UTILIZING GENERALIZED MEIR-KEELER CONTRACTION IN PERIODIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. This manuscript is divided into three segments. In the first segment, we formulate a unique common fixed point theorem satisfying generalized Meir-Keeler contraction on partially ordered metric spaces and also give an example to demonstrate the usability of our result. In the second segment of the article, some common coupled fixed point results are derived from our main results. In the last segment, we investigate the solution of some periodic boundary value problems. Our results generalize, extend and improve several well-known results of the existing literature.

1. INTRODUCTION

Banach contraction mapping principle has been generalized in several directions, One of these generalizations, known as the Meir-Keeler fixed point theorem [16]. In [17], Samet established the coupled fixed points of mixed strict monotone generalized Meir-Keeler operators and also established the existence and uniqueness results for coupled fixed point. Berinde and Pecurar [2] obtained more general coupled fixed point theorems for mixed monotone operators $F : X^2 \to X$ satisfying a generalized Meir-Keeler contractive condition.

On the other hand, the idea of the coupled fixed point was initiated by Guo and Lakshmikantham [11] in 1987. Following this paper, Gnana-Bhaskar and Lakshmikantham [3] introduced the notion of mixed monotone property for $F : X^2 \to X$ in a partially ordered metric space and utilized the same to prove some theorems on the existence and uniqueness of coupled fixed points. In 2009, Lakshmikantham and Ciric [15] generalized these results for nonlinear contraction mappings by introducing the notions of coupled coincidence point and mixed $g-$monotone property.

Subsequently, Choudhury and Kundu [4] introduced the notion of compatibility and using this notion, they improve the results of Lakshmikantham and Ciric [15], then after several authors established coupled fixed/coincidence point theorems by using this notion. Some of our fundamental sources are given in [1, 6, 7, 8, 9, 10, 12, 18].

In this manuscript, we prove a common fixed point theorem satisfying generalized Meir-Keeler contraction on partially ordered metric spaces. With the help of obtained result, we derive a common coupled fixed point result. As an application, we investigate the solution of periodic boundary value problems. We also give an example to show the fruitfulness of our results. We generalize, sharpen and modify the results of Berinde and Pecurar [2], Gnana-Bhaskar and Lakshmikantham [3], Lakshmikantham and Ciric [15], Meir and Keeler [16] and various well-known results of the existing literature.

2. PRELIMINARIES

Definition 2.1 ([11])**.** Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a *coupled fixed point* of F if

$$F(x, \ y) = x \text{ and } F(y, \ x) = y.$$

Definition 2.2 ([3])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ is a given mapping. We say that F has the *mixed monotone property* if for all $x, y \in X$, we have

$$x_1, \ x_2 \ \in \ X, \ x_1 \preceq x_2 \implies F(x_1, \ y) \preceq F(x_2, \ y),$$
$$y_1, \ y_2 \ \in \ X, \ y_1 \preceq y_2 \implies F(x, \ y_1) \succeq F(x, \ y_2).$$

Definition 2.3 ([15])**.** Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of mappings F and G if

$$F(x, \ y) = Gx \text{ and } F(y, \ x) = Gy.$$

Definition 2.4 ([15])**.** Let $F : X^2 \to X$ and $G : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *common coupled fixed point* of mappings F and G if

$$x = F(x, \ y) = Gx \text{ and } y = F(y, \ x) = Gy.$$

Definition 2.5 ([15])**.** Mappings $F : X^2 \to X$ and $G : X \to X$ are said to be *commutative* if

$$GF(x, \ y) = F(Gx, \ Gy), \text{ for all } (x, \ y) \in X^2.$$

Definition 2.6 ([15])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $G : X \to X$ are given mappings. We say that F has the *mixed G– monotone property* if for all $x, y \in X$, we have

$$x_1, \ x_2 \ \in \ X, \ Gx_1 \preceq Gx_2 \implies F(x_1, \ y) \preceq F(x_2, \ y),$$
$$y_1, \ y_2 \ \in \ X, \ Gy_1 \preceq Gy_2 \implies F(x, \ y_1) \succeq F(x, \ y_2).$$

If G is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.7 ([4])**.** Mappings $F : X^2 \to X$ and $G : X \to X$ are said to be *compatible* if

$$\lim_{n \to \infty} d(GF(x_n, \ y_n), \ F(Gx_n, \ Gy_n)) \ = \ 0,$$
$$\lim_{n \to \infty} d(GF(y_n, \ x_n), \ F(Gy_n, \ Gx_n)) \ = \ 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n \to \infty} F(x_n, \ y_n) \ = \ \lim_{n \to \infty} Gx_n = x \in X,$$
$$\lim_{n \to \infty} F(y_n, \ x_n) \ = \ \lim_{n \to \infty} Gy_n = y \in X.$$

These studies could be applied to initial value problems defined by differential or integral equations.

