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SOME RESULTS ON UNIQUENESS OF CERTAIN TYPE OF SHIFT POLYNOMIALS SHARING A SMALL FUNCTION

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Abstract. The purpose of the paper is to study the uniqueness problems of certain type of difference polynomials sharing a small function. With the concept of weakly weighted sharing and relaxed weighted sharing we obtain some results which extend and generalize some results due to P. Sahoo and G. Biswas [Tamkang Journal of Mathematics, 49(2)(2018), 85-97].

1. Introduction

By a meromorphic function we shall always mean a meromorphic function in the complex plane. We assume that the reader is familiar with the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [7, 8, 20]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$, possibly outside of a set of finite linear measure. We say that the meromorphic function $\alpha(z)$ is a small function of f, if $T(r, \alpha(z)) = S(r,f)$.

Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let f and g be two nonconstant meromorphic functions. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM (counting multiplicities). On the other hand, if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM (ignoring multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the

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set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}(r,a;f)$ the counting function of zeros of $f-a$ with multiplicity less or equal to k, and by $\overline{N}_{k)}(r,a;f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,a;f)$ be the counting function of zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,a;f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + ... + \overline{N}_{(k}(r,a;f).$$

Let $N_E(r,a;f,g)$ ($\overline{N}_E(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "IM".

We now explain in the following definition the notion of weakly weighted sharing which was introduced by Lin and Lin [10].

Definition 1.1 ([10])**.** Let f and g share a "IM" and k be a positive integer or ∞. $\overline{N}^E_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^0_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.

Definition 1.2 ([10])**.** Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. If

$$\overline{N}_{k)}(r,a;f) - \overline{N}^E_{k)}(r,a;f,g) = S(r,f),$$
$$\overline{N}_{k)}(r,a;g) - \overline{N}^E_{k)}(r,a;f,g) = S(r,g),$$
$$\overline{N}_{(k+1}(r,a;f) - \overline{N}^0_{(k+1}(r,a;f,g) = S(r,f),$$
$$\overline{N}_{(k+1}(r,a;g) - \overline{N}^0_{(k+1}(r,a;f,g) = S(r,g),$$

or if $k=0$ and

$$\overline{N}(r,a;f) - \overline{N}_0(r,a;f,g) = S(r,f),$$
$$\overline{N}(r,a;g) - \overline{N}_0(r,a;f,g) = S(r,g),$$

then we say f and g weakly share a with weight k and we write f and g share "(a, k)" to mean that f, g weakly share a with weight k.

Now it is clear from definition 1.2 that weakly weighted sharing is a scaling between IM and CM.

In 2007, A. Banerjee and S. Mukherjee [2] introduced a new type of sharing which is weaker than weakly weighted sharing and is defined as follows.

Definition 1.3 ([2])**.** We denote by $\overline{N}(r, a; f \mid= p; \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 1.4 ([2])**.** Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. Suppose that f and g share a "IM". If for $p \neq q$,

$$\sum_{p,q \leq k} \overline{N}(r, a; f \mid= p; g \mid= q) = S(r),$$

then we say that f and g *share* a with weight k in a relaxed manner and in that case we write f and g share $(a, k)^*$.

Recently, the topic of difference equation and difference product in the complex plane $\mathbb{C}$ has attracted many mathematicians, a large number of papers have focused on value distribution of differences and differences operator analogues of Nevanlinna theory (see [4, 5, 6, 9, 15]) and many people paid their attention to the uniqueness of differences and difference polynomials of meromorphic function and obtained many interesting results. K. Liu and L.Z. Yang [12] also considered the zeros of $f^n(z)f(z+c) - p(z)$ and $f^n \Delta_c f$, where $p(z)$ is a nonzero polynomial and obtain the following theorem. In this direction J.L. Zhang [21] considered the zeros of certain type of difference polynomials and proved the following result for small functions.

Theorem A. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, $\alpha(z)$ ($\not\equiv 0$) be a small function with respect to $f(z)$ and c be a nonzero complex constant. If $n \geq 2$ is an integer then $f^n(z)(f(z) - 1)f(z + c) - \alpha(z)$ has infinitely many zeros.*

In the same paper, Zhang also proved the following uniqueness result.

Theorem B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z) - 1)g(z + c)$ share $\alpha(z)$ CM, then $f(z) = g(z)$.*

In 2014, using the idea of weakly weighted sharing and relaxed weighted sharing C. Meng [14] obtained the following uniqueness theorems which improve and supplement Theorem B in different directions.

Theorem C. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n\geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z),2)$", then $f(z)\equiv g(z)$.*

Theorem D. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n\geq 10$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $(\alpha(z),2)^*$, then $f(z)\equiv g(z)$.*

Theorem E. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n\geq 16$ is an integer. If*

$$\overline{E}_{2)}\bigg(\alpha(z),f^n(z)(f(z)-1)f(z+c)\bigg)=\overline{E}_{2)}\bigg(\alpha(z),g^n(z)(g(z)-1)g(z+c)\bigg),$$

then $f(z)\equiv g(z)$.

In 2015, P. Sahoo [17] studied the uniqueness problem of difference polynomials of the form $f^n(z)(f^m(z)-1)f(z+c)$ and proved the following results which generalize Theorems C-E.

Theorem F. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n\geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share "$(\alpha(z),2)$", then $f(z)\equiv tg(z)$ where $t^m=1$.*

Theorem G. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant, n and $m(\geq 1)$ are integers such that $n\geq 2m+8$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $(\alpha(z),2)^*$, then $f(z)\equiv tg(z)$ where $t^m=1$.*

Theorem H. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$.*

Suppose that c *is a nonzero complex constant,* n *and* $m(\geq 1)$ *are integers such that* $n \geq 4m + 12$*. If*

$$\overline{E}_{2)}\Big(\alpha(z), f^n(z)(f^m(z)-1)f(z+c)\Big) = \overline{E}_{2)}\Big(\alpha(z), g^n(z)(g^m(z)-1)g(z+c)\Big),$$

then $f(z) \equiv tg(z)$ *where* $t^m = 1$.

Regarding Theorems F-H, one may ask the following question.

Question 1.1. What can be said about the relationship between two entire functions f and g if one replace $f^n(z)(f^m(z)-1)f(z+c)$ by $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ in Theorem F-G?

In 2018 P. Sahoo [19] answered the above question and proved the following results which generalize Theorems F-H.

Theorem I. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to both* $f(z)$ *and* $g(z)$*. Suppose that* c *is a nonzero complex constant,* $n, k(\geq 0)$ *and* $m(\geq 1)$ *are integers such that* $n \geq 2k + m + 6$*. If* $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ *and* $(g^n(z)(g^m(z)-1)g(z+c))^{(k)}$ *share "*$(\alpha(z), 2)$*", then* $f(z) \equiv tg(z)$ *where* $t^m = 1$.

Theorem J. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to both* $f(z)$ *and* $g(z)$*. Suppose that* c *is a nonzero complex constant,* $n, k(\geq 0)$ *and* $m(\geq 1)$ *are integers such that* $n \geq 3k + 2m + 8$*. If* $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ *and* $(g^n(z)(g^m(z)-1)g(z+c))^{(k)}$ *share* $(\alpha(z), 2)^*$*, then* $f(z) \equiv tg(z)$ *where* $t^m = 1$.

Theorem K. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0, \infty)$ *be a small function with respect to both* $f(z)$ *and* $g(z)$*. Suppose that* c *is a nonzero complex constant,* $n, k(\geq 0)$ *and* $m(\geq 1)$ *are integers such that* $n \geq 5k + 4m + 12$*. If*

$$\overline{E}_{2)}\Big(\alpha(z), (f^n(z)(f^m(z)-1)f(z+c))^{(k)}\Big) = \overline{E}_{2)}\Big(\alpha(z), (g^n(z)(g^m(z)-1)g(z+c))^{(k)}\Big),$$

then $f(z) \equiv tg(z)$ *where* $t^m = 1$.

