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A-HILBERT SCHEMES FOR $\frac{1}{r}(1^{n-1}, a)$

Seung-Jo Jung

Abstract. For a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$, the G-Hilbert scheme is a fine moduli space of G-clusters, which are 0-dimensional G-invariant subschemes Z with $H^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$. In many cases, the G-Hilbert scheme provides a good resolution of the quotient singularity $\mathbb{C}^n/G$, but in general it can be very singular. In this note, we prove that for a cyclic group $A \subset \mathrm{GL}(n, \mathbb{C})$ of type $\frac{1}{r}(1, \ldots, 1, a)$ with r coprime to a, A-Hilbert Scheme is smooth and irreducible.

1. Introduction

Let G be a finite group in $\mathrm{GL}(n, \mathbb{C})$. A 0-dimensional G-invariant subscheme $Z \subset \mathbb{C}^n$ is called a *G-cluster* if $H^0(\mathcal{O}_Z)$ is isomorphic to the regular representation $\mathbb{C}[G]$ of G as a $\mathbb{C}[G]$-module. Ito–Nakamura[5] introduced the *G-Hilbert scheme* $G\text{-Hilb}\,\mathbb{C}^n$ which is a fine moduli space of G-clusters. Furthermore, they proved that if $G \subset \mathrm{SL}(2, \mathbb{C})$, then $G\text{-Hilb}\,\mathbb{C}^n$ is the minimal resolution of $\mathbb{C}^2/G$. For various cases, $G\text{-Hilb}\,\mathbb{C}^n$ provides a good resolution of the quotient singularity $\mathbb{C}^n/G$. For example, for a finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, the G-Hilbert scheme $G\text{-Hilb}\,\mathbb{C}^3$ is a crepant resolution of $\mathbb{C}^3/G$ proved in [1].

For an abelian group A, in [7] Nakamura introduced the notion of A-graphs corresponding to torus-invariant A-clusters. Using A-graphs, he described Hilb^A which is an irreducible component of A-Hilb. Using his idea, in many cases, A-Hilb can be calculated.

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Let $A_n \subset \mathrm{GL}(n, \mathbb{C})$ be the finite group of type $\frac{1}{r}(1, \ldots, 1, a)$ with r coprime to a. We use the coordinate $x_1, \ldots, x_{n-1}, y$. The thing which makes this group special is that the weights of x_j's are the same. Thus we can deduce the calculation to the case where $n = 2$. In this note, from the well-knwon description of A-Hilb for the group of type $\frac{1}{r}(1, a)$ (e.g. [8]), we calculate A_n-Hilb $\mathbb{C}^n$.

The rest of the paper is organized as follows. Section 2 introduces the toric description due to Nakamura[7]. Section 3 is devoted to description of A-Hilb $\mathbb{C}^n$ for the group of type $\frac{1}{r}(1, \ldots, 1, a)$.

2. A-Hilb via Nakamura's A-graphs

2.1. Toric geometry for cyclic quotient singularities Let $A \subset \mathrm{GL}(n, \mathbb{C})$ be the finite group of type $\frac{1}{r}(a_1, \ldots, a_n)$, i.e. A is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon^{a_1}, \ldots, \epsilon^{a_n})$ where ϵ is a primitive r-th root of unity. Define the lattice

$$L = \mathbb{Z}^n + \mathbb{Z} \cdot \frac{1}{r}(a_1, \ldots, a_n)$$

which is an overlattice of $\overline{L} = \mathbb{Z}^n$ of finite index. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\overline{L} = \mathbb{Z}^n$. Consider the dual lattices $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The embedding of A into the torus $\mathbf{T} := (\mathbb{C}^\times)^n \subset \mathrm{GL}(n, \mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow A^\vee$$

where $A^\vee := \mathrm{Hom}(A, \mathbb{C}^\times)$ is the character group of A. The group A acts on a monomial $x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ by

$$g\colon x^m \mapsto \epsilon^{\rho(g)} x^m,$$

where $\rho = \mathrm{wt}(x^m)$. In this case, we call ρ the *weight of* x^m. As M is the kernel of the map wt, we have that x^m is A-invariant if and only if $m \in M$. Thus the dual lattices $\overline{M}$ and M can be identified with Laurent monomials and A-invariant Laurent monomials, respectively. Furthermore, define

$$\overline{M}_{\geq 0} := \{x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \in M \mid m_i \geq 0 \quad \forall i\}.$$

