J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.1.31 ISSN(Online) 2287-6081
Volume 29, Number 1 (February 2022), Pages 31–35

A SUM OF AN ALTERNATING SERIES INVOLVING CENTRAL BINOMIAL NUMBERS AND ITS THREE PROOFS

Yue-Wu Li [a] and Feng Qi [b,c,*]

Dedicated to retired Professor Bo-Yan Xi at Inner Mongolia University for Nationalities, China

Abstract. In the note, by virtue of Abel's theorem and Abel's limit theorem in the theory of power series, the author provides three proofs for a sum of an alternating series involving central binomial numbers.

1. An Alternation Series involving Central Binomial Numbers

On 29 June 2021, Vuk Stojiljkovic (University of Novi Sad, Serbia) asked on the ResearchGate of the alternating series

$$\sum_{k=1}^{\infty}\frac{(-1)^k\binom{2k}{k}}{2^{2k}k}. \tag{1}$$

In this note, we give an answer to the sum of the alternating series (1).

Theorem 1. *The sum*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k}=2\ln\bigl[2\bigl(\sqrt{2}-1\bigr)\bigr] \tag{2}$$

is valid.

We will provide three proofs of Theorem 1 in next section.

2. Three Proofs for the Sum of an Alternation Series

In this section, we provide three proofs for the sum in (2).

Received by the editors July 06, 2021. Accepted August 23, 2021.
2020 *Mathematics Subject Classification.* Primary 41A58; Secondary 11B65, 26A09.
Key words and phrases. Abel's theorem, Abel's limit theorem, sum, alternating series, central binomial number, arcsine, series expansion, binomial expansion, proof.
*Corresponding author.

First Proof. It is common knowledge [1, p. 81, 4.4.40] that the arcsine function $\arcsin z$ has the series expansion

$$\arcsin z = z + \sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

This series expansion can be reformulated [4, p. 58, (1.1)] as

$$\arcsin z = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \binom{2k}{k} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

Differentiating with respect to z, we obtain

$$\frac{1}{\sqrt{1-z^2}} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \binom{2k}{k} z^{2k} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{z}{2}\right)^{2k}, \quad |z| < 1.$$

Let $\left(\frac{z}{2}\right)^2 = -w$, that is, $z^2 = -4w$. Then

$$\frac{1}{\sqrt{1+4w}} = \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} w^k, \quad |w| < \frac{1}{4}. \tag{3}$$

We can rewrite this series expansion in the form

$$\frac{1}{w}\left(\frac{1}{\sqrt{1+4w}} - 1\right) = \sum_{k=1}^{\infty} (-1)^k \binom{2k}{k} w^{k-1}, \quad 0 < |w| < \frac{1}{4}. \tag{4}$$

Integrating on both sides with respect to $w \in (0, x] \subseteq \left(0, \frac{1}{4}\right)$, we acquire

$$\begin{aligned}
\int_0^x \frac{1}{w}\left(\frac{1}{\sqrt{1+4w}} - 1\right) \mathrm{d}\, w &= \int_0^x \frac{-4}{\sqrt{1+4w}\left(1+\sqrt{1+4w}\,\right)} \mathrm{d}\, w \\
&= -2 \int_0^x \frac{\mathrm{d} \ln\left(1+\sqrt{1+4w}\,\right)}{\mathrm{d}\, w} \mathrm{d}\, w \\
&= 2\left[\ln 2 - \ln\left(1+\sqrt{1+4x}\,\right)\right] \\
&= \sum_{k=1}^{\infty} (-1)^k \binom{2k}{k} \int_0^x w^{k-1} \mathrm{d}\, w \\
&= \sum_{k=1}^{\infty} (-1)^k \binom{2k}{k} \frac{x^k}{k}
\end{aligned}$$

for $0 < x < \frac{1}{4}$. Accordingly, by Abel's theorem [2, p. 234, Theorem 9.20], it follows that

$$\sum_{k=1}^{\infty} (-1)^k \binom{2k}{k} \frac{x^k}{k} = \begin{cases} 0, & x = 0 \\ 2 \ln \dfrac{\sqrt{1+4x}\, - 1}{2x}, & x \neq 0 \end{cases}$$

for $|x| < \frac{1}{4}$. By virtue of Abel's limit theorem in [2, p. 245, Theorem 9.31], taking $x = \frac{1}{4}$, we arrive at

$$\sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\frac{(1/4)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k} = 2\ln\bigl[2\bigl(\sqrt{2}-1\bigr)\bigr].$$

