J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.1.19 ISSN(Online) 2287-6081
Volume 29, Number 1 (February 2022), Pages 19–29

TOPOLOGICAL STRUCTURES IN COMPLETE CO-RESIDUATED LATTICES

Young-Hee Kim [a] and Yong Chan Kim [b,*]

Abstract. Information systems and decision rules with imprecision and uncertainty in data analysis are studied in complete residuated lattices. In this paper, we introduce the notions of Alexandrov pretopology (precotopology) and join-meet (meet-join) operators in complete co-residuated lattices. Moreover, their properties and examples are investigated.

1. Introduction

Pawlak [19, 20] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of Pawlak's rough sets, many researchers [1-12, 23,24] developed lower and upper approximation operators. Radzikowska et al.[21, 22] investigated (I, T)-generalized fuzzy rough set where T is a t-norm and I is an implication. J.S.Mi et al.[15] investigated (S, T)-generalized fuzzy rough set where T is a t-norm and $S(a, b) = 1 - T(1 - a, 1 - b)$ is an implication.

Ward et al. [27] introduced a complete residuated lattice which is an algebraic structure for many valued logic [3-5]. It is an important mathematical tool as algebraic structures for many valued logics [1-12,23,24]. Using this concepts, fuzzy rough sets, information systems and decision rules were investigated in complete residuated lattices [1, 2, 7, 24]. Moreover, Zheng et al. [28] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng et al.[10] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$ where $(L, \vee, \wedge, \&, 0, 1)$ is a

Received by the editors October 28, 2020. Accepted January 09, 2022.
2010 *Mathematics Subject Classification.* 03E72, 54A40,54B10.
Key words and phrases. complete co-residuated lattice, distance spaces, Alexandrov pretopology (precotopology), join-meet (meet-join) operators .
This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense [13].

Kim et al. [8-12, 16-18] studied the properties of fuzzy join and meet completeness, L-fuzzy upper and lower approximation spaces and Alexandrov L-topologies with fuzzy partially ordered spaces and fuzzy distance spaces in complete(co-)residuated lattices.

In this paper, we introduce the notions of Alexandrov pretopology (precotopology) and join-meet (meet-join) operators in complete co-residuated lattices. Moreover, their properties and examples are investigated.

2. Preliminaries

Definition 2.1 ([7, 29])**.** An algebra $(L, \wedge, \vee, \oplus, 0, 1)$ is called a *complete co-residuated lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice where 0 is the bottom element and 1 is the top element.

(C2) $a = a \oplus 0$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$.

(C3) $(\bigwedge_{i \in \Gamma} a_i) \oplus b = \bigwedge_{i \in \Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge \{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x)$, $(\alpha \oplus A)(x) = \alpha \oplus A(x)$, $\alpha_X(x) = \alpha$.

Put $n(x) = 1 \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law.*

Remark 2.2. (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice where

$$x \ominus y = \bigwedge \{z \in L \mid y \vee z \geq x\} = \begin{cases} 0, & \text{if } y \geq x, \\ x, & \text{if } y \not\geq x. \end{cases}$$

Put $n(x) = 1 \ominus x = 1$ for $x \neq 1$ and $n(1) = 0$. Then $n(n(x)) = 0$ for $x \neq 1$ and $n(n(1)) = 1$. Hence n does not satisfy a double negative law.

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0, 1], \leq, \oplus)$, is a complete co-residuated lattice [26].

(3) $([1,\infty],\leq,\vee,\oplus=\cdot,\wedge,1,\infty)$ is a complete co-residuated lattice where

$$x\ominus y=\bigwedge\{z\in[1,\infty]\mid yz\geq x\}=\begin{cases}1, & \text{if } y\geq x,\\ \frac{x}{y}, & \text{if } y\not\geq x.\end{cases}$$

$$\infty\cdot a=a\cdot\infty=\infty,\forall a\in[1,\infty],\infty\ominus\infty=1.$$

Put $n(x)=\infty\ominus x=\infty$ for $x\neq\infty$ and $n(\infty)=1$. Then $n(n(x))=1$ for $x\neq\infty$ and $n(n(\infty))=\infty$. Hence n does not satisfy a double negative law.