Let σ_+ be the cone in $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ generated by $e_1, e_2, \ldots, e_n$. Then:

(i) the cone σ_+ with the lattice L defines $X = \mathbb{A}^n/A$ as a toric variety.
(ii) the cone σ_+ with the lattice $\overline{L}$ defines $\mathbb{C}^n$ as a toric variety.
(iii) The quotient map $\mathbb{C}^n \to X$ is induced by the inclusion $\overline{L} \subset L$.

2.2. Nakamura's A-graphs For abelian group cases, Nakamura[7] provided a toric method to construct A-Hilb using *A-graphs*.

Definition 2.1. A set Γ of monomials in $\mathbb{C}[x_1, \dots, x_n]$ is called an *A-graph* if Γ satisfies:

(i) $1 \in \Gamma$.
(ii) The restriction map $\mathrm{wt}\,|_\Gamma \colon \Gamma \to A^\vee$ is bijective, i.e. for each weight $\rho \in A^\vee$, there exists a unique monomial $x^m \in \Gamma$ with $\mathrm{wt}(x^m) = \rho$.
(iii) For a monomial $x^m \in \Gamma$, if x^n divides x^m, then x^n is also in Γ.

For an A-graph Γ, let wt_Γ denote the composition of wt and $(\mathrm{wt}\,|_\Gamma)^{-1}$:

$$\mathrm{wt}_\Gamma := (\mathrm{wt}\,|_\Gamma)^{-1} \circ \mathrm{wt} \colon \overline{M} \to \Gamma,$$

i.e. $\mathrm{wt}_\Gamma(x^m)$ is the unique monomial in Γ whose weight is the same as x^m.

For an A-graph Γ, we can define an A-invariant ideal I_Γ generated by all monomials not in Γ, i.e.

$$I_\Gamma = \langle x^m \mid x^m \notin \Gamma \rangle.$$

This ideal defines an A-cluster $Z(\Gamma)$ whose corresponding ideal I_Γ. As I_Γ is a monomial ideal, Z is $\mathbf{T}$-invariant. In fact, the converse is true: for a $\mathbf{T}$-invariant A-cluster Z, there is an A-graph Γ such that I_Γ is the defining ideal of Z. This means that we have a bijection between the set of $\mathbf{T}$-invariant A-clusters and the set of A-graphs.

2.3. Deformation space $D(\Gamma)$ For an A-graph, we define the *deformation space* $D(\Gamma)$ of I_Z as follows. First, for each weight $\rho \in A^\vee$, $\mathbf{m}_\rho$ denote the unique monomial of weight ρ in Γ. The A-graph Γ fixes the monomial basis of the vector space $H^0(\mathcal{O}_{Z(\Gamma)})$. Let Z be an A-cluster such that $H^0(\mathcal{O}_Z)$ has monomial basis Γ. Giving the A-cluster structure on the vector space $H^0(\mathcal{O}_Z)$ is equivalent to have nr

parameters coming from the $\mathbb{C}[x_1,\dots,x_n]$-action

$$\{\lambda_{j,\rho}\,|\,1\le j\le n,\,\rho\in A^\vee\}$$

with

$$x_j\cdot \mathbf{m}_\rho=\lambda_{j,\rho}\mathbf{m}_{\mathrm{wt}_\Gamma(x_j\cdot\mathbf{m}_\rho)}\quad\text{in } H^0(\mathcal{O}_Z).$$

From $x_ix_{j'}\cdot\mathbf{m}_\rho = x_j\cdot x_{j'}\cdot\mathbf{m}_\rho$, these parameters satisfy "commutative" relations. From this description, we have an affine open set $D(\Gamma)$ in A-Hilb $\mathbb{C}^n$. (For details, see [6]) In general, it is hard to calculate $D(\Gamma)$. But $D(\Gamma)$ has an irreducible component which is described below.