Definition 2.8 ([3, 10])**.** A partially ordered metric space $(X, d, \preceq)$ is a metric space (X, d) provided with a partial order $\preceq$. A partially ordered metric space $(X, d, \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X, d, \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular. Let $F, G : X \to X$ be two mappings. We say that F is $(G, \preceq)$-*non-decreasing* if $Fx \preceq Fy$ for all $x, y \in X$ such that $Gx \preceq Gy$. If G is the identity mapping on X, we say that F is $\preceq$-*non-decreasing*.

Definition 2.9 ([5])**.** Two self-mappings G and F of a non-empty set X are said to be *commutative* if $GFx = FGx$ for all $x \in X$.

Definition 2.10 ([13])**.** Let $(X, d, \preceq)$ be a partially ordered metric space. Two mappings $F, G : X \to X$ are said to be *compatible* if

$$\lim_{n \to \infty} d(GFx_n, \ FGx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n \in X.$$

Definition 2.11 ([14]). Two self-mappings G and F of a non-empty set X are said to be *weakly compatible* if they commute at their coincidence points, that is, if $Gx = Fx$ for some $x \in X$, then $GFx = FGx$.

Definition 2.12 ([5]). Let X be a nonempty set. Mappings g and F, where $g : X \to X$ and $F : X^2 \to X$ are said to be *weakly compatible* if they commute at their coupled coincidence points, that is, if $F(x, y) = gx$ and $F(y, x) = gy$ for some $(x, y) \in X^2$, then $gF(x, y) = F(gx, gy)$ and $gF(y, x) = F(gy, gx)$.

3. FIXED POINT RESULTS

In this section we prove a unique fixed point result for mappings $f, g : X \to X$ in a partially ordered metric space $(X, d, \preceq)$, where X is a non-empty set. For simplicity, we denote $g(x)$ by gx where $x \in X$.

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space and $f, g : X \to X$ be two mappings such that f is $(g, \preceq)$-non-decreasing, $f(X) \subseteq g(X)$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$(3.1) \qquad \varepsilon \le d(gx, gy) \le \varepsilon + \delta(\varepsilon) \Rightarrow d(fx, fy) < \varepsilon,$$

for all $x, y \in X$ such that $gx \preceq gy$. There exists $x_0 \in X$ such that $gx_0 \preceq fx_0$. Also assume that, at least, one of the following conditions holds.

(a) (X, d) is complete, f and g are continuous and the pair (f, g) is compatible.

(b) $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular.

(c) (X, d) is complete, g is continuous and monotone non-decreasing, the pair (f, g) is compatible and $(X, d, \preceq)$ is non-decreasing-regular.

Then f and g have, at least, a coincidence point. Moreover, if for every $x, y \in X$ there exists a point $z \in X$ such that fz is comparable to fx and fy and also the pair (f, g) is weakly compatible. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Then $gx_0 \preceq fx_0 = gx_1$. Since f is $(g, \preceq)$-non-decreasing, $fx_0 \preceq fx_1$. Continuing in this manner, we get a sequence $\{x_n\}_{n\ge 0}$ such that $\{gx_n\}$

is $\preceq$-non-decreasing, $gx_{n+1} = fx_n \preceq fx_{n+1} = gx_{n+2}$ and

(3.2) $$gx_{n+1} = fx_n \text{ for all } n \geq 0.$$

Condition (3.1) implies the strict contractive condition

(3.3) $$d(fx, \ fy) < d(gx, \ gy),$$

for all $x, y \in X$ such that $gx \preceq gy$.

Let $d_n = d(gx_n, gx_{n+1})$ for all $n \geq 0$. First we claim that $\{d_n\} \to 0$. Now, by using the contractive condition (3.3), we have

$$d(gx_{n+1}, \ gx_{n+2}) = d(fx_n, \ fx_{n+1}) < d(gx_n, \ gx_{n+1}), \text{ that is, } d_{n+1} < d_n.$$

This shows that $\{d_n\}_{n \geq 0}$ is a decreasing sequence of non-negative numbers. Therefore, there exists some $\varepsilon \geq 0$ such that

$$\lim_{n \to \infty} d_n = \lim_{n \to \infty} d(gx_n, \ gx_{n+1}) = \varepsilon.$$

We shall now prove that $\varepsilon = 0$. Suppose, to the contrary, that $\varepsilon > 0$. Then there exists a positive integer p such that

$$\varepsilon < d_p < \varepsilon + \delta(\varepsilon) \Rightarrow d(fx_p, \ fx_{p+1}) < \varepsilon.$$