(4) $([0,\infty],\leq,\vee,\oplus=+,\wedge,0,\infty)$ is a complete co-residuated lattice where

$$\begin{aligned}&y\ominus x=\bigwedge\{z\in[0,\infty]\mid x+z\geq y\}\\&=\bigwedge\{z\in[0,\infty]\mid z\geq -x+y\}=(y-x)\vee 0,\\&\infty+a=a+\infty=\infty,\forall a\in[0,\infty],\infty\ominus\infty=0.\end{aligned}$$

Put $n(x)=\infty\ominus x=\infty$ for $x\neq\infty$ and $n(\infty)=0$. Then $n(n(x))=0$ for $x\neq\infty$ and $n(n(\infty))=\infty$. Hence n does not satisfy a double negative law.

(5) $([0,1],\leq,\vee,\oplus,\wedge,0,1)$ is a complete co-residuated lattice where

$$\begin{aligned}&x\oplus y=(x^p+y^p)^{\frac{1}{p}}\wedge 1,\ \ 1\leq p<\infty,\\&x\ominus y=\bigwedge\{z\in[0,1]\mid (z^p+y^p)^{\frac{1}{p}}\geq x\}\\&=\bigwedge\{z\in[0,1]\mid z\geq (x^p-y^p)^{\frac{1}{p}}\}=(x^p-y^p)^{\frac{1}{p}}\vee 0,\end{aligned}$$

Put $n(x)=1\ominus x=(1-x^p)^{\frac{1}{p}}$ for $1\leq p<\infty$. Then $n(n(x))=x$ for $x\in[0,1]$. Hence n satisfies a double negative law.

(6) Let $P(X)$ be the collection of all subsets of X. Then $(P(X),\subset,\cup,\cap,\oplus=\cup,\emptyset,X)$ is a complete co-residuated lattice where

$$\begin{aligned}&A\ominus B=\bigwedge\{C\in P(X)\mid B\cup C\supset A\}\\&=A\cap B^c=A-B.\end{aligned}$$

Put $n(A)=X\ominus A=A^c$ for each $A\subset X$. Then $n(n(A))=A$. Hence n satisfies a double negative law.

Lemma 2.3 ([11])**.** *Let $(L,\wedge,\vee,\oplus,\ominus,0,1)$ be a complete co-residuated lattice. For each $x,y,z,x_i,y_i\in L$, we have the following properties.*

(1) *If $y\leq z$, $x\oplus y\leq x\oplus z$, $y\ominus x\leq z\ominus x$ and $x\ominus z\leq x\ominus y$.*

(2) *$(\bigvee_{i\in\Gamma}x_i)\ominus y=\bigvee_{i\in\Gamma}(x_i\ominus y)$ and $x\ominus(\bigwedge_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\ominus y_i)$.*

(3) *$(\bigwedge_{i\in\Gamma}x_i)\ominus y\leq\bigwedge_{i\in\Gamma}(x_i\ominus y)$*

(4) *$x\ominus(\bigvee_{i\in\Gamma}y_i)\leq\bigwedge_{i\in\Gamma}(x\ominus y_i)$.*

(5) *$x\ominus x=0$, $x\ominus 0=x$ and $0\ominus x=0$. Moreover, $x\ominus y=0$ iff $x\leq y$.*

(6) *$y\oplus(x\ominus y)\geq x$, $y\geq x\ominus(x\ominus y)$ and $(x\ominus y)\oplus(y\ominus z)\geq x\ominus z$.*

(7) *$x\ominus(y\oplus z)=(x\ominus y)\ominus z=(x\ominus z)\ominus y$.*

(8) $x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ *and* $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.

(9) $x \oplus y = 0$ *iff* $x = 0$ *and* $y = 0$.

(10) $(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ *and* $(x \ominus y) \oplus z \geq x \ominus (y \ominus z)$.

(11) *If* L *satisfies a double negative law and* $n(x) = 1 \ominus x$, *then* $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ *and* $x \ominus y = n(y) \ominus n(x)$.

Definition 2.4 ([11])**.** Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x, x) = 0$ for all $x \in X$,

(M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$, for all $x, y, z \in X$,

(M3) If $d_X(x, y) = d_X(y, x) = 0$, then $x = y$.

The pair (X, d_X) is called a *distance space*.

