J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
http://dx.doi.org/10.7468/jksmeb.2022.29.2.161 ISSN(Online) 2287-6081
Volume 29, Number 2 (May 2022), Pages 161–170

ALPHA INVARIANT ALONG CURVES FOR GENERAL POLARIZATIONS OF DEL PEZZO SURFACES OF DEGREE 2

Joonyeong Won

Abstract. For an arbitrary ample divisor A in smooth del Pezzo surface S of degree 2, we completely compute alpha invariant along curves when the ample divisor A is birational type.

All considered varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic 0 throughout this article.

1. Introduction

In recent years the notion of K-stability has been of great importance in the study of the existence of canonical metrics on complex varieties. The first important invariant for K-stability is alpha invariant which is introduced by Tian [14] gives a numerical criterion for the existence of Kähler-Einstein metrics on Fano manifolds equivalantly K-polystability of it. The paper [14] proved that if X is a smooth Fano variety of dimension n with canonical divisor K_X, the lower bound $\alpha(X, -K_X) > \frac{n}{n+1}$ implies that X admits a Kähler-Einstein metric in $c_1(X)$.

On the other hand, the Yau-Tian-Donaldson conjecture states that the existence of a constant scalar curvature Kähler metric in $c_1(A)$ for a polarised manifold (X, A) is equivalent to the algebro-geometric notion of K-stability, a certain version of stability notion of geometric invariant theory. This conjecture has recently been proven when the divisor A is anticanonical ([4], [5], [6], [15]). Odaka and Sano [11]

Received by the editors February 23, 2022. Accepted March 16, 2022.
2010 *Mathematics Subject Classification.* 14J45.
Key words and phrases. K-stability, alpha invariant, del Pezzo surface.
The author was supported by the National Research Foundation of Korea (NRF-2020R1A2C1A 01008018).

have given a direct algebraic proof that $\alpha(X, -K_X) > \frac{n}{n+1}$ implies that $(X, -K_X)$ is K-stable. Futhermore, Dervan [8] generalizes this result by giving a sufficient condition for general polarisations of Fano varieties to be K-stable.

Theorem 1.1 ([8]). *Let (X, A) be a polarised $\mathbb{Q}$-Gorenstein log canonical variety of dimension n with canonical divisor K_X. And let $\nu(A) = \frac{-K_X \cdot A^{n-1}}{A^n}$. Suppose that*

(i) $\alpha(X, A) > \frac{n}{n+1}\nu(A)$ *and*

(ii) $-K_X - \frac{n}{n+1}\nu(A)A$ *is nef.*

Then (X, A) is K-stable.

For anti-canonically polarised smooth del Pezzo surfaces, the classification is completely done by Cheltsov [1]. The paper [1] implies that general anticanonically polarized smooth del pezzo surfaces of degree ≤ 3 are K-stable. Generalizing this, Dervan verifies K-stability for certain polarizations (S, A_λ), where S is del Pezzo surface of degree 1 and $A_\lambda = -K_S + \lambda(\text{exceptional curve})$. The computation of α-invariant is valuable in its own sake, and initiated by the results of Dervan, the papers [9], [3] study the α-invariant for all polarizations of del Pezzo surfaces of degree 1. By the computation, it turns out that condition (ii) is stronger than condition (i) in Theorem 1.1 for del Pezzo surfaces of degree 1. The articles [9], [3] proves

Theorem 1.2. *Let (S, A) be a polarized smooth del Pezzo surface of degree 1. Suppose that $-K_S - \frac{2(-K_S \cdot A)}{3(A)^2}A$ is nef. Then (S, A) is K-stable.*

Indeed the paper [3] showes that same property holds for del Pezzo surfaces of degree 2. In separate meanings, main ingredient of the paper [9] is computation of the alpha invariants along curves. The invariant is valuable in its own manner. And this will provides complete computations of alpha invariants. Thus main purpose of present article is as follows

Main Theorem 1.3. Let S be a smooth del Pezzo surface of degree 2 and A be an ample divisor of S. If the ample divisor A is birational type, that is, $\mu A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{7} a_i E_i$, where E_i is exceptional -1-curves, then

- When $s_A > 4$, $\alpha_c(S, A) = \frac{1}{2+a_1}$;
- When $s_A \leq 3$, $\alpha_c(S, A) = \frac{3}{3+3a_1+s_A}$.