It follows from (3.2) that

$$d_{p+1} = d(gx_{p+1}, \ gx_{p+2}) < \varepsilon,$$

which is a contradiction. Thus $\varepsilon = 0$ and hence

(3.4) $$\lim_{n \to \infty} d_n = \lim_{n \to \infty} d(gx_n, \ gx_{n+1}) = 0.$$

Now we claim that $\{gx_n\}_{n \geq 0}$ is a Cauchy sequence in X. Let $\varepsilon > 0$ be arbitrary and $\delta(\varepsilon)$ be the corresponding value from the hypothesis of our theorem. By (3.4), there exists a positive integer k such that

(3.5) $$d_k = d(gx_k, \ gx_{k+1}) < \delta(\varepsilon).$$

For fixed number k, consider the set

$$A_k = \{x \in X : gx_k \preceq gx \text{ and } d(gx_k, \ gx) < \varepsilon + \delta(\varepsilon)\}.$$

By (3.5), $A_k \neq \phi$. We claim that

(3.6) $$gx \in A_k \Rightarrow fx \in A_k.$$

Let $gx \in A_k$. Then

(3.7) $$d(gx_k, \ gx) < \varepsilon + \delta(\varepsilon) \Rightarrow d(fx_k, \ fx) < \varepsilon.$$

Now, by (3.2), (3.5) and (3.7), we have

$$
\begin{aligned}
d(gx_k,\ fx) &\leq\ d(gx_k,\ fx_k) + d(fx_k,\ fx) \\
&\leq\ d(gx_k,\ gx_{k+1}) + d(fx_k,\ fx) \\
&<\ \varepsilon + \delta(\varepsilon).
\end{aligned}
$$

Thus $fx \in A_k$. Again

$$
\begin{aligned}
d(gx_k,\ gx_{k+1}) &\leq\ d(gx_k,\ fx) + d(fx,\ gx_{k+1}) \\
&<\ 2(\varepsilon + \delta(\varepsilon)).
\end{aligned}
$$

Thus $gx_{k+1} \in A_k$ and so by induction $gx_n \in A_k$, for all $n > k$. This implies that for all $n,\ m > k$, we have

$$
\begin{aligned}
d(gx_n,\ gx_m) &\leq\ d(gx_n,\ gx_k) + d(gx_k,\ gx_m) \\
&<\ 2(\varepsilon + \delta(\varepsilon)) = 4\varepsilon.
\end{aligned}
$$

This demonstrate that $\{gx_n\}_{n \geq 0}$ is a Cauchy sequence in X. We now claim that f and g have a coincidence point distinguishing between cases $(a) - (c)$.

First suppose (a) holds, that is, (X, d) is complete, f and g are continuous and the pair (f, g) is compatible. Since (X, d) is complete, there exists $x \in X$ such that $\{gx_n\} \to x$ and by (3.2), we also have that $\{fx_n\} \to x$. As f and g are continuous and so $\{fgx_n\} \to fx$ and $\{ggx_n\} \to gx$. Since the pair (f, g) is compatible, we conclude that

$$
d(gx,\ fx) = \lim_{n \to \infty} d(ggx_{n+1},\ fgx_n) = \lim_{n \to \infty} d(gfx_n,\ fgx_n) = 0,
$$

that is, x is a coincidence point of f and g.

Suppose now that (b) holds, that is, $(g(X), d)$ is complete and $(X, d, \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X), d)$, so there exists $y \in g(X)$ such that $\{gx_n\} \to y$. Let $x \in X$ be any point such that $y = gx$. In this case $\{gx_n\} \to gx$. Since $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gx, we have $gx_n \preceq gx$ for all $n \geq 0$.

Applying the contractive condition (3.3), we have

$$
d(gx_{n+1},\ fx) = d(fx_n,\ fx) < d(gx_n,\ gx).
$$

Taking $n \to \infty$ in the above inequality and by using the fact that $\{gx_n\} \to gx$, we get $d(gx,\ fx) = 0$, that is, x is a coincidence point of f and g.

Finally suppose that (c) holds, that is, (X, d) is complete, g is continuous and monotone non-decreasing, the pair (f, g) is compatible and $(X, d, \preceq)$ is non-decreasing-regular. Since (X, d) is complete, there exists $x \in X$ such that $\{gx_n\} \to x$. By (3.2), we also have that $\{fx_n\} \to x$. As g is continuous, then $\{ggx_n\} \to gx$. Moreover, the pair (f, g) is compatible and so we have

$$\lim_{n \to \infty} d(ggx_{n+1}, \ fgx_n) = \lim_{n \to \infty} d(gfx_n, \ fgx_n) = 0.$$

As $\{ggx_n\} \to gx$ the previous property means that $\{fgx_n\} \to gx$.

