J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.1.113 ISSN(Online) 2287-6081
Volume 29, Number 1 (February 2022), Pages 113–124

ON A CLASS OF ANALYTIC FUNCTION RELATED TO SCHWARZ LEMMA

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Abstract. In this paper, we plan to introduce the class of the analytic functions called $\mathcal{P}(b)$ and to investigate the various properties of the functions belonging this class. The modulus of the second coefficient c_2 in the expansion of $f(z) = z+c_2z^2+...$ belonging to the given class will be estimated from above. Also, we estimate a modulus of the second angular derivative of $f(z)$ function at the boundary point α with $f'(\alpha) = 1 - b$, $b \in \mathbb{C}$, by taking into account their first nonzero two Maclaurin coefficients.

1. Introduction

Let k be an analytic function in the unit disc $U = \{z : |z| < 1\}$, $k(0) = 0$ and $k : U \to U$. In accordance with the classical Schwarz lemma, for any point z in the unit disc U, we have $|k(z)| \leq |z|$ for all $z \in U$ and $|k'(0)| \leq 1$. In addition, if the equality $|k(z)| = |z|$ holds for any $z \neq 0$, or $|k'(0)| = 1$, then k is a rotation; that is $k(z) = ze^{i\theta}$, θ real ([5], p.329). Schwarz lemma has important applications in engineering [12, 13]. In this study, the Shwarz lemma will be presented for the following class $\mathcal{P}(b)$ which will be given.

Let $\mathcal{A}$ denote the class of functions $f(z) = z + \sum\limits_{p=2}^{\infty} c_p z^p$ that are analytic in U. Also, let $\mathcal{P}(b)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\sum_{p=2}^{\infty} p\,|c_p| \leq |b|\,, \tag{1.1}$$

where $0 \neq b \in \mathbb{C}$. In this paper, we study some of the propeties of the clases $\mathcal{P}(b)$. Namely, the modulus of the second coefficient c_2 in the expansion of $f(z) = z + c_2z^2 + ...$ belonging to the given class will be estimated from above. That is, we obtain the bound of c_2.

Received by the editors November 28, 2021. Accepted January 17, 2022.
2010 *Mathematics Subject Classification.* 30C80.
Key words and phrases. analytic function, Schwarz lemma.

Let $f \in \mathcal{P}(b)$ and consider the following function

$$\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1}. \tag{1.2}$$

It is an analytic function in U and $\varphi(0) = 0$. Now, let us show that $|\varphi(z)| < 1$ in U. Now let us check the difference of the modules of the numerator and denominator of the function $\varphi(z)$ given in (1.2). Therefore, we take

$$\left|f'(z) - 1\right| - \left|f'(z) + 2b - 1\right| = \left|\sum_{p=2}^{\infty} pc_p z^{p-1}\right| - \left|2b + c\sum_{p=2}^{\infty} pc_p z^{p-1}\right|$$
$$\leq \sum_{p=2}^{\infty} p\left|c_p\right|\left|z\right|^{p-1} - 2\left|b\right| + \sum_{p=2}^{\infty} p\left|c_p\right|\left|z\right|^{p-1} < 2\sum_{p=2}^{\infty} p\left|c_p\right| - 2\left|b\right|.$$

Since $\sum\limits_{p=2}^{\infty} p\left|c_p\right| \leq |b|$, we obtain

$$\left|f'(z) - 1\right| - \left|f'(z) + 2b - 1\right| \leq 0$$

and

$$\left|\frac{f'(z) - 1}{f'(z) + 2b - 1}\right| < 1.$$

Therefore, from the Schwarz lemma, we obtain

$$\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1} = \frac{2c_2 z + 3c_3 z^2 + ...}{2c_2 z + 3c_3 z^2 + ... + 2b}$$
$$\frac{\varphi(z)}{z} = \frac{2c_2 + 3c_3 z + ...}{2c_2 z + 3c_3 z^2 + ... + 2b}$$
$$\left|\varphi'(0)\right| = \left|\frac{c_2}{b}\right| \leq 1$$

and

$$\left|c_2\right| \leq \left|b\right|.$$

We thus obtain the following lemma.