Remark 2.5 ([11])**.** (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a pseudo-quasi-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space. For $\tau \subset L^X$, we define a function $d_\tau : \tau \times \tau \to L$ as $d_\tau(A, B) = \bigvee_{x \in X}(A(x) \ominus B(x))$. Then (τ, d_τ) is a distance space.

3. Topological Structures in Complete Co-residuated Lattices

In this section, we assume $(L, \wedge, \vee, \oplus, \ominus, 0, 1)$ is a complete co-residuated lattice with a double negative law $n(x) = 1 \ominus x$.

Definition 3.1. (1) A subset $\tau \subset L^X$ is called an *Alexandrov pretopology* on X iff it satisfies the following conditions:

(O1) $\alpha_X \in \tau$.

(O2) If $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i \in I} A_i \in \tau$.

(O3) If $A \in \tau$ and $\alpha \in L$, then $A \ominus \alpha \in \tau$.

(2) A subset $\eta \subset L^X$ is called an *Alexandrov precotopology* on X iff it satisfies the following conditions:

(CO1) $\alpha_X \in \eta$.

(CO2) If $A_i \in \eta$ for all $i \in I$, then $\bigwedge_{i \in I} A_i \in \eta$.

(CO3) If $A \in \eta$ and $\alpha \in L$, then $\alpha \oplus A \in \eta$.

A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X iff it is both Alexandrov pretopology and Alexandrov precotopology on X.

Definition 3.2. A map $\mathcal{K} : L^X \to L^X$ is called a *meet-join operator* if it satisfies the following conditions:

(K1) $\mathcal{K}(\alpha_X) = n(\alpha_X)$,

(K2) $\mathcal{K}(A) \leq n(A)$, for $A \in L^X$,

(K3) $\mathcal{K}(A \oplus \alpha) \geq \mathcal{K}(A) \ominus \alpha$ for each $\alpha \in L, A \in L^X$ and $\mathcal{K}(B) \leq \mathcal{K}(A)$ for $A \leq B$.

The pair $(X, \mathcal{K})$ is called a *meet-join space*.

Definition 3.3. A map $\mathcal{D} : L^X \to L^X$ is called a *join-meet operator* if it satisfies the following conditions:

(D1) $\mathcal{D}(\alpha_X) = n(\alpha_X)$,

(D2) $n(A) \leq \mathcal{D}(A)$, for $A \in L^X$,

(D3) $\alpha \oplus \mathcal{D}(A) \geq \mathcal{D}(A \ominus \alpha)$ for each $\alpha \in L, A \in L^X$ and $\mathcal{D}(A) \geq \mathcal{D}(B)$ for $A \leq B$.

The pair $(X, \mathcal{D})$ is called a *join-meet space*.

Theorem 3.4. *Let $\mathcal{M} : L^X \to L^X$ be a map. The following statements are equivalent.*

(1) $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{M}(B), \mathcal{M}(A))$ *for all* $A, B \in L^X$.

(2) $\mathcal{M}(A) \ominus \alpha \leq \mathcal{M}(A \oplus \alpha)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{M}(B) \leq \mathcal{M}(A)$ *for* $A \leq B$.

(3) $\alpha \oplus \mathcal{M}(A) \geq \mathcal{M}(A \ominus \alpha)$ *for each* $\alpha \in L, A \in L^X$ *and* $\mathcal{M}(B) \leq \mathcal{M}(A)$ *for* $A \leq B$.

Proof. (1) $\Rightarrow$ (2). If $A \leq B$, then $d_{L^X}(A, B) = 0$ and $d_{L^X}(\mathcal{M}(B), \mathcal{M}(A)) = 0$. Thus $\mathcal{M}(B) \leq \mathcal{M}(A)$. Since $\alpha \geq d_{L^X}(\alpha \oplus A, A) \geq d_{L^X}(\mathcal{M}(A), \mathcal{M}(\alpha \oplus A))$, we have $\mathcal{M}(\alpha \oplus A) \geq \mathcal{M}(A) \ominus \alpha$.

(2)$\Rightarrow$ (1). Let $\alpha = d_{L^X}(A, B)$. Since $B \oplus d_{L^X}(A, B) \geq A$, $\mathcal{M}(A) \geq \mathcal{M}(d_{L^X}(A, B) \oplus B) \geq \mathcal{M}(B) \ominus d_{L^X}(A, B)$. Then $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{M}(B), \mathcal{M}(A))$.

