J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2022.29.2.201
Volume 29, Number 2 (May 2022), Pages 201–206

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

FACTORIAL NODAL COMPLETE INTERSECTION 3-FOLDS IN $\mathbb{P}^5$

Kyusik Hong

Abstract. Let X be a nodal complete intersection 3-fold defined by a hypersurface in $\mathbb{P}^5$ of degree n and a smooth quadratic hypersurface in $\mathbb{P}^5$. Then we show that X is factorial if it has at most $n^2 - n + 1$ nodes and contains no 2-planes, where $n = 3, 4$.

1. Introduction

All varieties are assumed to be projective, normal and defined over $\mathbb{C}$. A variety is called nodal if all its singular points are only ordinary double points, i.e., nodes. