$.

Let $A \le B$ be given. Define $\Phi_2 : L^X \to L$ as $\Phi_2(A) = \Phi_2(B) = 0$ and $\Phi_2(C) = 1$, otherwise. Then $(\sqcap_{L^X}\Phi_2)(x) = \bigwedge_{D\in L^X}(\Phi_2(D) \oplus D(x)) = A(x) \wedge B(x) = A(x)$. Since $\mathcal{D}^{\to}(\Phi_2)(B) = \bigvee_{B=\mathcal{D}(A)}\Phi_2(A)$ and $\mathcal{D}(\sqcap_{L^X}\Phi_2) \le \sqcap_{L^X}\mathcal{D}^{\to}(\Phi_2)$ for $\Phi_2 \in L^{L^X}$,

$$\begin{aligned}
&\sqcap_{L^X}\mathcal{D}^{\to}(\Phi_2)(x) = \textstyle\bigvee_{C=\mathcal{D}(A)\in L^X}(\mathcal{D}^{\to}(\Phi_2)(C) \oplus C(x))\\
&= (\Phi_2(A) \oplus \mathcal{D}(A)(x)) \wedge (\Phi_2(B) \oplus \mathcal{D}(B)(x)) = \mathcal{D}(A)(x) \wedge \mathcal{D}(B)(x)\\
&\ge \mathcal{D}(\sqcap_{L^X}\Phi_1)(x) = \mathcal{D}(A \wedge B)(x) = \mathcal{D}(A)(x).
\end{aligned}$$

Hence $\mathcal{D}(A) \le \mathcal{D}(B)$.

($\Leftarrow$) $\mathcal{D}(\sqcap_{L^X}\Phi) \le \sqcap_{L^X}\mathcal{D}^{\to}(\Phi)$, since

$$\begin{aligned}
&\sqcap_{L^X}\mathcal{D}^{\to}(\Phi) = \textstyle\bigwedge_{A\in L^X}(\Phi(A) \oplus \mathcal{D}(A))\\
&\ge \textstyle\bigwedge_{A\in L^X}\mathcal{D}(\Phi(A) \oplus A) \ge \mathcal{D}(\bigwedge_{A\in L^X}(\Phi(A) \oplus A)) = \mathcal{D}(\sqcap_{L^X}\Phi).
\end{aligned}$$

□

Theorem 3.13. *Let $\mathcal{D} : L^X \to L^X$ be a map with $d_{L^X}(A, B) \ge d_{L^X}(\mathcal{D}(A), \mathcal{D}(B))$ for all $A, B \in L^X$. Then followings hold.*

(1) *$\tau_D = \{A \in L^X \mid A \le \mathcal{D}(A)\}$ is an Alexandrov fuzzy pretopology, that is, τ_D is a fuzzy join complete lattice.*

(2) *$\eta_D = \{A \in L^X \mid \mathcal{D}(A) \le A\}$ is an Alexandrov fuzzy precotopology, that is, η_D is a fuzzy meet complete lattice.*

Proof. (1) (O1) For each $A \in \tau_D$, by Theorem 3.11, $\mathcal{D}(\alpha \ominus A) \ge \alpha \ominus \mathcal{D}(A) \ge \alpha \ominus A$. Hence $(\alpha \ominus A) \in \tau_D$.

(O2) For each $A_i \in \tau_D$ for $i \in \Gamma$, $\mathcal{D}(\bigvee_{i\in\Gamma} A_i) \ge \bigvee_{i\in\Gamma}\mathcal{D}(A_i) \ge \bigvee_{i\in\Gamma} A_i$. Hence $\bigvee_{i\in\Gamma} A_i \in \tau_D$.

(2) (O1) For each $A \in \eta_D$, by Theorem 3.11, $\mathcal{D}(\alpha \oplus A) \le \alpha \oplus \mathcal{D}(A) \le \alpha \oplus A$. Hence $(\alpha \oplus A) \in \eta_D$.

(O2) For each $A_i \in \eta_D$ for $i \in \Gamma$, $\mathcal{D}(\bigwedge_{i\in\Gamma} A_i) \le \bigwedge_{i\in\Gamma}\mathcal{D}(A_i) \le \bigwedge_{i\in\Gamma} A_i$. Hence $\bigwedge_{i\in\Gamma} A_i \in \eta_D$. □

Example 3.14. Let X be a set and $R \in L^{X\times X}$. For each $A \in L^X$, define $D_1, D_2 : L^X \to L^X$ as follows:

$$D_1(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x)),\ D_2(A)(y) = \bigvee_{x\in X}(R(x,y) \ominus A(x)).$$

For each $A, B \in L^X$, the followings hold.