Regarding the results of P. Sahoo stated above it is natural to ask the following question which is the motive of the present paper.

Question 1.2. What can be said about the relationship between two entire functions $f(z)$ and $g(z)$ if one replace the difference polynomial $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ by $(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ in Theorems I-K, where $f(z)$ is a transcendental entire function of finite order, $c_j (j=1,2,...,s)$, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, s and $\mu_j (j=1,2,...,s)$ are integer?

For the sake of simplicity we also use the notation $\sigma = \sum_{j=1}^{s} \mu_j$.

In the paper, our main concern is to find the possible answer of the above question. We prove following theorems which extend and generalize Theorems I-K. The following theorems are the main results of the paper.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, c_j $(j=1,2,...,s)$ be finite complex constants and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq \max\{2k+m+\sigma+5, \sigma+2s+3\}$. If $(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$ share "$(\alpha,2)$", then $f(z)\equiv tg(z)$ for some constant t such that $t^{n+\sigma}=t^m=1$.*

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, c_j $(j=1,2,...,s)$ be finite complex constants and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq \max\{3k+2m+2\sigma+6, \sigma+2s+3\}$. If $(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$ share $(\alpha,2)^*$, then the conclusions of theorem 1.1 hold.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, c_j $(j=1,2,...,s)$ be finite complex constants and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq \max\{5k+4m+4\sigma+8, \sigma+2s+3\}$. If $\overline{E}_{2)}\left(\alpha(z), \left(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)}\right) = \overline{E}_{2)}\left(\alpha(z), \left(g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)}\right)$, then the conclusions of Theorem 1.1 hold.*

2. Preliminaries

Let F and G be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by H the function as follows:

$$H=\left(\frac{F''}{G'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Lemma 2.1 ([4])**.** *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$T(r,f(z+c))=T(r,f)+S(r,f).$$

Lemma 2.2 ([13])**.** *Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then*

$$\begin{aligned}N(r,0;f(z+c))&\leq N(r,0;f)+S(r,f),\\ N(r,\infty;f(z+c))&\leq N(r,\infty;f)+S(r,f),\\ \overline{N}(r,0;f(z+c))&\leq \overline{N}(r,0;f)+S(r,f),\\ \overline{N}(r,\infty;f(z+c))&\leq \overline{N}(r,\infty;f)+S(r,f),\end{aligned}$$

outside of possible exceptional set with finite logarithmic measure.

Lemma 2.3 ([3])**.** *Let f be an entire function of finite order and*

$$F=f^n(z)(f^m(z)-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j}.$$

Then

$$T(r,F)=(n+m+\sigma)T(r,f)+S(r,f).$$

Lemma 2.4 ([22])**.** *Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f). \tag{2.1}$$

and

$$N_p\left(r,0;f^{(k)}\right)\leq k\overline{N}(r,\infty;f)+N_{p+k}(r,0;f)+S(r,f). \tag{2.2}$$

Lemma 2.5 ([2])**.** *Let F and G be two nonconstant meromorphic functions that share "$(1,2)$" and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G) \\ &\quad -\sum_{p=3}^{\infty}\overline{N}\big(r,0;\frac{G'}{G}\mid\geq p\big)+S(r,F)+S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.6 ([2])**.** *Let F and G be two nonconstant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G) \\ &\quad +\overline{N}(r,0;F)+\overline{N}(r,\infty;F)-m(r,1;G) \\ &\quad +S(r,F)+S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.7 ([11])**.** *Let F and G be two nonconstant entire functions, and $p \geq 2$ an integer. If $\overline{E}_{p)}(1,F)=\overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \leq N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G),$$

and the same inequality is true for $T(r,G)$.