2.3.1. LOCAL CHARTS AND G-GRAPHS For an A-graph Γ, define $S(\Gamma)$ to be the subsemigroup of M generated by $\dfrac{x^n\cdot x^m}{\mathrm{wt}_\Gamma(x^n\cdot x^m)}$ for all $x^n\in\overline{M}_{\ge0}$ and $x^m\in\Gamma$. Define a cone $\sigma(\Gamma)$ in $L_\mathbb{R}=\mathbb{R}^n$ as follows:

$$\begin{aligned}\sigma(\Gamma)&=S(\Gamma)^\vee\\&=\left\{u\in L_\mathbb{R}\mid \left\langle u,\frac{x^n\cdot x^m}{\mathrm{wt}_\Gamma(x^n\cdot x^m)}\right\rangle\ge 0,\quad \forall x^n\in\overline{M}_{\ge0},\quad x^m\in\Gamma\right\}.\end{aligned}$$

Observe that:

(i) $\left(\overline{M}_{\ge0}\cap M\right)\subset S(\Gamma)$,
(ii) $\sigma(\Gamma)\subset\sigma_+$,
(iii) $S(\Gamma)\subset\left(\sigma(\Gamma)^\vee\cap M\right)$.

Note that $S(\Gamma)$ is finitely generated as a semigroup. Thus we can define an affine toric variety associated to the semigroup $S(\Gamma)$. Define two affine toric varieties:

$$\begin{aligned}U(\Gamma)&:=\operatorname{Spec}\mathbb{C}[S(\Gamma)],\\U^\nu(\Gamma)&:=\operatorname{Spec}\mathbb{C}[\sigma^\vee(\Gamma)\cap M].\end{aligned}$$

Note that $U^\nu(\Gamma)$ is the normalization of $U(\Gamma)$ and that the torus $\operatorname{Spec}\mathbb{C}[M]$ of $U(\Gamma)$ is isomorphic to $(\mathbb{C}^\times)^n/A$.

Even though A-Hilb $\mathbb{C}^n$ does not need to be irreducible, A-Hilb $\mathbb{C}^n$ has a unique irreducible component Hilb^A containing the torus $(\mathbb{C}^\times)^n/A$ by Craw–Maclagan–Thomas [3]. The irreducible component Hilb^A is called the *birational component* of A-Hilb $\mathbb{C}^n$.

Theorem 2.2 ([3, 7, 6])**.** *Let A be an abelian group in* $\mathrm{GL}(n, \mathbb{C})$*. Then:*

(i) *The birational component* Hilb^A *is a not-necessarily-normal toric variety birational to* $\mathbb{C}^n/A$*.*
(ii) *The variety* Hilb^A *is covered by* $U(\Gamma)$ *for all A-graphs* Γ*.*

Remark 2.3 ([6])**.** Since each irreducible component of A-Hilb $\mathbb{C}^n$ should have $\mathbf{T}$-invariant points, to prove A-Hilb $\mathbb{C}^n$ is irreducible, it suffices to show the following for each A-graph Γ.

(i) $\sigma(\Gamma)$ is an n-dimensional cone, which implies that $U(\Gamma)$ contains a $\mathbf{T}$-invariant point corresponding to I_Γ.
(ii) The deformation space $D(\Gamma)$ of I_Γ is equal to $U(\Gamma)$.

3. $\frac{1}{r}(1^{n-1}, a)$ Cases

In this section, we calculate A-Hilb for $\frac{1}{r}(1, \ldots, 1, a)$ via finding all A-graphs.