2. Preliminaries and Notations

2.1. α-invariant For a polarized smooth Fano variety (X, A), its α-invariant can be defined as

$$\alpha(X, A) = \sup \left\{ c \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, cD) \text{ is log canonical for every} \\ \text{effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} A. \end{array} \right\}.$$

For every effective $\mathbb{Q}$-divisor B on X, the number

$$\mathrm{lct}(X, B) = \sup \{ c \in \mathbb{Q} \mid \text{the log pair } (X, cD) \text{ is log canonical} \}$$

is called the *log canonical threshold* of B. Note that

$$\alpha(X, A) = \inf \{ \mathrm{lct}(X, B) | \; B \text{ is an effective } \mathbb{Q}\text{-divisor such that } B \sim_{\mathbb{Q}} A \}$$

Tian introduced α-invariant of smooth Fano varieties in [14] and proved

Theorem 2.1 ([14, Theorem 2.1]). *Let X be a smooth Fano variety of dimension n. If $\alpha(X, -K_X) > \frac{n}{n+1}$, then X admits a Kähler-Einstein metric.*

We will use α-invariant for curves α_c to give a bound of the α-invariant.

$$\alpha_c(X, A) = \sup \left\{ c \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, cD) \text{ is log canonical along all curves} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} A. \end{array} \right\}.$$

If the variety is a surface, then the number $\alpha_c(X, A)$ is a reciprocal of the maximal multiplicity along a curve of a divisor B, where B is $\mathbb{Q}$-lineary equivalent to A.

The present article deals with a del Pezzo surface S of degree 1 and makes application of Theorem 1.1. So the slope $\nu(A)$ is always denoted by $\frac{-K_S \cdot A}{A^2}$

2.2. del Pezzo surfaces of degree 2 Let S be a smooth del Pezzo surface of degree 2. The variety can be obtained by blowing up $\mathbb{P}^2$ at seven points in general position. Let $\pi : S \to \mathbb{P}^2$ be such a blow up and $E_1, \ldots, E_7$ be its exceptional curves. Denote the point $\pi(E_i)$ by P_i.

Let h be the divisor class in S corresponding to $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and e_i be the class of the exceptional curves E_i, where $i = 1, \ldots, 7$. Since the classes $h, e_1, \ldots, e_7$ form an orthogonal basis of the Picard group of S, for a divisor A on S we may write $[A] = \beta h + \sum_{i=1}^{7} \beta_i e_i$, where β and β_i's are constants. It is well known that the divisor A is ample if and only if the intersection number $A \cdot C$ is positive for all -1-curves C and the curve C corresponds to one of following classes

- e_i ;
- $h - e_i - e_j$ for $i \neq j$;

- $2h - e_i - e_j - e_k - e_l - e_m$ for $i \neq j \neq k \neq l \neq m$;
- $[-K_S] + e_i - e_j$ for $i \neq j$;

That is, relations attained by the intersection number define the ample cone of S. The Mori cone $\overline{\mathbb{NE}}(S)$ of the surface S is polyhedral. In addition, it is generated by all the (-1)-curves on S.

From now on, the divisor A is always assumed to be ample, unless otherwise stated. The following method to express the divisor A in terms of $-K_S$ and (-1)-curves is adopted from [7], [12]. For the log pair (S, A), we define an invariant of (S, A) by

$$\mu := \inf \left\{ \lambda \in \mathbb{Q}_{>0} \;\middle|\; \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective} \right\}.$$

The invariant μ is always obtained by a positive rational number. Let $\Delta_{(S,A)}$ be the smallest extremal face of the boundary of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu A$.