Lemma 1.1. *If $f \in \mathcal{P}(b)$, then we have the inequality*

$$\left|c_2\right| \leq \left|b\right|. \tag{1.3}$$

Now let us consider the following function by taking into account of the be critical points, which are different from zero, of the function $f(z) - z$,

$$r(z) = \frac{\varphi(z)}{\prod\limits_{i=1}^{n} \frac{z - s_i}{1 - \overline{s_i} z}}.$$

Since $r(z)$ function satisfies the conditions of the Schwarz lemma, we obtain

$$\begin{aligned} r(z) &= \frac{f'(z)-1}{f'(z)+2b-1}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}} \\ &= \frac{2c_2z+3c_3z^2+...}{2c_2z+3c_3z^2+...+2b}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}}, \\ \frac{r(z)}{z} &= \frac{2c_2+3c_3z+...}{2c_2z+3c_3z^2+...+2b}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}}, \end{aligned}$$

$$\left|r'(0)\right| = \frac{|c_2|}{|b|\prod\limits_{i=1}^{n}|s_i|} \leq 1$$

and

$$|c_2| \leq |b|\prod_{i=1}^{n}|s_i|.$$

We thus obtain the following lemma.

Lemma 1.2. *Let $f \in \mathcal{P}(b)$ and $s_1, s_2, ..., s_n$ be critical points of the function $f(z)-z$ in U that are different from zero. Then we have the inequality*

$$|c_2| \leq |b|\prod_{i=1}^{n}|s_i|.$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which are called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows [10, 15]:

Lemma 1.3. *If $k(z)$ extends continuously to some boundary point $\alpha \in \partial U = \{z : |z| = 1\}$ with $|\alpha| = 1$, and if $|k(\alpha)| = 1$ and $k'(\alpha)$ exists, then*

$$\left|k'(\alpha)\right| \geq \frac{2}{1+|k'(0)|} \tag{1.4}$$

and

$$\left|k'(\alpha)\right| \geq 1. \tag{1.5}$$

Moreover, the equality in (1.4) *holds if and only if $k(z) = z\frac{z-a}{1-az}$ for some $a \in (-1, 0]$. Also, the equality in* (1.3) *holds if and only if $k(z) = ze^{i\theta}$.*

Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 2, 3, 4, 6, 7, 8, 9, 10, 11].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [14]).

Lemma 1.4 (Julia-Wolff lemma)**.** *Let k be an analytic function in U, $k(0) = 0$ and $k(U) \subset U$. If, in addition, the function k has an angular limit $k(\alpha)$ at $\alpha \in \partial U$, $|k(\alpha)| = 1$, then the angular derivative $k'(\alpha)$ exists and $1 \leq |k'(\alpha)| \leq \infty$.*

Corollary 1.5. *The analytic function k has a finite angular derivative $k'(\alpha)$ if and only if k' has the finite angular limit $k'(\alpha)$ at $\alpha \in \partial U$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{P}(b)$ class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

Theorem 2.1. *Let $f \in \mathcal{P}(b)$. Assume that, for some $\alpha \in \partial U$, f has an angular limit $f(\alpha)$ at the points α, $f'(\alpha) = 1 - b$. Then we have the inequality*

$$\left|f''(\alpha)\right| \geq \frac{|b|}{2}. \tag{2.1}$$

Proof. Consider the function

$$\varphi(z) = \frac{f'(z) - 1}{f'(z) + 2b - 1}.$$

Also, since $f'(\alpha) = 1 - b$, we have $|\varphi(\alpha)| = 1$. Therefore, from (1.5), we obtain

$$1 \leq \left|\varphi'(\alpha)\right| = \frac{2\left|f''(\alpha)\right||b|}{|b|^2} = \frac{2\left|f''(\alpha)\right|}{|b|}$$

and

$$\left|f''(\alpha)\right| \geq \frac{|b|}{2}.$$

□

The inequality (2.1) can be strengthened from below by taking into account, $c_2 = \frac{f''(0)}{2}$, the first coefficient of the expansion of the function $f(z) = z + c_2z^2 + c_3z^3 + \ldots$.