Since $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to x, we deduce that $gx_n \preceq x$, which, by the monotonicity of g, implies $ggx_n \preceq gx$. By using the contractive condition (3.4), we get

$$d(fgx_n, \ fx) < d(ggx_n, \ gx).$$

On taking $n \to \infty$ in the above inequality, by using the fact that $\{ggx_n\} \to gx$ and $\{fgx_n\} \to gx$, we get $d(gx, fx) = 0$, that is, x is a coincidence point of f and g.

Suppose x and y are coincidence points of g and f, that is, $gx = fx$ and $gy = fy$. Now, we show that $gx = gy$. By the assumption, there exists $z \in X$ such that fz is comparable with fx and fy. Put $z_0 = z$ and choose $z_1 \in X$ so that $gz_0 = fz_1$. One can inductively define sequences $\{gz_n\}$ where $gz_{n+1} = fz_n$ for all $n \geq 0$. Hence $fx = gx$ and $fz = fz_0 = gz_1$ are comparable. Suppose that $gz_1 \preceq gx$, we shall prove that $gz_n \preceq gx$ for each $n \geq 0$. In fact, we will use mathematical induction. Since $gz_1 \preceq gx$, our claim is true for $n = 1$.

Suppose that $gz_n \preceq gx$ holds for some $n > 1$. Since f is g-non-decreasing with respect to $\preceq$, we have $gz_{n+1} = fz_n \preceq fx = gx$ and this proves our claim.

Let $e_n = d(gx, gz_n)$ for all $n \geq 0$. Since $gz_n \preceq gx$, by using conditions (3.3) and (3.2), we have

(3.8) $d(gx, \ gz_{n+1}) = d(fx, \ fz_n) < d(gx, \ gz_n)$, that is, $e_{n+1} < e_n$.

This shows that the sequence $\{e_n\}_{n \geq 0}$ is a decreasing sequence of positive numbers. Then there exists $e \geq 0$ such that

(3.9) $\lim_{n \to \infty} e_n = \lim_{n \to \infty} d(gx, \ gz_n) = e.$

If possible, suppose $e > 0$. On taking $n \to \infty$ in (3.8) and by using (3.9), we get a contradiction. Hence

(3.10) $\lim_{n \to \infty} e_n = \lim_{n \to \infty} d(gx, \ gz_n) = 0.$

Similarly, one can obtain that

$$(3.11) \qquad \lim_{n \to \infty} d(gy, \ gz_n) = 0.$$

Hence, by (3.10) and (3.11), we get

$$(3.12) \qquad gx = gy.$$

Since $gx = fx$, by weak compatibility of g and f, we have $ggx = gfx = fgx$. Let $u = gx$, then $gu = fu$, that is, u is a coincidence point of g and f. Then from (3.12) with $y = u$, it follows that $gx = gu$, that is, $u = gu = fu$. Therefore, u is a common fixed point of g and f. To prove the uniqueness, assume that v is another common fixed point of g and f. Then by (3.12) we have $v = gv = gu = u$. Hence the common fixed point of g and f is unique. $\qquad \square$

Putting $g = I$ (the identity mapping) in Theorem 3.1, we get the following Corollary.

Corollary 3.2. *Let $(X, \ d, \ \preceq)$ be a partially ordered metric space. Suppose $f : X \to X$ is a non-decreasing mapping and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq d(x, \ y) \leq \varepsilon + \delta(\varepsilon) \Rightarrow d(fx, \ fy) < \varepsilon,$$

for all $x, y \in X$ such that $x \preceq y$. There exists $x_0 \in X$ such that $x_0 \preceq fx_0$. Then f has a fixed point.

Example 3.1. Suppose $X = \mathbb{R}$, furnished with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers. Let $f, g : X \to X$ be defined as

$$fx = \frac{x^2}{3} \text{ and } gx = x^2.$$

First, we shall show that the mappings f and g satisfy the contractive condition of Theorem 3.1. Let $x, y \in X$ with $gx \preceq gy$ such that

$$\varepsilon \leq d(gx, \ gy) \leq \varepsilon + \delta(\varepsilon), \text{ that is, } \varepsilon \leq \left|x^2 - y^2\right| \leq \varepsilon + \delta(\varepsilon).$$

Then

$$d(fx, \ fy) = \left|\frac{x^2}{3} - \frac{y^2}{3}\right| \leq \frac{1}{3}\left|x^2 - y^2\right| \leq \frac{1}{3}(\varepsilon + \delta(\varepsilon)) < \varepsilon.$$

Thus the contractive condition of Theorem 3.1 is satisfied for all $x, y \in X$. Moreover, all the other conditions of Theorem 3.1 are satisfied and $u = 0$ is a unique common fixed point of f and g.