Let $\phi\colon S \to Z$ be the contraction given by the face $\Delta_{(S,A)}$. Then either ϕ is a birational morphism or a conic bundle with $Z \cong \mathbb{P}^1$. In the former case $\Delta_{(S,A)}$ is generated by r disjoint (-1)-curves contracted by ϕ, where $r \leq 8$. In the later case, $\Delta_{(S,A)}$ is generated by the (-1)-curves in the eight reducible fibers of ϕ. Each reducible fiber consists of two (-1)-curves that intersect transversally at one point.

Suppose that ϕ is birational. Let $E_1, \ldots, E_r$ be all (-1)-curves contained in $\Delta_{(S,A)}$. These are disjoint and generate the face $\Delta_{(S,A)}$. Therefore,

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i$$

for some positive rational numbers $a_1, \ldots, a_r$. We have $a_i < 1$ for every i because $A \cdot E_i > 0$. Vice versa, for every positive rational numbers $a_1, \ldots, a_r < 1$, the divisor

$$-K_S + \sum_{i=1}^{r} a_i E_i$$

is ample.

Suppose that ϕ is a conic bundle. Then there are a 0-curve B and seven disjoint (-1)-curves $E_1, E_2, E_3, E_4, E_5, E_6, E_7$, each of which is contained in a distinct fiber of ϕ, such that

$$K_S + \mu A \sim_{\mathbb{Q}} aB + \sum_{i=1}^{7} a_i E_i$$

for some positive rational number a and non-negative rational numbers $a_1, \dots, a_7 < 1$. In particular, these curves generate the face $\Delta_{(S,A)}$. Vice versa, for every positive rational number a and non-negative rational numbers $a_1, \dots, a_7 < 1$ the divisor

$$-K_S + aB + \sum_{i=1}^{7} a_i E_i$$

is ample.

The followings describe the notations that we will use in the rest of the present paper. Unless otherwise mentioned, these notations are fixed from now until the end of the paper.

- When the morphism ϕ is birational, $\mu A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{7} a_i E_i$.
  Fixing the order $a_1 \geq a_2 \geq \cdots \geq a_8$, $s_A = \sum_{i=2}^{7} a_i$.
- When the morphism ϕ is conic bundle, $\mu A \sim_{\mathbb{Q}} -K_S + aB + \sum_{i=1}^{7} a_i E_i$.
  Fixing the order $a_1 \geq a_2 \geq \cdots \geq a_7$, $s_A = \sum_{i=2}^{7} a_i$.
- L_i is a -1-curve corresponding to a class $h - e_1 - e_i$.
- C is a -1-curve corresponding to a class $3h - 2e_1 - \sum_{j=7}^{8} e_j$.

3. Log Canonical Thresholds along Curves

$$\mu A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{7} a_i E_i + aB,$$

where $a = 0$ if ϕ is birational and $a_7 = 0$ if ϕ is conic bundle.

Under the notations of Section 2, by choosing six exceptional curves $D_1, \dots, D_5$, where $\{D_1, \dots, D_5\} \subset \{E_1, \dots, E_5, E_6\}$, we obtain the birational morphism $S \to S_7$, where S_7 is a del Pezzo surface of degree 7. And there exist two disjoint -1-curves D_6 and D_7 in S_7, We have the birational morphism $\pi : S \to \mathbb{P}^2$ defined by contraction of $D_1, \dots, D_7$. Let $d_1, \dots, d_5, d_6, d_7$ be divisor classes corresponding to $D_1, \dots, D_7$ respectively. If the morphism ϕ is birational, then simply we have that $\{D_1, \dots, D_7\} = \{E_1, \dots, E_7\}$.