Lemma 2.8 ([18])**.** *Let f and g be two entire functions and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$, be integers, and let $F=(f^n(z)(f(z)-1)^m\prod_{j=1}^{s}f(z+c_j)^{\mu_j})^{(k)}$, $G=(g^n(z)(g(z)-1)^m\prod_{j=1}^{s}g(z+c_j)^{\mu_j})^{(k)}$. If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r;0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n \leq 2k+m+\sigma+2$.*

Lemma 2.9. *Let $f(z)$, $g(z)$ be two transcendental entire functions of finite order and $c_j (j=1,2,...,s)$ be finite complex constants. Let $m(\geq 1)$ and $n(\geq 1)$ be integers such that $n \geq \sigma+2s+3$. If*

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{s}f(z+c_j)^{\mu_j} \equiv g^n(z)(g^m(z)-1)\prod_{j=1}^{s}g(z+c_j)^{\mu_j},$$

then $f(z) \equiv tg(z)$ for some constant t such that $t^m=t^{n+\sigma}=1$.

Proof. Proof of Lemma follows from [1, Lemma 12]. □

3. Proofs of the Main Theorems

Proof of Theorem 1.1. Let $F = \frac{F_1^k}{\alpha(z)}$ and $G = \frac{G_1^k}{\alpha(z)}$ where $F_1 = f^n(z)(f^m(z) - 1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$, $G_1 = g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}$. Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. If possible we may assume that $H \not\equiv 0$. Using (2.1) and Lemma 2.3 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ &\leq T(r,(F_1)^{(k)}) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

From this we get

$$(n+m+\sigma)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f). \tag{3.1}$$

Again by (2.2) we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{3.2}$$

Similarly

$$N_2(r,0;G) \leq N_{k+2}(r,0;G_1) + S(r,g). \tag{3.3}$$

Using (3.2), (3.3) and Lemma 2.5 we obtain from (3.1)

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) \\ &\quad +S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (k+m+\sigma+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{3.4}$$

Similarly,

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+2)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.5}$$

(3.4) and (3.5) together give

$$(n-2k-m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the assumption that

$$n \geq 2k+m+\sigma+5.$$

Thus, we must have $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.6}$$

where $A(\neq 0)$ and B are constants. From (3.6) it is obvious that F, G share the value 1 CM and hence they share "$(1,2)$". Therefore $n \geq 2k + m + \sigma + 5$. We now discuss the following three cases separately.

Case 1. Suppose that $B \neq 0$ and $A = B$. Then from (3.6) we obtain

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.7}$$

If $B = -1$, then from (3.7) we obtain $FG = 1$. Then

$$\left(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)} \left(g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right)^{(k)} = \alpha^2.$$

Since the number of zeros of $\alpha(z)$ is finite, it follows that f as well as g has finitely many zeros. We put $f(z) = h(z)e^{\beta(z)}$, where $h(z)$ is a nonzero polynomial and $\beta(z)$ is a nonconstant polynomial. Now replacing $\sum_{j=1}^{s} \mu_j \beta(z+c_j)$ by $\gamma(z)$ and $\prod_{j=1}^{s} h(z+c_j)^{\mu_j}$ by $\nu(z)$ we deduce that

$$\begin{aligned}
& \left(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right)^{(k)} \\
= \; & \left(h^n(z)e^{n\beta(z)}(h^m(z)e^{m\beta(z)}-1)\prod_{j=1}^{s} h(z+c_j)^{\mu_j} e^{\mu_j\beta(z+c_j)}\right)^{(k)} \\
= \; & \left(h^n(z)\nu(z)e^{n\beta(z)+\gamma(z)}(h^m(z)e^{m\beta(z)}-1)\right)^{(k)} \\
= \; & \left(h^{n+m}(z)\nu(z)e^{(n+m)\beta(z)+\gamma(z)} - h^n(z)\nu(z)e^{n\beta(z)+\gamma(z)}\right)^{(k)} \\
= \; & e^{(n+m)\beta(z)+\gamma(z)} P_1(\beta(z),\gamma(z),h(z),\nu(z),...,\beta^{(k)}(z),\gamma^{(k)}(z),h^{(k)}(z),\nu^{(k)}(z)) \\
& - e^{n\beta(z)+\gamma(z)} P_2(\beta(z),\gamma(z),h(z),\nu(z),...,\beta^{(k)}(z),\gamma^{(k)}(z),h^{(k)}(z),\nu^{(k)}(z)) \\
= \; & e^{n\beta(z)+\gamma(z)}(P_1 e^{m\beta(z)} - P_2).
\end{aligned}$$