4. Coupled Fixed Point Results

In this section, we derive some coupled fixed point results for mappings S, T : $X^2 \to X^2$ in partially ordered metric space $(X^2, \Delta_2, \sqsubseteq)$, where X is a non-empty set with the help of the results established in the previous section. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). Let $(X, \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:

$$W \sqsubseteq V \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } W = (u, \ v), \ V = (x, \ y) \in X^2.$$

Definition 4.1 ([1]). Let (X, d) be a metric space. Define $\Delta_n : X^n \times X^n \to [0, +\infty)$, for $A = (a_1, a_2, ..., a_n)$, $B = (b_1, b_2, ..., b_n) \in X^n$, by

$$\Delta_n(A, \ B) = \frac{1}{n} \sum_{i=1}^{n} d(a_i, \ b_i).$$

Then Δ_n is metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) is complete.

Let $F : X^2 \to X$ and $G : X \to X$ be two mappings. Define the mappings S, $T : X^2 \to X^2$, for all $V = (x, y) \in X^2$, by

$$S(V) = (F(x, \ y), \ F(y, \ x)) \text{ and } T(V) = (Gx, \ Gy).$$

Lemma 4.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose $F : X^2 \to X$, $G : X \to X$ and $S, T : X^2 \to X^2$ are mappings. Then*

(1) If $(X, d, \preceq)$ is regular, then $(X^2, \Delta_2, \sqsubseteq)$ is regular.

(2) If F is $d-$continuous, then S is Δ_2-continuous.

(3) F has the mixed monotone property with respect to $\preceq$ if and only if S is $\sqsubseteq -$non-decreasing.

(4) F has the mixed $G-$monotone property with respect to $\preceq$ if and only if then S is $(T, \sqsubseteq)-$non-decreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $Gx_0 \preceq F(x_0, y_0)$ and $Gy_0 \succeq F(y_0, x_0)$, then there exists a point $V_0 = (x_0, y_0) \in X^2$ such that $T(V_0) \sqsubseteq S(V_0)$.

(6) If $F(X^2) \subseteq G(X)$, then $S(X^2) \subseteq T(X^2)$.

(7) If F and G are commuting in $(X, d, \preceq)$, then S and T are also commuting in $(X^2, \Delta_2, \sqsubseteq)$.

(8) If F and G are compatible in $(X, d, \preceq)$, then S and T are also compatible in $(X^2, \Delta_2, \sqsubseteq)$.

(9) If F and G are weak compatible in $(X, d, \preceq)$, then S and T are also weak compatible in $(X^2, \Delta_2, \sqsubseteq)$.

(10) A point $V = (x, y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of S and T.

(11) A point $V = (x, y) \in X^2$ is a coupled fixed point of F if and only if it is a fixed point of S.

Proof. Items (1) (2), (3), (4), (5), (6), (10) and (11) are obvious.

(7) Let $V = (x, y) \in X^2$ be arbitrary. Since F and G are commuting, by the definition of S and T, we have $ST(V) = S(Gx, Gy) = (F(Gx, Gy), F(Gy, Gx)) = (GF(x, y), GF(y, x)) = T(F(x, y), F(y, x)) = TS(V)$, which shows that S and T are commuting.

(8) Let $\{V_n\}$ be any sequence in X^2 such that $S(V_n) \overset{\Delta_2}{\to} V$ and $T(V_n) \overset{\Delta_2}{\to} V$, where $V_n = (x_n, y_n)$ and $V = (x, y)$. Then

$$(F(x_n, y_n), F(y_n, x_n)) \overset{\Delta_2}{\to} (x, y) \Rightarrow F(x_n, y_n) \overset{d}{\to} x \text{ and } F(y_n, x_n) \overset{d}{\to} y,$$

and

$$(Gx_n, Gy_n) \overset{\Delta_2}{\to} (x, y) \Rightarrow Gx_n \overset{d}{\to} x \text{ and } Gy_n \overset{d}{\to} y.$$

Therefore

$$\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} Gx_n = x \in X,$$
$$\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} Gy_n = y \in X.$$

Since the pair $\{F, G\}$ is compatible, we have

$$\lim_{n \to \infty} d(F(Gx_n, Gy_n), GF(x_n, y_n)) = 0,$$
$$\lim_{n \to \infty} d(F(Gy_n, Gx_n), GF(y_n, x_n)) = 0.$$