(1) $\Rightarrow$ (3). If $A \leq B$, then $\mathcal{M}(B) \leq \mathcal{M}(A)$. Since $\alpha \geq d_{L^X}(A, A \ominus \alpha) \geq d_{L^X}(\mathcal{M}(A \ominus \alpha), \mathcal{M}(A))$, we have $\mathcal{M}(A \ominus \alpha) \leq \mathcal{M}(A) \oplus \alpha$.

(3)$\Rightarrow$ (1). Let $\alpha = d_{L^X}(A, B)$. Then $d_{L^X}(A, B) \geq d_{L^X}(\mathcal{M}(B), \mathcal{M}(A))$ from:

$$\mathcal{M}(B) \leq \mathcal{M}(A \ominus d_{L^X}(A, B)) \leq \mathcal{M}(A) \oplus d_{L^X}(A, B).$$

□

Theorem 3.5. *Let (X, η) be an Alexandrov precotopological space. Define $\mathcal{D}_\eta : L^X \to L^X$ by*

$$\mathcal{D}_\eta(A) = \bigwedge_{B \in \eta} (d_{L^X}(n(A), B) \oplus B).$$

Then the following properties hold.

(1) $\mathcal{D}_\eta(A) = \bigwedge_{i\in\Gamma}\{A_i \mid n(A) \leq A_i, A_i \in \eta\}$.

(2) *$\mathcal{D}_\eta$ is a join-meet operator on X such that $\mathcal{D}_\eta(n(\mathcal{D}_\eta(A))) = \mathcal{D}_\eta(A)$.*

(3) $d_{L^X}(\mathcal{D}_\eta(A), \mathcal{D}_\eta(C)) \leq d_{L^X}(C, A)$.

(4) *$\eta_{\mathcal{D}_\eta} = \eta$ where $\eta_{\mathcal{D}_\eta} = \{A \in L^X \mid A = \mathcal{D}_\eta(n(A))\}$.*

(5) *If $\mathcal{D}$ is a join-meet operator on X, then $\mathcal{D}_{\eta_\mathcal{D}} \geq \mathcal{D}$. Moreover, the equality holds if $\mathcal{D}(n(\mathcal{D}(A))) = \mathcal{D}(A)$ for each $A \in L^X$.*

Proof. (1) Put $D_1(A) = \bigwedge_{i\in\Gamma}\{A_i \mid n(A) \leq A_i, A_i \in \eta\}$. Since

$$\mathcal{D}_\eta(A) = \bigwedge_{C\in\eta} (d_{L^X}(n(A), C) \oplus C) \in \eta$$

and

$$\bigwedge_{A_i\in\eta} (d_{L^X}(n(A), A_i) \oplus A_i) \geq n(A), \mathcal{D}_\eta(A) \geq D_1(A).$$

Since $D_1(A) \in \eta$, $\mathcal{D}_\eta(A) \leq d_{L^X}(n(A), D_1(A)) \oplus D_1(A) = D_1(A)$. Hence $\mathcal{D}_\eta = D_1$.

(2) (D1) For all $x \in X$, since $n(\alpha_X) = n(\alpha)_X \in \eta$,

$$\begin{aligned}\mathcal{D}_\eta(\alpha_X)(x) &= \bigwedge_{C\in\eta} d_{L^X}(n(\alpha_X), C) \oplus C(x)\\ &\leq d_{L^X}(n(\alpha_X), n(\alpha_X)) \oplus n(\alpha_X)(x) = n(\alpha_X)(x).\end{aligned}$$

(D2) For each $A \in L^X$, $d_{L^X}(n(A), B) \oplus B \geq n(A)$. Hence $\mathcal{D}_\eta(A) \geq n(A)$.

(D3) If $A \leq B$, then $\mathcal{D}_\eta(A) \geq \mathcal{D}_\eta(B)$.

For each $A, B \in L^X$, we have

$$\begin{aligned}\alpha \oplus \mathcal{D}_\eta(A) &= \alpha \oplus \bigwedge_{i\in\Gamma}\{A_i \mid n(A) \leq A_i, A_i \in \eta\}\\ &= \bigwedge_{i\in\Gamma}\{\alpha \oplus A_i \mid n(A) \leq A_i, A_i \in \eta\}\\ &\geq \bigwedge_{i\in\Gamma}\{\alpha \oplus A_i \mid \alpha \oplus n(A) = n(A \ominus \alpha) \leq \alpha \oplus A_i, \alpha \oplus A_i \in \eta\}\\ &\geq \mathcal{D}_\eta(A \ominus \alpha).\end{aligned}$$

Since $\mathcal{D}_\eta(A) \in \eta$,

$$\begin{aligned}\mathcal{D}_\eta(n(\mathcal{D}_\eta(A))) &= \bigwedge_{C\in\eta} (d_{L^X}(\mathcal{D}_\eta(A), C) \oplus C)\\ &\leq d_{L^X}(\mathcal{D}_\eta(A), \mathcal{D}_\eta(A)) \oplus \mathcal{D}_\eta(A) = \mathcal{D}_\eta(A).\end{aligned}$$

(3) For each $A, C \in L^X$, we have

$$\begin{aligned}
&d_{L^X}(\mathcal{D}_\eta(A), \mathcal{D}_\eta(C)) = \bigvee_{x\in X} (\mathcal{D}_\eta(A)(x) \ominus \mathcal{D}_\eta(C)(x))\\
&= \bigvee_{x\in X} \Big(\bigwedge_{B\in\eta} (d_{L^X}(n(A), B) \oplus B(x)) \ominus \bigwedge_{E\in\eta} (d_{L^X}(n(C), E) \oplus E(x))\Big)\\
&\leq \bigvee_{x\in X} \bigvee_{E\in\eta} \Big(d_{L^X}(n(A), E) \oplus E(x)) \ominus (d_{L^X}(n(C), E) \oplus E(x))\Big)\\
&\leq \bigvee_{x\in X} \Big(d_{L^X}(n(A), E) \ominus d_{L^X}(n(C), E))\\
&\leq d_{L^X}(n(A), n(C)) = d_{L^X}(C, A).
\end{aligned}$$

(4) $\eta_{\mathcal{D}_\eta} = \eta$ where $\eta_{\mathcal{D}_\eta} = \{A \in L^X \mid A = \mathcal{D}_\eta(n(A))\}$.

$$\begin{aligned}
&A \in \eta,\ A = \mathcal{D}_\eta(n(A)), A \in \eta_{\mathcal{D}_\eta}\\
&A \in \eta_{\mathcal{D}_\eta}, A = \mathcal{D}_\eta(n(A)) \in \eta, A \in \eta.
\end{aligned}$$

(5) For each $A \in L^X$,

$$\begin{aligned}
&\mathcal{D}_{\eta_\mathcal{D}}(A) = \bigwedge_{i\in\Gamma} \{A_i \mid n(A) \leq A_i, A_i \in \eta_\mathcal{D}\}\\
&= \bigwedge_{i\in\Gamma} \{\mathcal{D}(n(A_i)) \mid n(A) \leq A_i, A_i \in \eta_\mathcal{D}\} \geq \mathcal{D}(A)\\
&(A \geq n(A_i) \Rightarrow \mathcal{D}(A) \leq \mathcal{D}(n(A_i))).
\end{aligned}$$

If $\mathcal{D}(n(\mathcal{D}(A))) = \mathcal{D}(A)$ for each $A \in L^X$,

$$\begin{aligned}
&\mathcal{D}_{\eta_\mathcal{D}}(A)(x) = \bigwedge_{C\in\eta_\mathcal{D}} (d_{L^X}(A, C) \oplus C(x))\\
&\leq d_{L^X}(A, \mathcal{D}(A)) \oplus \mathcal{D}(A)(x) = \mathcal{D}(A)(x).
\end{aligned}$$

□

Theorem 3.6. *Let (X, τ) be an Alexandrov pretopological space. Define $\mathcal{K}_\tau : L^X \to L^X$ by*

$$\mathcal{K}_\tau(A) = \bigvee_{B_i\in\tau} (B_i \ominus d_{L^X}(B_i, n(A)))$$

Then the following properties hold.

(1) $\mathcal{K}_\tau(A) = \bigvee\{B_i \in \tau \mid B_i \leq n(A)\}$.