If the morphism ϕ is conic bundle and factors through $\mathbb{F}_1$, then we have that $\{D_1, \dots, D_6\} = \{E_1, \dots, E_6\}$. And if the morphism ϕ is conic bundle and factors through $\mathbb{P}^1 \times \mathbb{P}^1$, then we have that the divisor $D' = \{E_1, \dots, E_6\} \setminus \{D_1, \dots, D_5\}$ corresponds to a class $h - d_6 - d_7$.

Note that the morphism π depends on E_i's. We call D_i as π-exceptional curve if it belongs to the set that defines the morphism described previously.

For an effective divisor D on a surface X, define a value $\sigma(D)$ to be

$$\mathrm{Max}\{a_i | D = \sum a_i D_i, \text{ where } D_i \text{ is an irreducible curve}\}.$$

Definition 3.1. On an algebraic surface X, we call the maximal multiplicity of divisor A as

$$\sup\{\sigma(D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ and } D \sim_{\mathbb{Q}} A\}.$$

Lemma 3.2. *Suppose that the maximal multiplicity α of μA is greater than one. Then either the maximal multiplicity is attained on an π-exceptional curve D_i or we have the following inequality*

$$\frac{2 + s_A + 2a_1 - a_6 - a_7 + 3a}{3} \geq \alpha.$$

Proof. Note that any effective divisor on S is generated by -1-curves of 240 types and $K_1, K_2 \in |-K_S|$ (cf. [13]). Suppose that an effective divisor $\alpha C + \Gamma$ is $\mathbb{Q}$-linearly equivalent to μA, where C is an irreducible curve and the support of Γ does not contain C. The curve C is linearly equivalent to $\sum b_i B_i$, where b_i is integer and B_i is -1-curve or K_i and $b_i = 1$ for all i by the maximality of α. . So we have that $\alpha(\sum B_i) + \Gamma \sim_{\mathbb{Q}} \mu A$.

Now consider the intersection

$$3 = \mu A \cdot h \geq \alpha \sum B_k \cdot h,$$

where $h = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. It means that $B_k \cdot h \leq 2$. Thus B_k is a π-exceptional curve or one of curves L_{ij} and C_{ijlmn} which correspond to classes $h - d_i - d_j$ and $2h - d_i - d_j - d_l - d_m - d_n$.

Now suppose that $\alpha L_{ij} + \Omega \sim_{\mathbb{Q}} \mu A$, where the support of Ω does not contain L_{ij}. Let C_{ip} and D_j be a curve corresponding to a class $3h - 2d_p - \sum_{k \neq p} d_k + d_i$ and d_j, where $p \notin \{i, j\}$. In the cases that ϕ is birational or ϕ is conic bundle which factor through $\mathbb{F}_1$, then the following inequality holds

$$2 + s_A + 2a_1 - a_6 - a_7 + 3a \geq 2 + s_A + 2a_p - a_i - a_j + 3a \geq (C_{ip} + D_j) \cdot \mu A \geq 3\alpha.$$

When the morphism ϕ is conic bundle and factors through $\mathbb{P}^1 \times \mathbb{P}^1$, Suppose that $\{D_1, \ldots, D_6\} = \{E_1, \ldots, E_6\}$. If $\{i, j\} \subset \{1, \ldots, 6\}$, then we have the same inequality

$$2 + s_A + 2a_1 - a_6 - a_7 + 3a \geq 2 + s_A + 2a_p - a_i - a_j + 3a \geq (C_{ip} + D_j) \cdot \mu A \geq 3\alpha.$$

If $\{i, j\} \subset \{6, 7\}$, that is, $L_{67} = E_6$, then contract $E_2, \ldots, E_5, E_6$, we have the morphism $S \to S_7$. And there are two disjoint -1-curves D_6, D_7 such that E_1

meets D_6, D_7. The contraction π' of $E_2, \dots, E_4, E_5, D_6, D_7$ defines the morphism $S \to \mathbb{P}^2$. Thus L_{67} is π'-exceptional. Now assume that $i = 5, j = 7$. then contract $E_1, \dots, E_4, E_6$, we have the morphism $S \to S_7$. And there is two disjoint -1-curves D_6, D_7 such that E_5 meets D_6, D_7. In addition L_{57} is one of D_6, D_7. The contraction π'' of $E_1, \dots, E_4, E_6, D_6, D_7$ defines the morphism $S \to \mathbb{P}^2$. Thus L_{57} is π''-exceptional.