In particular,

$$\lim_{n \to \infty} \Delta_2(ST(V_n), TS(V_n))$$
$$= \lim_{n \to \infty} \Delta_2(S(Gx_n, Gy_n), T(F(x_n, y_n), F(y_n, x_n)))$$
$$= \lim_{n \to \infty} \Delta_2((F(Gx_n, Gy_n), F(Gy_n, Gx_n)), (GF(x_n, y_n), GF(y_n, x_n)))$$
$$= \lim_{n \to \infty} \frac{d(F(Gx_n, Gy_n), GF(x_n, y_n)) + d(F(Gy_n, Gx_n), GF(y_n, x_n))}{2}$$
$$= 0.$$

Hence, the mappings S and T are compatible in $(X^2, \Delta_2, \sqsubseteq)$.

(9) Let $V = (x, y) \in X^2$ be a coincidence point S and T. Then $S(V) = T(V)$, that is, $(F(x, y), F(y, x)) = (Gx, Gy)$, that is, $F(x, y) = Gx$ and $F(y, x) = Gy$. Since F and G are weak compatible, by the definition of S and T, we have $ST(V) = S(Gx, Gy) = (F(Gx, Gy), F(Gy, Gx)) = (GF(x, y), GF(y, x)) = T(F(x, y), F(y, x)) = TS(V)$, which shows that S and T commute at their coincidence point, that is, S and T are weak compatible. $\square$

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose $F : X^2 \to X$ and $G : X \to X$ are two mappings such that F has the mixed $G-$monotone property with respect to $\preceq$ on X and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(Gx,\ Gu) + d(Gy,\ Gv)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon,$$

for all $x, y, u, v \in X$ with $Gx \preceq Gu$ and $Gy \succeq Gv$. Suppose that $F(X^2) \subseteq G(X)$, G is continuous and monotone non-decreasing and the pair $\{F, G\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

Suppose that there exist two elements $x_0, y_0 \in X$ with

$$Gx_0 \preceq F(x_0,\ y_0) \text{ and } Gy_0 \succeq F(y_0,\ x_0).$$

Then F and G have a coupled coincidence point. Furthermore, suppose that for every $(x, y), (z, w) \in X^2$, there exists a point $(u, v) \in X^2$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, w), F(w, z))$, and also the pair (F, G) is weakly compatible. Then F and G have a unique common coupled fixed point.

Proof. One can easily obtain that the contractive condition means that, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \Delta_2(T(V),\ T(W)) \leq \varepsilon + \delta(\varepsilon) \Rightarrow \Delta_2(S(V),\ S(W)) < \varepsilon,$$

for all $V = (x, y), W = (u, v) \in X^2$ with $T(V) \sqsubseteq T(W)$. Thus it is only necessary to utilize Theorem 3.1 to the mappings $f = S$ and $g = T$ in the partially ordered metric space $(X^2, \Delta_2, \sqsubseteq)$ taking into account all items of Lemma 4.1. $\square$

Putting $G = I$ (the identity mapping) in Theorem 4.1, we get the following Corollary.

Corollary 4.2. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Suppose $F : X^2 \to X$ has mixed monotone property with respect to $\preceq$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(x,\ u) + d(y,\ v)}{2} \leq \varepsilon + \delta(\varepsilon)$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon,$$

for all $x, y, u, v \in X$, with $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) *F is continuous or*

(b) *$(X, d, \preceq)$ is regular.*

Suppose that there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

5. APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

In this segment, first we obtain the solution of the following first-order periodic problem:

$$(5.1) \qquad \begin{cases} u'(t) = f(t,\ u(t)),\ t \in [0,\ T], \\ \qquad u(0) = u(T), \end{cases}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Take into account of the space $X = C(I, \mathbb{R})$ $(I = [0,\ T])$ of all continuous functions from I to $\mathbb{R}$, which is a regular complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{t \in I} |x(t) - y(t)|,\ \text{ for all } x,\ y \in X,$$

with a partial order, for all $x, y \in X$, given by

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),\ \text{ for all } t \in I.$$

Definition 5.1. A lower solution of (5.1) is a function $\alpha \in C^1(I, \mathbb{R})$ such that

$$\begin{aligned} \alpha'(t) &\leq f(t,\ \alpha(t)) \text{ for } t \in I, \\ \alpha(0) &= \alpha(T) = 0. \end{aligned}$$

Theorem 5.1. *Consider the problem* (5.1) *with continuous function* $f : I \times \mathbb{R} \to \mathbb{R}$ *and suppose there exists* $\lambda > 0$ *such that for* $x, y \in \mathbb{R}$ *with* $x \succeq y$,

$$0 \leq f(t, x) + \lambda x - f(t, y) - \lambda y \leq \frac{\lambda}{2}(x - y).$$

Then the existence of a lower solution of (5.1) *gives us the existence of a solution of* (5.1).