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|f''(\alpha)\right| \geq \frac{4\left|b\right|^2}{2\left|b\right| + \left|f''(0)\right|}. \tag{2.2}$$

Proof. Let $w(z)$ function be the same as (1.2). So, from (1.4), we obtain

$$\frac{2}{1+\left|\varphi'(0)\right|} \leq \left|\varphi'(\alpha)\right| = \frac{2\left|f''(\alpha)\right|}{\left|b\right|}.$$

Since

$$\left|\varphi'(0)\right| = \left|\frac{c_2}{b}\right|,$$

we take

$$\frac{2}{1+\left|\frac{c_2}{b}\right|} \leq \frac{2\left|f''(\alpha)\right|}{\left|b\right|},$$

$$\left|f''(\alpha)\right| \geq \frac{2\left|b\right|^2}{\left|b\right| + \left|\frac{f''(0)}{2}\right|}$$

and

$$\left|f''(\alpha)\right| \geq \frac{4\left|b\right|^2}{2\left|b\right| + \left|f''(0)\right|}$$

□

The inequality (2.2) can be strengthened as below by taking into account $c_3 = \frac{f'''(0)}{3!}$ which is the coefficient in the expansion of the function $f(z) = z + c_2z^2 + c_3z^3 + ...$

Theorem 2.3. *Let $f \in \mathcal{P}(b)$. Assume that, for some $\alpha \in \partial U$, f has an angular limit $f(\alpha)$ at the points α, $f'(\alpha) = 1 - b$. Then we have the inequality*

$$\left|f''(\alpha)\right| \geq \frac{\left|b\right|}{2}\left(1 + \frac{4\left(\left|b\right| - \left|c_2\right|\right)^2}{2\left(\left|b\right|^2 - \left|c_2\right|^2\right) + \left|3bc_3 - 2c_2^2\right|}\right). \tag{2.3}$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem 2.1 and $h(z) = z$. By the maximum principle, for each $z \in U$, we have the inequality $\left|\varphi(z)\right| \leq \left|h(z)\right|$. So,

$$\begin{aligned} v(z) &= \frac{\varphi(z)}{h(z)} = \frac{1}{z}\left(\frac{f'(z)-1}{f'(z)+2b-1}\right) \\ &= \frac{1}{z}\frac{2c_2z + 3c_3z^2 + ...}{2c_2z + 3c_3z^2 + ... + 2b} \\ &= \frac{2c_2 + 3c_3z + ...}{2c_2z + 3c_3z^2 + ... + 2b} \end{aligned}$$

is analytic function in U and $|v(z)| \leq 1$ for $z \in U$. In particular, we have

$$|v(0)| = \frac{|c_2|}{|b|} \leq 1 \tag{2.4}$$

and

$$\left|v'(0)\right| = \frac{\left|3bc_3 - 2c_2^2\right|}{2\,|b|^2}.$$

The auxiliary function

$$t(z) = \frac{v(z) - v(0)}{1 - \overline{v(0)}v(z)}$$

is analytic in U, $t(0) = 0$, $|t(z)| < 1$ for $|z| < 1$ and $|t(\alpha)| = 1$ for $\alpha \in \partial U$. From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |t'(0)|} &\leq \left|t'(\alpha)\right| = \frac{1 - |v(0)|^2}{\left|1 - \overline{v(0)}v(\alpha)\right|^2}\left|v'(\alpha)\right| \\ &\leq \frac{1 + |v(0)|}{1 - |v(0)|}\left\{\left|\varphi'(\alpha)\right| - \left|h'(\alpha)\right|\right\} \\ &= \frac{|b| + |c_2|}{|b| - |c_2|}\left(\frac{2\,|f''(\alpha)|}{|b|} - 1\right). \end{aligned}$$