$$\begin{aligned}
&d_{L^X}(D_1(A), D_1(B))\\
&= \textstyle\bigvee_{y\in X}((\bigwedge_{x\in X}(R(x,y) \oplus A(x))) \ominus (\bigwedge_{x\in X}(R(x,y) \oplus B(x))))
\end{aligned}$$

$$\begin{array}{l}
\leq \bigvee_{y\in X}\bigvee_{x\in X}((R(x,y)\oplus A(x))\ominus(\bigwedge_{x\in X}(R(x,y)\oplus B(x))))\\
\leq \bigvee_{y\in X}\bigvee_{x\in X}((R(x,y)\oplus A(x))\ominus(R(x,y)\oplus B(x))))\\
\leq \bigvee_{x\in X}(A(x)\ominus B(x)) = d_{L^X}(A,B),\\
d_{L^X}(D_2(A),D_2(B))\\
= \bigvee_{y\in X}((\bigvee_{x\in X}(R(x,y)\ominus A(x)))\ominus(\bigvee_{x\in X}(R(x,y)\ominus B(x))))\\
\leq \bigvee_{y\in X}((R(x,y)\ominus A(x))\ominus(\bigvee_{x\in X}(R(x,y)\ominus B(x))))\\
\leq \bigvee_{y\in X}\bigvee_{x\in X}((R(x,y)\ominus A(x))\ominus(R(x,y)\ominus B(x))))\\
\leq \bigvee_{x\in X}(A(x)\ominus B(x)) = d_{L^X}(A,B).
\end{array}$$

For each $i\in\{1,2\}$, by Theorems 3.12, 3.13 and 3.14, the followings hold.

(1) $\alpha\oplus\mathcal{D}_i(A)\geq\mathcal{D}_i(\alpha\oplus A)$ for each $\alpha\in L, A\in L^X$ and $\mathcal{D}_i(A)\leq\mathcal{D}_i(B)$ for $A\leq B$.

(2) $\mathcal{D}_i(\alpha\ominus A)\geq\alpha\ominus\mathcal{D}_i(A)$ for each $\alpha\in L, A\in L^X$ and $\mathcal{D}_i(A)\leq\mathcal{D}_i(B)$ for $A\leq B$.

(3) $\sqcup_{L^X}\mathcal{D}_i^{\rightarrow}(\Phi)\leq\mathcal{D}_i(\sqcup_{L^X}\Phi)$ for each $\Phi\in L^{L^X}$ where $\mathcal{D}_i^{\rightarrow}(\Phi)(B)=\bigvee_{B=\mathcal{D}_i(A)}\Phi(A)$.

(4) $\mathcal{D}_i(\sqcap_{L^X}\Phi)\leq\sqcap_{L^X}\mathcal{D}_i^{\rightarrow}(\Phi)$ for each $\Phi\in L^{L^X}$.

(5) $\tau_{D_i}=\{A\in L^X\mid A\leq\mathcal{D}_i(A)\}$ is an Alexandrov fuzzy pretopology, that is, τ_{D_i} is a fuzzy join complete lattice.

(6) $\eta_{D_i}=\{A\in L^X\mid \mathcal{D}_i(A)\leq A\}$ is an Alexandrov fuzzy precotopology, that is, η_{D_i} is a fuzzy meet complete lattice.

Example 3.15. Let $X=\{x,y,z\}$, $A\in[0,\infty]^X$ with $A(x)=8, A(y)=3, A(z)=9$.

(1) Define an Alexandrov pretopology as

$$\tau_X=\{\alpha\ominus A\mid\alpha\in[0,\infty]\}.$$

By Theorem 3.7(1), (τ_X,d_{τ_X}) is a distance space. For each $\Phi:\tau_X\to[0,\infty]$, since $\bigvee_{C\in\tau_X}(\Phi(C)\ominus C)=\bigvee_{\alpha\in[0,\infty]}(\Phi(\alpha\ominus A)\ominus(\alpha\ominus A))=\bigvee_{\alpha\in[0,\infty]}((\Phi(\alpha\ominus A)\oplus\alpha)\ominus A))\in\tau_X$, it follows that

$$\begin{array}{ll}
d_\tau(B,\sqcup_{\tau_X}\Phi) & =\bigvee_{C\in\tau_X}(\Phi(C)\ominus d_{\tau_X}(B,C))\\
& =\bigvee_{C\in\tau_X}d_{\tau_X}(B,\Phi(C)\ominus C)=d_{\tau_X}(B,\bigvee_{C\in\tau_X}(\Phi(C)\ominus C))\\
& =d_{\tau_X}(B,\bigvee_{\alpha\in[0,\infty]}((\Phi(\alpha\ominus A)\oplus\alpha)\ominus A))).
\end{array}$$

By Theorem 3.2(2), (τ_X,d_{τ_X}) is a fuzzy join complete lattice.

(2) Define an Alexandrov precotopology as

$$\eta_X=\{\alpha\oplus A\mid\alpha\in[0,\infty]\}.$$

By Theorem 3.7(1), (η_X,d_{η_X}) is a distance space. For each $\Psi:\eta_X\to[0,\infty]$, since $\bigwedge_{C\in\tau_X}(\Psi(C)\oplus C)\in\eta_X=\bigwedge_{\alpha\in[0,\infty]}((\Psi(\alpha\oplus A)\oplus\alpha)\oplus A))\in\eta_X$, we have

$$\begin{aligned} d_{\eta_X}(\sqcap_{\eta_X}\Psi, B) &= \bigvee_{C\in\eta_X}(\Psi(C)\ominus d_{\eta_X}(C,B)) \\ &= d_{\eta_X}(\bigwedge_{C\in\eta_X}(\Psi(C)\oplus C), B) \\ &= d_{\eta_X}(\bigwedge_{\alpha\in[0,\infty]}((\Psi(\alpha\oplus A)\oplus\alpha)\oplus A)), B). \end{aligned}$$

By Theorem 3.2(3), (η_X, d_{η_X}) is a fuzzy meet complete lattice.

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