Proof. It is noticeable that, problem (5.1) is equivalent to the following integral equation

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds,$$

where $G(t, s)$ is the Green function given by

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Define the mapping $F : X \to X$ as follows:

$$F(x)(t) = \int_0^T G(t, s)[f(s, x(s)) + \lambda x(s)]ds.$$

If $x_1 \succeq x_2$, then by using our assumption, we have $f(t, x_1) + \lambda x_1 \geq f(t, x_2) + \lambda x_2$. Since $G(t, s) > 0$, for $t \in I$, one can obtain

$$
\begin{aligned}
F(x_1)(t) &= \int_0^T G(t, s)[f(s, x_1(s)) + \lambda x_1(s)]ds \\
&\geq \int_0^T G(t, s)[f(s, x_2(s)) + \lambda x_2(s)]ds \\
&= F(x_2)(t).
\end{aligned}
$$

Consequently F is a non-decreasing mapping. Let $x, y \in X$ with $x \preceq y$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

(5.2) $$\varepsilon \leq d(x, y) \leq \varepsilon + \delta(\varepsilon).$$

Now, by (5.2), we have

$$d(Fx, \ Fy)$$
$$= \ \sup_{t \in I} |F(x)(t) - F(y)(t)|$$
$$= \ \sup_{t \in I} \left| \int_0^T G(t, \ s)[f(s, \ x(s)) + \lambda x(s) - f(s, \ y(s)) - \lambda y(s)]ds \right|$$
$$\leq \ \sup_{t \in I} \left| \int_0^T G(t, \ s) \cdot \frac{\lambda}{2}(x(s) - y(s))ds \right|$$
$$\leq \ \frac{\lambda}{2}d(x, \ y) \sup_{t \in I} \left| \int_0^T G(t, \ s)ds \right|$$
$$\leq \ \frac{\lambda}{2}d(x, \ y) \sup_{t \in I} \left| \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}ds \right|$$
$$\leq \ \frac{1}{2}d(x, \ y)$$
$$\leq \ \frac{1}{2}(\varepsilon + \delta(\varepsilon)) < \varepsilon.$$

Thus the contractive condition of Corollary 3.2 is satisfied. Finally, suppose that $\alpha \in X$ is a lower solution of (5.1), then

$$\alpha'(s) + \lambda\alpha(s) \leq f(s, \ \alpha(s)) + \lambda\alpha(s), \text{ for } t \in I.$$

Multiplying by $G(t, \ s)$ and then integrating, we get

$$\int_0^T \alpha'(s)G(t, \ s)ds + \lambda \int_0^T \alpha(s)G(t, \ s)ds \leq F(\alpha)(t), \text{ for } t \in I.$$

Then, for all $t \in I$, we have

$$\int_0^t \alpha'(s)\frac{e^{(T+s-t)\lambda}}{e^{\lambda T} - 1}ds + \int_t^T \alpha'(s)\frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}ds + \lambda \int_0^T \alpha(s)G(t, \ s)ds \leq F(\alpha)(t).$$

Using integration by parts and $\alpha(0) = \alpha(T) = 0$, we get

$$\alpha(t) \leq F(\alpha)(t) \text{ for all } t \in I.$$

This proves that $\alpha \preceq F(\alpha)$. Thus all the hypothesis of Corollary 3.2 are satisfied. Consequently, F has a fixed point $x \in X$ which is the solution of (5.1) in $X = C(I, \mathbb{R})$. $\hfill\square$

Next, we investigate the solution of the following two-point boundary value problem.

(5.3)
$$\begin{cases} -x''(t) = \theta(t, \ x(t), \ x(t)), \ x \in (0, \ +\infty), \ t \in [0, \ 1], \\ \qquad\qquad x(0) = x(1) = 0. \end{cases}$$

Theorem 5.2. *With the following assumptions*

(a) $\theta : [0, \ 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(b) *Suppose that for all $t \in I$, $x_1 \succeq x_2$ and $y_1 \preceq y_2$,*

$$0 \leq \theta(t, \ x_1, \ y_1) - \theta(t, \ x_2, \ y_2) \leq (x_1 - x_2) + (y_1 - y_2).$$