$\alpha C_{ijlmn} + \Omega \sim_{\mathbb{Q}} \mu A$, where the support of Ω does not contain C_{ijlmn}. Let C_{ip} and D_j be a curve corresponding to a class $3h - 2d_p - \sum_{k \neq p} d_k + d_i$ and d_j, where $p \notin \{i, j, l, m, n\}$. Then in any case the following inequality holds

$$2 + s_A + 2a_1 - a_6 - a_7 + 3a \geq 2 + s_A + 2a_p - a_i - a_j + 3a \geq (C_{ip} + D_j) \cdot \mu A \geq 3\alpha.$$

□

In all lemmas of the present section, the maximal multiplicity is obtained on an exceptional curve by Lemma 3.2. In other words, we can always find a divisor whose multiplicity along E_i has at least the value of the bound in Lemma 3.2. For all π-exceptional curves, the processes to find out maximal multiplicity along the curve are the same. Thus we will consider only maximal multiplicity along single exceptional curve E_1 which computes $\alpha_c(S, A)$ according to the order of a_i's.

3.1. Birational morphism case. We assume that the contraction $\phi : S \to Z$ by the face $\Delta_{(S,A)}$ is birational. There are r disjoint (-1)-curves $E_1, \dots, E_r$ that generate the face $\Delta_{(S,A)}$, where $1 \leq r \leq 7$. In addition, we can find $(7 - r)$ disjoint (-1)-curves $E_{r+1}, \dots, E_7$ on S that intersect none of the (-1)-curves $E_1, \dots, E_r$. We are then can obtain a birational morphism $\pi : S \to \mathbb{P}^2$ by contracting the eight disjoint (-1)-curves $E_1, \dots, E_7$ on S to $\mathbb{P}^2$. Furthermore, we may write

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{7} a_i E_i,$$

where a_i's are rational numbers with $0 < a_i < 1$ for $i = 1, \dots, r$ and $a_i = 0$ for $i = r + 1, \dots, 7$.

Lemma 3.3. *If $s_A \geq 3$, then $\alpha_c(S, \mu A) = \frac{1}{2+a_1}$.*

Proof. There exist an effective divisor μA such that

$$\mu A \sim_{\mathbb{Q}} (1 - a_2)l_2 + \cdots + (1 - a_7)l_7 + (s_A - a_6 - 2)l_6 + (2 + a_1)E_1 + (s_A - 3)E_7.$$

Therefore we have that $\alpha_c(S, \mu A) \leq \frac{1}{2+a_1}$.

Now for $\eta < \frac{1}{2+a_1}$, suppose that the pair $(S, \eta D)$ is not log canonical along an irreducible curve C, where the effective divisor D is $\mathbb{Q}$-linearly equivalent to μA. Then we write $D = \alpha C + \Omega$, where the support of Ω does not contain C. Since the inequality $\frac{2+s_A+2a_1-a_6-a_7+3a}{3} \leq 2 + a_1$ holds, the curve C is one of E_i by Lemma 3.2.

We write $\mu A \sim_{\mathbb{Q}} D = \alpha E_i + \sum_{h \neq i} b_h E_h + \Omega$, where the support of Ω does not contain E_i and E_h's. Then we have

$$12 + 6a_i = (\sum_{p \neq i}^{7} L_p + \sum_{h \neq i}^{7} E_h) \cdot D \geq 6\alpha,$$

where the -1-curve L_p corresponds to a class of $h - e_p - e_i$, so that $2 + a_i \geq \alpha$. It is a contradiction, thus $\alpha_c(S, \mu A) = \frac{1}{2+a_1}$. □