Obviously $P_1 e^{m\beta(z)} - P_2$ has infinite number of zeros, which contradicts with the fact that g is an entire function.

If $B \neq -1$, from (3.7), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Using (2.1), (2.2) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{B+1}; G\right) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) \\ &\quad -(n+m+\sigma)T(r,g) + S(r,g). \end{aligned}$$

This gives

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n-2k-m-\sigma-2)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n \geq 2k+m+\sigma+5$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (3.6) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, 0; F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (3.6) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, it follows that $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. Now by Lemma 2.8, it can be shown that $n \leq 2k+m+\sigma+2$, which is a contradiction. Thus $A = 1$ and then $F = G$. Then

$$(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k)} = (g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k)}$$

Integrating once we obtain

$$(f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j})^{(k-1)} = (g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j})^{(k-1)} + c_{k-1}$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using lemma 2.8, it follows that $n \leq 2k+m+\sigma+2$, a contradiction. Hence $c_{k-1} = 0$. Repeating the process k-times, we deduce that

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} \equiv g^n(z)(g^m(z)-1)\prod_{j=1}^{s} g(z+c_j)^{\mu_j},$$

which by Lemma 2.9 gives $f(z) \equiv tg(z)$ for some constant t such that $t^m = t^{n+s} = 1$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 1.1. Then F and G are transcendental meromorphic functions that share $(\alpha,2)^*$ except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H\not\equiv 0$. Using (2.2) for $p=1$, (3.3) and Lemmas 2.1 and 2.6 we obtain from (3.1)

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\overline{N}(r,0;F)\\ &\quad +\overline{N}(r,\infty;F)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r;0;G_1)+N_{k+1}(r,0;F_1)\\ &\quad +S(r,f)+S(r,g)\\ &\leq (2k+2m+2\sigma+3)T(r,f)+(k+m+\sigma+2)T(r,g)\\ &\quad +S(r,f)+S(r,g) \qquad (3.8)\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}(n+m+\sigma)T(r,g) &\leq (2k+2m+2\sigma+3)T(r,g)+(k+m+\sigma+2)T(r,f)\\ &\quad +S(r,f)+S(r,g). \qquad (3.9)\end{aligned}$$

(3.8) and (3.9) together give

$$(n-3k-2m-2\sigma-5)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g).$$

contradicting with the fact that $n\geq 3k+2m+2\sigma+6$. Thus we must have $H\equiv 0$. Then the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 1.1. Then F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1,F)=\overline{E}_{2)}(1,G)$ except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H\not\equiv 0$. Using (2.2), (3.3) and Lemmas 2.1 and 2.7 we obtain from (3.1)

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &\quad +N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\ &\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\ &\quad +N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\ &\leq (3k+3m+3\sigma+4)T(r,f)+(2k+2m+2\sigma+3)T(r,g)\\ &\quad +S(r,f)+S(r,g) \qquad (3.10)\end{aligned}$$

In a similar manner we obtain

$$(n+m+\sigma)T(r,g) \leq (3k+3m+3\sigma+4)T(r,g)+(2k+2m+2\sigma+3)T(r,f) + S(r,f)+S(r,g). \tag{3.11}$$

(3.10) and (3.11) together give

$$(n-5k-4m-4\sigma-7)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

contradicting with the fact that $n \geq 5k+4m+4\sigma+8$. Thus we must have $H \equiv 0$. Then the result follows from the proof of Theorem 1.1. This completes the proof of Theorem 1.3. □

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