In conclusion, we obtain the sum (2). The first proof is complete. □

Second Proof. It is easy to see that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{1}{2^{2k}}\binom{2k}{k} = \sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\frac{1}{k}\left(\frac{1}{4}\right)^k.$$

Let

$$f(x) = \sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\frac{x^k}{k}.$$

Then

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}x^{k-1} \\ &= \frac{1}{x}\sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}x^k \\ &= \frac{1}{x}\left[\sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}x^k - 1\right] \\ &= \begin{cases} -2, & x = 0 \\ \dfrac{1}{x}\left(\dfrac{1}{\sqrt{4x+1}} - 1\right), & x \ne 0 \end{cases} \end{aligned}$$

for $0 < |x| < \frac{1}{4}$, where we used the series expansion (3). Integrating in $x \in (0, t] \subseteq \bigl(0, \frac{1}{4}\bigr)$ on both sides, we find

$$\begin{aligned} \int_0^t f'(x)\,\mathrm{d}\,x &= f(t) - f(0) \\ &= \int_0^t \frac{1}{x}\left(\frac{1}{\sqrt{4x+1}} - 1\right)\mathrm{d}\,x \\ &= \begin{cases} 0, & t = 0; \\ 2\ln\dfrac{\sqrt{4t+1}-1}{2t}, & t \ne 0. \end{cases} \end{aligned}$$

As a result, we acquire

$$f(t)=\sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\frac{x^k}{k}=\begin{cases}0, & t=0\\ 2\ln\dfrac{\sqrt{4t+1}-1}{2t}, & t\ne0\end{cases}$$

for $|t|<\frac{1}{4}$. By virtue of Abel's limit theorem in [2, p. 245, Theorem 9.31], letting $t=\frac{1}{4}$, we discover

$$f\left(\frac{1}{4}\right)=\sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\frac{(1/4)^k}{k}=\sum_{k=1}^{\infty}\frac{(-1)^k}{2^{2k}k}\binom{2k}{k}=2\ln\bigl[2\bigl(\sqrt{2}-1\bigr)\bigr].$$

Consequently, we conclude the sum (2). The second proof is complete. □

Third Proof. It is well known [3, p. 108, 4.6.7] that the binomial expansion is

$$(1+z)^{\alpha}=1+\frac{\alpha}{1!}z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots \tag{5}$$

for $\alpha\in\mathbb{C}$ and $|z|<1$. Taking $\alpha=-\frac{1}{2}$ in the binomial expansion (5), we acquire

$$\begin{aligned}\frac{1}{\sqrt{1+z}}&=1-\frac{1!!}{2}\frac{z}{1!}+\frac{3!!}{2^2}\frac{z^2}{2!}-\frac{5!!}{2^3}\frac{z^3}{3!}+\cdots\\ &=1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k}\frac{z^k}{k!}\\ &=1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k)!!(2k-1)!!}{2^k(2k)!!}\frac{z^k}{k!}\\ &=1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k)!}{(k!)^2}\frac{z^k}{2^{2k}}\\ &=1+\sum_{k=1}^{\infty}(-1)^k\binom{2k}{k}\left(\frac{z}{4}\right)^k\end{aligned}$$

for $|z|<1$. Setting $z=4w$, we produce

$$\frac{1}{\sqrt{1+4w}}=\sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}w^k,\quad |w|<\frac{1}{4}.$$

This series expansion is equivalent to the series expansion (4). The rest of this proof is same as texts after the series expansion (4) in the first proof. The third proof is thus complete. □

Acknowledgment

The authors thank anonymous referees for their careful corrections to, valuable comments on, and helpful suggestions to the original version of this paper.

The first author, Dr. Yue-Wu Li, was partially supported by the Doctors Foundation of Hulunbuir University (Grant No. 2018BS12) in China.

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[a]School of Mathematics and Statistics, Hulunbuir University, Hulunbuir 021008, Inner Mongolia, China
Email address: `yuewul@126.com`
URL: `https://orcid.org/0000-0002-8858-6768`

[b]School of Mathematical Sciences, Tiangong University, Tianjin 300160, China

[c]Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
Email address: `qifeng618@gmail.com`
URL: `https://orcid.org/0000-0001-6239-2968`