Investigating the maximal multiplicity along E_i is same as the proof of the previous lemma. If $\mu A \sim_{\mathbb{Q}} D = \alpha_1 E_1 + \Omega$, where the support of Ω does not contain E_1. By computing an intersection number with a suitable divisors, we have an upper bound $f_1(a_1, \ldots, a_7)$ of α_1. Similarly, if $\mu A \sim_{\mathbb{Q}} D = \alpha_i E_i + \Omega_i$, where the support of Ω_i does not contain E_i. By symmetry of E_i, that is, substitutions of vectors $(a_1, \ldots, a_7)$ and $(E_1, \ldots, E_7)$ by $(a_i, a_1, \ldots, \hat{a}_i, \ldots, a_7)$ and $(E_i, E_1, \ldots, \hat{E}_i, \ldots, E_7)$, respectively in the intersection form, we obtain an upper bound $f_i = f_1(a_i, a_1, \ldots, \hat{a}_i, \ldots, a_7)$ of α_i. And by the order of $(a_1, \ldots, a_7)$, we always have that $f_1 \geq f_i$ so that the maximum is attained on E_1 among them. From now on we only consider the maximal multiplicity along E_1 of μA.

Lemma 3.4. *If $s_A \leq 3$, then $\alpha_c(S, \mu A) = \frac{3}{3+3a_1+s_A}$.*

Proof. There exists an effective divisor

$$\mu A \sim_{\mathbb{Q}} \frac{3 - s_A}{3} C + \sum_{i=2}^{7} a_i L_i + \frac{3 + 3a_1 + s_A}{3} E_1 + \sum_{i=2}^{7} (2a_i + \frac{s_A}{3}) E_i.$$

Thus we obtain that $\alpha_c(S, \mu A) \leq \frac{3}{3+3a_1+s_A}$.

Assume $\eta < \frac{3}{3+3a_1+s_A}$, suppose that the pair $(S, \eta D)$ is not log canonical along an irreducible curve C, where the effective divisor D is $\mathbb{Q}$-linearly equivalent to μA. Then we write $D = \alpha C + \Omega$, where the support of Ω does not contain C. Since the inequality $\frac{2+s_A+2a_1-a_6-a_7+3a}{3} \leq \frac{1+2a_1+s_A}{2}$ holds, the curve C is one of E_1 by Lemma 3.2.

If we set an effective divisor D as $D = \alpha E_1 + bZ + \sum_{i=2}^{8} c_i C_i + \Omega$, where the support of Ω does not contain $E_1, C_2, \dots, C_8$ and Z, then we have the following inequality

$$9 + 9a_1 + 3s_A = (-K_S + C + \sum_{i=2}^{7} L_i) \cdot D \geq 9\alpha.$$

Thus $\alpha_c(S, \mu A) = \frac{3}{3+3a_1+s_A}$. □

Combining with Lemma 3.3 and Lemma 3.4, we have the following theorem.

Theorem 3.5. *Let S be a smooth del Pezzo surface of degree 2 and A be an ample divisor of S . If the morphism ϕ is birational, $\mu A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{7} a_i E_i$, then*

- *When $s_A > 4$, $\alpha_c(S, A) = \frac{1}{2+a_1}$;*
- *When $3 \geq s_A$, $\alpha_c(S, A) = \frac{3}{3+3a_1+s_A}$.*

Conjecture 3.6. *If S be a smooth del Pezzo surface of degree 2 , then we have $\alpha(S, A) = \alpha_c(S, A)$ for any ample divisor A in S.*

Although similar conjecture provided for del Pezzo surface of degree 1 by the paper [9], the paper [2] gave counter-example of the conjecture so the conjecture is false. However we expect the conjecture is true in del Pezzo surface of degree 2 besides degree 1 case.

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DEPARTMENT OF MATHEMATICS, EWHA WOMANS UNIVERSITY, SEOUL 03760, KOREA
Email address: `leonwon@ewha.ac.kr`