(c) *There exists $(u, \ v) \in C^2(I, \ \mathbb{R}) \times C^2(I, \ \mathbb{R})$ such that*

$$\begin{cases} -u''(t) \leq \theta(t, \ u(t), \ v(t)), \ t \in [0, \ 1], \\ -v''(t) \geq \theta(t, \ v(t), \ u(t)), \ t \in [0, \ 1], \\ \quad u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

problem (5.3) *has a unique solution in $C^2(I, \ \mathbb{R})$.*

Proof. It is notice that the solution (in $C^2(I, \ \mathbb{R})$) of problem (5.3) is equivalent to the solution (in $C(I, \ \mathbb{R})$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t, \ s)\theta(s, \ x(s), \ x(s))ds \text{ for } t \in [0, \ 1],$$

where $G(t, \ s)$ is the Green function of differential operator $-\frac{d^2}{dt^2}$ with Dirichlet boundary condition $x(0) = x(1) = 0$, that is,

$$G(t, \ s) = \begin{cases} t(1 - s), \ 0 \leq t \leq s \leq 1, \\ s(1 - t), \ 0 \leq s \leq t \leq 1. \end{cases}$$

Define now the mapping $F : X^2 \to X$ by

$$F(x, \ y)(t) = \int_0^1 G(t, \ s)\theta(s, \ x(s), \ y(s))ds, \ t \in [0, \ 1] \text{ and } x, \ y \in X.$$

It follows, from (b), that F has the mixed monotone property with respect to the partial order $\preceq$ in X. Let $x, \ y, \ u, \ v \in X$ with $x \succeq u$ and $y \preceq v$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

(5.4) $$\varepsilon \leq \frac{d(x, \ u) + d(y, \ v)}{2} \leq \varepsilon + \delta(\varepsilon).$$

Now, from (b) and (5.4), we have

$$d(F(x,\ y),\ F(u,\ v))$$

$$=\ \sup_{t\in I}|F(x,\ y)(t) - F(u,\ v)(t)|$$

$$=\ \sup_{t\in I}\int_0^1 G(t,\ s)[\theta(s,\ x(s),\ y(s)) - \theta(s,\ u(s),\ v(s))]ds$$

$$\leq\ \sup_{t\in I}\int_0^1 G(t,\ s)\cdot(x(s) - u(s) + y(s) - v(s))ds$$

$$\leq\ (d(x,\ u) + d(y,\ v))\sup_{t\in I}\int_0^1 G(t,\ s)ds.$$

Thus

$$(5.5)\qquad d(F(x,\ y),\ F(u,\ v)) \leq (d(x,\ u) + d(y,\ v))\sup_{t\in I}\int_0^1 G(t,\ s)ds.$$

It is noticeable that

$$\int_0^1 G(t,\ s)ds = -\frac{t^2}{2} + \frac{t}{2},$$

and that

$$\sup_{t\in[0,\ 1]}\int_0^1 G(t,\ s)ds = \frac{1}{8}.$$

These facts and (5.5) give us

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{1}{8}(d(x,\ u) + d(y,\ v))$$

Similarly, we can obtain

$$d(F(y,\ x),\ F(v,\ u)) \leq \frac{1}{8}(d(x,\ u) + d(y,\ v))$$

By summing up the above two inequalities, dividing by 2 and using (5.4), we get

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2}$$

$$\leq\ \frac{1}{8}\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right)$$

$$\leq\ \frac{1}{8}(\varepsilon + \delta(\varepsilon)) < \varepsilon.$$

Thus the contractive condition of Corollary 4.2 is satisfied. Now, let $(u, v) \in C^2(I, \mathbb{R}) \times C^2(I, \mathbb{R})$ be a solution to (5.3), then

$$-u''(s) \leq \theta(s,\ u(s),\ v(s)),\ s \in [0,\ 1].$$

Multiplying by $G(t, s)$, we get

$$\int_0^1 -u''(s)G(t,\ s)ds \leq F(u,\ v)(t),\ t \in [0,\ 1].$$

Then, for all $t \in [0, 1]$, we have

$$-(1-t)\int_0^t su''(s)ds - t\int_t^1 (1-s)u''(s)ds \le F(u, v)(t).$$

Using integration by parts and $u(0) = u(1) = 0$, for all $t \in [0, 1]$, we get

$$-(1-t)(tu'(t) - u(t)) - t(-(1-t)u'(t) - u(t)) \le F(u, v)(t).$$

Thus, we have

$$u(t) \preceq F(u, v)(t), \text{ for } t \in [0, 1].$$

It follows that $u \preceq F(u, v)$. Similarly, one can show that $v \succeq F(v, u)$. Thus all the hypothesis of Corollary 4.2 are satisfied. Consequently, F has a coupled fixed point $(x, y) \in X^2$ which is the solution of (5.3) in $X = C(I, \mathbb{R})$. $\qquad\square$

Remark 5.1. Using the same criterion, we can obtain tripled, quadruple and in general, multidimensional fixed point theorems from Theorem 3.1.

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