3.1. Hirzebruch–Jung continued fraction We review the Hirzebruch–Jung continued fraction of rational numbers (see e.g. [8]). Let r, a be positive integers such that $a < r$. Assume further that r and a are coprime. An expression $\frac{r}{a} = [a_1, a_2, \ldots, a_k]$ is called the *Hirzebruch-Jung continued fraction* of $\frac{r}{a}$ if

$$\frac{r}{a} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \frac{1}{a_k}}}.$$

Consider the lattice

$$L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, a),$$

which is an overlattice of $\mathbb{Z}^2$ of finite index. Define

$$\begin{aligned} w_0 &= (0, 1), \\ w_1 &= \frac{1}{r}(1, a), \\ w_{i+1} &= a_i w_i - w_{i-1} \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

By construction, we get $w_{k+1} = (1,0)$. Define for $i = 0, 1, \ldots, k+1$, let

$$w_i = \frac{1}{r}(\alpha_i, \beta_i). \tag{3.1}$$

The key picture is that $\{w_i, w_{i+1}\}$ form a $\mathbb{Z}$-basis of L, i.e. the cone

$$\sigma_i := \mathrm{Cone}(w_i, w_{i+1})$$

defines a smooth affine toric variety. Moreover the following holds.

Theorem 3.2 (e.g. [8, 2])**.** *Let Σ be the minimal toric fan containing all σ_i's for $0 \le i \le k$. Then the toric variety corresponding to the fan Σ is the minimal resolution of the quotient singularity of type $\frac{1}{r}(1, a)$.*

Example 3.3. Let G be the group of type $\frac{1}{5}(1,2)$. Then the Hirzebruch–Jung continued fraction of $\frac{5}{2}$ is

$$\frac{5}{2} = [3, 2].$$

Then in the notation above,

$$w_0 = (0,1), \quad w_1 = \frac{1}{5}(1,2), \quad w_2 = \frac{1}{5}(3,1), \quad w_3 = (1,0).$$

Let Σ be the minimal toric fan containing the following three cones:

$$\sigma_0 = \mathrm{Cone}(w_0, w_1), \quad \sigma_1 = \mathrm{Cone}(w_1, w_2), \quad \sigma_2 = \mathrm{Cone}(w_2, w_3).$$

Then the fan Σ defines the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. From toric geometry, the minimal resolution can be covered by three affine toric open sets corresponding to the three 2-dimensional toric cones. For example, the toric cone $\sigma_1 = \mathrm{Cone}(w_1, w_2)$ corresponds to the affine toric variety $\mathrm{Spec}\,\mathbb{C}[\frac{x^2}{y}, \frac{y^3}{x}]$ which is smooth.

3.2. A-graphs for $n = 2$ Let A_2 be the group of type $\frac{1}{r}(1, a)$ with coordinates x, y. By [4], it is well-known that A_2-Hilb $\mathbb{C}^2$ is the minimal resolution of $\mathbb{C}^2/A_2$. Therefore the toric variety in Theorem 3.2 is isomorphic to A_2-Hilb $\mathbb{C}^2$. From this fact, we can find all A_2-graphs.

Proposition 3.4. *For A_2 of type $\frac{1}{r}(1,a)$ with $(r,a)=1$ and $\frac{r}{a}=[a_1,a_2,\dots,a_k]$, the number of A_2-graphs is $(k+1)$. More precisely, for each $0\le i\le k$, the toric cone σ_i corresponds to an A_2-graph Γ_i with*

$$S(\Gamma_i)=\sigma_i^\vee\cap M.$$

Moreover, the A_2-Hilb $\mathbb{C}^2$ is irreducible.