Since

$$t'(z) = \frac{1 - |v(0)|^2}{\left(1 - \overline{v(0)}v(z)\right)^2}v'(z)$$

and

$$\left|t'(0)\right| = \frac{|v'(0)|}{1 - |v(0)|^2} = \frac{\frac{\left|3bc_3 - 2c_2^2\right|}{2|b|^2}}{1 - \left(\frac{|c_2|}{|b|}\right)^2} = \frac{\left|3bc_3 - 2c_2^2\right|}{2\left(|b|^2 - |c_2|^2\right)},$$

we obtain

$$\frac{2}{1 + \frac{\left|3bc_3 - 2c_2^2\right|}{2\left(|b|^2 - |c_2|^2\right)}} \leq \frac{|b| + |c_2|}{|b| - |c_2|}\left(\frac{2\,|f''(\alpha)|}{|b|} - 1\right),$$

$$\frac{4\left(|b|^2 - |c_2|^2\right)}{2\left(|b|^2 - |c_2|^2\right) + \left|3bc_3 - 2c_2^2\right|}\frac{|b| - |c_2|}{|b| + |c_2|} \leq \frac{2\,|f''(\alpha)|}{|b|} - 1$$

and

$$\left|f''(\alpha)\right| \geq \frac{|b|}{2}\left(1 + \frac{4\left(|b| - |c_2|\right)^2}{2\left(|b|^2 - |c_2|^2\right) + \left|3bc_3 - 2c_2^2\right|}\right).$$

□

If $f(z)-z$ have critical points different from $z=0$, taking into account these critical points, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.4. *Let $f\in\mathcal{P}(b)$ and $s_1,s_2,...,s_n$ be critical points of the function $f(z)-z$ in D that are different from zero. Assume that, for some $\alpha\in\partial U$, f has an angular limit $f(\alpha)$ at the points α, $f'(\alpha)=1-b$. Then we have the inequality*

$$(2.5)\ \left|f''(\alpha)\right|\geq\frac{|b|}{2}\left(1+\sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}\right.$$
$$\left.+\frac{4\left(|b|\prod\limits_{i=1}^{n}|s_i|-|c_2|\right)^2}{2\left(\left(|b|\prod\limits_{i=1}^{n}|s_i|\right)^2-|c_2|^2\right)+\prod\limits_{i=1}^{n}|s_i|\left|3bc_3-2c_2^2+2bc_2\sum\limits_{i=1}^{n}\frac{1-|s_i|^2}{s_i}\right|}\right).$$

Proof. Let $\varphi(z)$ be as in (1.2) and $s_1,s_2,...,s_n$ be critical points of the function $f(z)-z$ in U that are different from zero. Also, consider the function

$$B(z)=z\prod_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}.$$

By the maximum principle for each $z\in U$, we have

$$|\varphi(z)|\leq|B(z)|.$$

Consider the function

$$\begin{aligned}n(z)&=\frac{\varphi(z)}{B(z)}=\left(\frac{f'(z)-1}{f'(z)+2b-1}\right)\frac{1}{z\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}}\\&=\frac{2c_2z+3c_3z^2+...}{2c_2z+3c_3z^2+...+2b}\frac{1}{z\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}}\\&=\frac{2c_2+3c_3z+...}{2c_2z+3c_3z^2+...+2b}\frac{1}{\prod\limits_{i=1}^{n}\frac{z-s_i}{1-\overline{s_i}z}}.\end{aligned}$$

$n(z)$ is analytic in U and $|n(z)|<1$ for $|z|<1$. In particular, we have

$$|n(0)|=\frac{|c_2|}{|b|\prod\limits_{i=1}^{n}|s_i|}$$

and

$$\left|n'(0)\right| = \frac{\left|3bc_3 - 2c_2^2 + 2bc_2 \sum\limits_{i=1}^{n} \frac{1-|s_i|^2}{s_i}\right|}{2\left|b\right|^2 \prod\limits_{i=1}^{n} |s_i|}.$$

The auxiliary function

$$g(z) = \frac{n(z) - n(0)}{1 - \overline{n(0)}n(z)}$$

is analytic in U, $|g(z)| < 1$ for $|z| < 1$ and $g(0) = 0$. For $\alpha \in \partial U$ and $f'(\alpha) = 1 - b$ we take $|g(\alpha)| = 1$.