(2) *$\mathcal{K}_\tau$ is a meet-join operator on X such that $\mathcal{K}_\tau(n(\mathcal{K}_\tau(A))) = \mathcal{K}_\tau(A)$.*

(3) $d_{L^X}(\mathcal{K}_\tau(A), \mathcal{K}_\tau(C)) \leq d_{L^X}(C, A)$.

(4) *For each $A, C \in L^X$, $\tau_{\mathcal{K}_\tau} = \tau$ where $\tau_{\mathcal{K}_\tau} = \{A \in L^X \mid A = \mathcal{K}_\tau(n(A))\}$.*

(5) *If $\mathcal{K}$ is a meet-join operator on X, then $\mathcal{K}_{\tau_\mathcal{K}} \leq \mathcal{K}$. Moreover, the equality holds if $\mathcal{K}(n(\mathcal{K}(A))) = \mathcal{K}(A)$ for each $A \in L^X$.*

Proof. (1) Since τ is an Alexandrov pretopology on X, $\bigvee_{C\in\tau}(C\ominus d_{L^X}(C,n(A))\in\tau$. Put $K_1(A)=\bigvee_{i\in\Gamma}\{A_i\mid A_i\leq n(A),A_i\in\tau\}$. Since $A_i\leq n(A)\oplus d_{L^X}(A_i,n(A))$ iff $A_i\ominus d_{L^X}(A_i,n(A))\leq n(A)$, by $A_i\ominus d_{L^X}(A_i,n(A))\in\tau$, $K_1(A)\geq\mathcal{K}_\tau(A)$.

Since $K_1(A)\in\tau$, $\mathcal{K}_\tau(A)\geq K_1(A)\ominus d_{L^X}(K_1(A),n(A))=K_1(A)\ominus 0=K_1(A)$.

(2) (K1) For each $x\in X$,

$$\begin{aligned}\mathcal{K}_\tau(\alpha_X)(x)&=\bigvee_{B\in L^X}\Big(B\ominus d_{L^X}(B,n(\alpha_X))\Big)\\&\geq n(\alpha_X)\ominus d_{L^X}(n(\alpha_X),n(\alpha_X))=n(\alpha_X)(x),\end{aligned}$$

(K2) It follows $A_i\ominus d_{L^X}(A_i,n(A))\leq n(A)$.

(K3) For each $A,C\in L^X$,

$$\begin{aligned}&\mathcal{K}_\tau(A)\ominus\alpha=\bigvee\{B_i\in\tau\mid B_i\leq n(A)\}\ominus\alpha\\&=\bigvee\{B_i\ominus\alpha\in\tau\mid B_i\leq n(A)\}\\&\leq\bigvee\{B_i\ominus\alpha\in\tau\mid B_i\ominus\alpha\leq n(A)\ominus\alpha=n(A\oplus\alpha)\}\\&\leq\mathcal{K}_\tau(A\oplus\alpha).\end{aligned}$$

Since $\mathcal{K}_\tau(A)\in\tau$, $\mathcal{K}_\tau(A)=\mathcal{K}_\tau(n(\mathcal{K}_\tau(A)))$ from:

$$\mathcal{K}_\tau(n(\mathcal{K}_\tau(A)))\geq\mathcal{K}_\tau(A)\ominus d_{L^X}(\mathcal{K}_\tau(A),\mathcal{K}_\tau(A))=\mathcal{K}_\tau(A).$$

(3) For each $A,C\in L^X$,

$$\begin{aligned}&d_{L^X}(\mathcal{K}_\tau(A),\mathcal{K}_\tau(C))=\bigvee_{x\in X}\Big(\mathcal{K}_\tau(A)(x)\ominus\mathcal{K}_\tau(C)(x)\Big)\\&=\bigvee_{x\in X}\Big(\bigvee_{B\in\tau}(B(x)\ominus d_{L^X}(B,n(A)))\ominus\bigvee_{D\in\tau}(D(x)\ominus d_{L^X}(D,n(C)))\Big)\\&\leq\bigvee_{x\in X}\bigvee_{B\in\tau}\Big((B(x)\ominus d_{L^X}(B,n(A))))\\&\ominus((B(x)\ominus d_{L^X}(B,n(C)))\Big)\ \text{(by Lemma 2.3(8))}\\&\leq\bigwedge_{B\in\tau}(d_{L^X}(B,n(C))\ominus d_{L^X}(B,n(A)))\ \text{(put } B=n(A))\\&\leq d_{L^X}(n(A),n(C))=d_{L^X}(C,A)\ \text{(by Lemma 2.3(11))}.\end{aligned}$$

(4) It is similarly proved as Theorem 3.5(4).