In particular, we have

$$\Gamma_0=\{1,y,\dots,y^{r-1}\},\quad \Gamma_k=\{1,x,\dots,x^{r-1}\}.$$

Remark 3.5. From toric geometry (e.g. [2, 8]), for each $0\le i\le k$, we have $S(\Gamma_i)=\mathbb{C}\left[\frac{x^{\beta_i}}{y^{\alpha_i}},\frac{y^{\alpha_{i+1}}}{x^{\beta_{i+1}}}\right]$. Moreover, the A_2-graph Γ_i contains at least the following monomials

$$1,x,\dots,x^{\beta_i-1},y,\dots,y^{\alpha_{i+1}-1}.$$

Example 3.6. Let G be the group of type $\frac{1}{5}(1,2)$. Since the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ is isomorphic to G-Hilb $\mathbb{C}^2$. Thus the toric fan of G-Hilb $\mathbb{C}^2$ has three 2-dimensional cones:

$$\sigma_0=\mathrm{Cone}(w_0,w_1),\quad \sigma_1=\mathrm{Cone}(w_1,w_2),\quad \sigma_2=\mathrm{Cone}(w_2,w_3).$$

The corresponding G-graphs are

$$\Gamma_0=\{1,y,y^2,y^3,y^4\},$$
$$\Gamma_1=\{1,x,y,xy,y^2\},$$
$$\Gamma_2=\{1,x,x^2,x^3,x^4\}.$$

The $\mathbf{T}$-invariant G-cluster corresponding to Γ_1 is defined by the ideal

$$I_{\Gamma_1}=\langle x^2,xy^2,y^3\rangle.$$

Here the deformation of $Z(I_{\Gamma_1})$ is given by the parameters λ,ν with

$$x^2=\lambda y,\quad y^3=\nu x,\quad xy^2=\lambda\nu.$$

This gives an affine open set $U(\Gamma_1)=\mathrm{Spec}\,\mathbb{C}[\lambda,\nu]$ which is in G-Hilb $\mathbb{C}^2$. For a point $(\lambda,\nu)\in\mathbb{C}^2\simeq U(\Gamma_1)$, it parametrises the G-cluster given by

$$I_{(\lambda,\mu)}:=\langle x^2-\lambda y,y^3-\nu x,xy^2-\lambda\nu\rangle$$

which is a G-invariant ideal of $\mathbb{C}[x,y]$.

3.3. ***A*-graphs for $n \geq 3$** Let A_n be the group of type $\frac{1}{r}(1,\dots,1,a)$ acting on $\mathbb{C}^n$ with $(r,a)=1$. We use the coordinates $x_1,\dots,x_{n-1},y$.

Theorem 3.7. *Let $A_n \subset \mathrm{GL}(n,\mathbb{C})$ be the group of type $\frac{1}{r}(1,\dots,1,a)$ with $(r,a)=1$. If the $\frac{r}{a}=[a_1,a_2,\dots,a_k]$, then the following hold.*

(i) *The number of A_n-graphs is $nk-k+1$.*

(ii) *A_n-$\mathrm{Hilb}\,\mathbb{C}^n$ is smooth and irreducible. Thus A_n-$\mathrm{Hilb}\,\mathbb{C}^n$ is a resolution of the quotient singularity $\mathbb{C}^n/A_n$.*

Proof. First consider the A_n-graph Γ^0 containing only powers of y. This means Γ^0 does not contain any other x_j's. Then Γ^0 should be

$$\Gamma^0=\{1,y,\dots,y^{r-1}\}.$$

For this A_n-graph, we have

$$S(\Gamma^0)=\mathbb{C}\left[\frac{x_1}{y},\dots,\frac{x_{n-1}}{y},y^r\right] \quad \text{and} \quad U(\Gamma^0)\simeq\mathbb{C}^n.$$