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |g'(0)|} &\leq \left|g'(\alpha)\right| = \frac{1 - |n(0)|^2}{\left|1 - \overline{n(0)}n(\alpha)\right|} \left|n'(\alpha)\right| \\ &\leq \frac{1 + |n(0)|}{1 - |n(0)|} \left(\left|\varphi'(\alpha)\right| - \left|B'(\alpha)\right|\right). \end{aligned}$$

It can be seen that

$$\left|g'(0)\right| = \frac{|n'(0)|}{1 - |n(0)|^2}$$

and

$$\begin{aligned} \left|g'(0)\right| &= \frac{\dfrac{\left|3bc_3 - 2c_2^2 + 2bc_2 \sum\limits_{i=1}^{n} \frac{1-|s_i|^2}{s_i}\right|}{2|b|^2 \prod\limits_{i=1}^{n} |s_i|}}{1 - \left(\dfrac{|c_2|}{|b| \prod\limits_{i=1}^{n} |s_i|}\right)^2} \\ &= \prod_{i=1}^{n} |s_i| \frac{\left|3bc_3 - 2c_2^2 + 2bc_2 \sum\limits_{i=1}^{n} \frac{1-|s_i|^2}{s_i}\right|}{2\left(\left(|b| \prod\limits_{i=1}^{n} |s_i|\right)^2 - |c_2|^2\right)}. \end{aligned}$$

Also,we have

$$\left|B'(\alpha)\right| = 1 + \sum_{i=1}^{n} \frac{1 - |s_i|^2}{|1 - s_i|^2}, \ \alpha \in \partial U.$$

Therefore, we obtain

$$\frac{2}{1+\prod_{i=1}^{n}|s_i|\frac{\left|3bc_3-2c_2^2+2bc_2\sum_{i=1}^{n}\frac{1-|s_i|^2}{s_i}\right|}{2\left(\left(|b|\prod_{i=1}^{n}|s_i|\right)^2-|c_2|^2\right)}}$$

$$\leq \frac{|b|\prod_{i=1}^{n}|s_i|+|c_2|}{|b|\prod_{i=1}^{n}|s_i|-|c_2|}\left(\frac{2\left|f''(\alpha)\right|}{|b|}-1-\sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}\right),$$

$$\frac{4\left(\left(|b|\prod_{i=1}^{n}|s_i|\right)^2-|c_2|^2\right)}{2\left(\left(|b|\prod_{i=1}^{n}|s_i|\right)^2-|c_2|^2\right)+\prod_{i=1}^{n}|s_i|\left|3bc_3-2c_2^2+2bc_2\sum_{i=1}^{n}\frac{1-|s_i|^2}{s_i}\right|}$$

$$\leq \frac{|b|\prod_{i=1}^{n}|s_i|+|c_2|}{|b|\prod_{i=1}^{n}|s_i|-|c_2|}\left(\frac{2\left|f''(\alpha)\right|}{|b|}-1-\sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}\right),$$

$$\frac{4\left(|b|\prod_{i=1}^{n}|s_i|-|c_2|\right)^2}{2\left(\left(|b|\prod_{i=1}^{n}|s_i|\right)^2-|c_2|^2\right)+\prod_{i=1}^{n}|s_i|\left|3bc_3-2c_2^2+2bc_2\sum_{i=1}^{n}\frac{1-|s_i|^2}{s_i}\right|}$$

$$\leq \frac{2\left|f''(\alpha)\right|}{|b|}-1-\sum_{i=1}^{n}\frac{1-|s_i|^2}{|1-s_i|^2}$$

and so, we get inequality (2.5). □

If $f(z)-z$ has no critical points different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.5. *Let $f\in\mathcal{P}(b)$, $f(z)-z$ has no critical points in U except $z=0$ and $c_2>0$. Assume that, for some $\alpha\in\partial U$, f has an angular limit $f(\alpha)$ at the points α, $f'(\alpha)=1-b$. Then we have the inequality*

$$\left|f''(\alpha)\right|\geq\frac{|b|}{2}\left(1-\frac{4c_2\,|b|\,c_2\ln^2\left(\frac{c_2}{|b|}\right)}{4c_2\,|b|\ln\left(\frac{c_2}{|b|}\right)-\left|3bc_3-2c_2^2\right|}\right). \tag{2.6}$$

Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z) - z$ has no critical points in U except $z = 0$, we denote by $\ln v(z)$ the analytic branch of the logarithm normed by the condition

$$\ln v(0) = \ln\left(\frac{c_2}{|b|}\right) < 0.$$

The auxiliary function

$$d(z) = \frac{\ln v(z) - \ln v(0)}{\ln v(z) + \ln v(0)}$$

is analytic in the unit disc U, $|d(z)| < 1$, $d(0) = 0$ and $|d(\alpha)| = 1$ for $\alpha \in \partial U$.

From (1.4), we obtain

$$\begin{aligned} \frac{2}{1 + |d'(0)|} &\leq \left|d'(\alpha)\right| = \frac{|2\ln v(0)|}{|\ln v(\alpha) + \ln v(0)|^2}\left|\frac{v'(\alpha)}{v(\alpha)}\right| \\ &= \frac{-2\ln v(0)}{\ln^2 v(0) + \arg^2 v(\alpha)}\left\{\left|\varphi'(\alpha)\right| - 1\right\}. \end{aligned}$$

Replacing $\arg^2 v(\alpha)$ by zero, then

$$\frac{1}{1 - \frac{\frac{\left|3bc_3 - 2c_2^2\right|}{2|b|^2}}{\frac{2c_2}{|b|}\ln\left(\frac{c_2}{|b|}\right)}} \leq \frac{-1}{\ln\left(\frac{c_2}{|b|}\right)}\left\{\frac{2\left|f''(\alpha)\right|}{|b|} - 1\right\}$$

and

$$1 - \frac{4\,|b|\,c_2 \ln^2\left(\frac{c_2}{|b|}\right)}{4\,|b|\,c_2 \ln\left(\frac{c_2}{|b|}\right) - \left|3bc_3 - 2c_2^2\right|} \leq \frac{2\left|f''(\alpha)\right|}{|b|}.$$

Thus, we obtain the inequality (2.6). □

The following theorem shows the relationship between the coefficients c_2 and c_3 in the Maclaurin expansion of the $f(z) = z + c_2 z^2 + c_3 z^3 + ...$ function.

Theorem 2.6. *Let $f \in \mathcal{P}(b)$, $f(z) - z$ has no critical points in U except $z = 0$ and $c_2 > 0$. Then we have the inequality*

$$\left|3bc_3 - 2c_2^2\right| \leq 4\left|bc_2 \ln\left(\frac{c_2}{|b|}\right)\right|.$$

Proof. Let $d(z)$ be the same as in the proof of Theorem 5. Here, $d(z)$ is analytic in the unit disc U, $|d(z)| < 1$, $d(0) = 0$. Therefore, the function $d(z)$ satisfies the assumptions of the Schwarz Lemma. Thus, we obtain

$$\begin{aligned} 1 &\geq \left|d'(0)\right| = \frac{|2\ln v(0)|}{|\ln v(0)+\ln v(0)|^2}\left|\frac{v'(0)}{v(0)}\right| \\ &= \frac{-1}{2\ln v(0)}\left|\frac{v'(0)}{v(0)}\right| \\ &= -\frac{\frac{\left|3bc_3-2c_2^2\right|}{2|b|^2}}{\frac{2c_2}{|b|}\ln\left(\frac{c_2}{|b|}\right)} \end{aligned}$$

and

$$\left|3bc_3-2c_2^2\right| \leq 4\left|bc_2\ln\left(\frac{c_2}{|b|}\right)\right|.$$

□

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