(5)

$$\begin{aligned}&\mathcal{K}_{\tau_\mathcal{K}}(A)=\bigvee\{B_i\in\tau_\mathcal{K}\mid B_i\leq n(A)\}\\&=\bigvee\{\mathcal{K}(n(B_i))\in\tau_\mathcal{K}\mid n(B_i)\geq A,\mathcal{K}(n(B_i))\leq\mathcal{K}(A)\}\leq\mathcal{K}(A).\end{aligned}$$

If $\mathcal{K}(n(\mathcal{K}(A))) = \mathcal{K}(A)$ for each $A \in L^X$, then $\mathcal{K}(A) \in \tau_{\mathcal{K}}$. Thus

$$\begin{aligned}\mathcal{K}_{\tau_{\mathcal{K}}}(A)(x) &= \bigvee_{B\in\tau_{\mathcal{K}}} (B(x) \ominus d_{L^X}(B, n(A))) \\ &\geq \mathcal{K}(A)(x) \ominus d_{L^X}(\mathcal{K}(A), n(A)) = \mathcal{K}(A)(x).\end{aligned}$$

□

Example 3.7. Let $X = \{x, y, z\}$ and $([0,1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined as $n(x) = 1 - x$,

$$x \oplus y = (x + y) \wedge 1, \;\; x \ominus y = (x - y) \vee 0.$$

Put $A \in [0,1]^X$ with $A(x) = 0.6, A(y) = 0.3, A(z) = 0.5$.

(1) Define an Alexandrov pretopology

$$\tau_X = \{(A \ominus \alpha) \vee \beta_X) \mid \alpha, \beta \in L\}.$$

By Theorem 3.6(4), $\tau_{\mathcal{K}_{\tau_X}} = \tau$ where $\tau_{\mathcal{K}_{\tau_X}} = \{A \in [0,1]^X \mid A = \mathcal{K}_{\tau_X}(n(A))\}$. Since $0.2 \oplus A = (0.8, 0.5, 0.7) \notin \tau_X = \tau_{\mathcal{K}_{\tau_X}}$, $\tau_X = \tau_{\mathcal{K}_{\tau_X}}$ is not an Alexandrov precotopology. For $B = (0.2, 0.4, 0.3) \in [0,1]^X$, $\mathcal{K}_{\tau_X}(B) = \bigvee\{A_i \in \tau_X \mid A_i \leq n(B)\} = 0.6_X$.

(2) Define an Alexandrov precotopology

$$\eta_X = \{(A \oplus \alpha) \wedge \beta_X) \mid \alpha, \beta \in L\}.$$

By Theorem 3.5(4), $\eta_{\mathcal{D}_{\eta_X}} = \eta$ where $\eta_{\mathcal{D}_{\eta_X}} = \{A \in [0,1]^X \mid A = \mathcal{D}_{\tau_X}(n(A))\}$. Since $A \ominus 0.2 = (0.4, 0.1, 0.3) \notin \eta_X = \eta_{\mathcal{D}_{\eta_X}}$, $\eta_X = \eta_{\mathcal{D}_{\eta_X}}$ is not an Alexandrov pretopology. For $B = (0.2, 0.4, 0.3) \in [0,1]^X$,

$$\mathcal{D}_{\eta_X}(B) = \bigwedge\{A_i \in \eta_X \mid n(B) \leq A_i\} = (0.9, 0.6, 0.8) \wedge 0.8_X = (0.8, 0.6, 0.8).$$

References

1. R. Bělohlávek: Fuzzy Relational Systems. Kluwer Academic Publishers, New York, 2002.
2. P. Chen & D. Zhang: Alexandroff co-topological spaces. Fuzzy Sets and Systems **161** (2010), 2505-2514.
3. P. Hájek: Metamathematices of Fuzzy Logic. Kluwer Academic Publishers, Dordrecht, 1998.
4. U. Höhle & E.P. Klement: Non-classical logic and their applications to fuzzy subsets. Kluwer Academic Publishers, Boston, 1995.