Suppose that Γ is an A_n-graph containing x_j for some $1\leq j\leq n-1$. Since x_l's are of the same weight, Γ cannot contain any x_l with $j\neq l$. Thus Γ consists of monomials in x_j and y. Thus Γ can be seen as an A_2-graph with considering the coordinate x_j,y. Thus Γ should correspond to one of A_2-graphs in Proposition 3.4, say Γ_i for some $1\leq i\leq k$. Since $S(\Gamma_i)=\mathbb{C}\left[\frac{x^{\beta_i}}{y^{\alpha_i}},\frac{y^{\alpha_{i+1}}}{x^{\beta_{i+1}}}\right]$, we have

$$S(\Gamma)=\mathbb{C}\left[\frac{x_1}{x_j},\dots,\frac{x_{n-1}}{x_j},\frac{x_j^{\beta_i}}{y^{\alpha_i}},\frac{y^{\alpha_{i+1}}}{x_j^{\beta_{i+1}}}\right] \quad \text{and} \quad U(\Gamma)\simeq\mathbb{C}^n.$$

Therefore, there are $(nk-k+1)$ A_n-graphs which yield a smooth affine toric open cover of $\mathrm{Hilb}^{A_n}\mathbb{C}^n$. Note that for each A_n-graph Γ, the toric cone $\sigma(\Gamma)$ is a smooth n-dimensional cone.

To complete the proof, it remains to show the irreducibility of A_n-$\mathrm{Hilb}\,\mathbb{C}^n$. First note that A_2-$\mathrm{Hilb}\,\mathbb{C}^2$ is irreducible. From Remark 2.3, it is sufficient to show the deformation space $D(\Gamma)$ is equal to $U(\Gamma)$ for each Γ. For the A_n-graph Γ^0, it is straightforward to see $D(\Gamma^0)=U(\Gamma^0)$.

Let Γ be an A_n-graph containing x_j for some $1 \le j \le n-1$, which corresponds to an A_2-graph Γ_i. Note that the action of x_j and y is already given by that of x encoded in the A_2-graph Γ_i. So the remaining deformation parameters are the parameters corresponding to the action of x_l for $l \ne j$, which are denoted by

$$\{\lambda_{l,\rho} \,|\, l \ne j, \rho \in A_n^\vee\}$$

in Section 2.3. Note that the weight of x_l is equal to x_j which is denoted by ρ_1. For the trivial character $\rho_0 \in A_n^\vee$, the parameters λ_{l,ρ_0} are given by

$$(*) \qquad x_l \cdot 1 = \lambda_{l,\rho_0}\, x_j.$$

With parameters $\{\lambda_{j,\rho} \,|\, \rho \in A_n^\vee\}$, these parameters λ_{l,ρ_0} determine other parameters as follows: Let $\rho \in A_n^\vee$. There exists a unique monomial $\mathbf{m}_\rho \in \Gamma$ of weight ρ. The parameter $\lambda_{l,\rho}$ is given by $x_l \cdot \mathbf{m}_\rho = \lambda_{l,\rho}\, \mathbf{m}_{\rho\rho_1}$. Here using Equation (*), we have

$$x_l \cdot \mathbf{m}_\rho = \lambda_{l,\rho_0}\, x_j \cdot \mathbf{m}_\rho = \lambda_{l,\rho_0}\lambda_{j,\rho}\, \mathbf{m}_{\rho\rho_1}.$$

This induces $\lambda_{l,\rho} = \lambda_{l,\rho_0}\lambda_{j,\rho}$ because $\mathbf{m}_{\rho\rho_1}$ is a base of the vector space $H^0(\mathcal{O}_{Z(\Gamma)})$. This implies that from 2-dimensional cases,

$$D(\Gamma) \simeq D(\Gamma_i) \times \mathbb{C}^{n-2} \simeq \mathbb{C}^n.$$

This implies that A_n-Hilb $\mathbb{C}^n$ is irreducible. □

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DEPARTMENT OF MATHEMATICS EDUCATION, INSTITUTE OF PURE AND APPLIED MATHEMATICS, JEONBUK NATIONAL UNIVERSITY, JEONJU 54896, KOREA
Email address: seungjo@jbnu.ac.kr