5. U. Höhle & S.E. Rodabaugh: Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory. The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, 1999.
6. F. Jinming: I-fuzzy Alexandrov topologies and specialization orders. Fuzzy Sets and Systems **158** (2007), 2359-2374.
7. Q. Junsheng & Hu. Bao Qing: On $(\odot, \&)$ -fuzzy rough sets based on residuated and co-residuated lattices. Fuzzy Sets and Systems **336** (2018), 54-86.
8. Y.C. Kim: Join-meet preserving maps and Alexandrov fuzzy topologies. Journal of Intelligent and Fuzzy Systems **28** (2015), 457-467.
9. ______: Join-meet preserving maps and fuzzy preorders. Journal of Intelligent and Fuzzy Systems **28** (2015), 1089-1097.
10. ______: Categories of fuzzy preorders, approximation operators and Alexandrov topologies. Journal of Intelligent and Fuzzy Systems **31** (2016), 1787-1793.
11. J.M. Ko & Y.C. Kim: Preserving maps and approximation operators in complete co-residuated lattices. Journal of the korean Insitutute of Intelligent Systems. **30** (2020), 389-398.
12. ______: Fuzzy complete lattices, Alexandrov L-fuzzy topologies and fuzzy rough sets. Journal of Intelligent and Fuzzy Systems **38** (2020), 3253-3266.
13. H. Lai & D. Zhang: Fuzzy preorder and fuzzy topology. Fuzzy Sets and Systems **157** (2006), 1865-1885.
14. Z.M. Ma & B.Q. Hu: Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets. Information Sciences **218** (2013), 194-204.
15. J.S. Mi, Y. Leung, H.Y. Zhao & T. Feng: Generalized fuzzy rough sets determined by a trianglar norm. Information Sciences **178** (2008), 3203-3213.
16. J.M. Oh & Y.C. Kim: Distance functions, upper approximation operators and Alexandrov fuzzy topologies. Journal of Intelligent and Fuzzy Systems **40** (2021), no. 6, 11927-11939.
17. ______: Fuzzy Galois connections on L-toplogies. Journal of Intelligent and Fuzzy Systems **40** (2021), 251-270.
18. ______: Various fuzzy connections and fuzzy concepts in complete co-residuated lattices. International Journal of Approximate Reasoning **142** (2022), 451-468.
19. Z. Pawlak: Rough sets. Internat. J. Comput. Inform. Sci. **11** (1982), 341-356.
20. Z. Pawlak: Rough sets: Theoretical Aspects of Reasoning about Data, System Theory. Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
21. A.M. Radzikowska & E.E. Kerre: A comparative study of fuzy rough sets. Fuzzy Sets and Systems **126** (2012), 137-155.
22. A.M. Radzikowska & E.E. Kerre: Characterisation of main classes of fuzzy relations using fuzzy modal operators. Fuzzy Sets and Systems **152** (2005), 223-247.

23. S.E. Rodabaugh & E.P. Klement: Topological and Algebraic Structures in Fuzzy Sets. The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.
24. Y.H. She & G.J. Wang: An axiomatic approach of fuzzy rough sets based on residuated lattices. Computers and Mathematics with Applications **58** (2009), 189-201.
25. S. P. Tiwari & A.K. Srivastava: Fuzzy rough sets, fuzzy preorders and fuzzy topologies. Fuzzy Sets and Systems **210** (2013), 63-68.
26. E. Turunen: Mathematics Behind Fuzzy Logic. A Springer-Verlag Co., 1999.
27. M. Ward & R.P. Dilworth: Residuated lattices. Trans. Amer. Math. Soc. **45** (1939), 335-354.
28. W.Z. Wu, Y. Leung & J.S. Mi: On charterizations of (Φ, T)-fuzzy approximation operators. Fuzzy Sets and Systems **154** (2005), 76-102.
29. M.C. Zheng & G.J. Wang: Coresiduated lattice with applications. Fuzzy systems and Mathematics **19** (2005), 1-6.

[a]Ingenium College of Liberal Arts-Mathematics, Kwangwoon University, Seoul 01897, Korea
Email address: yhkim@kw.ac.kr

[b]Mathematics Department, Gangneung-Wonju National University, Gangneung 25457, Korea
Email address: yck@gwnu.ac.kr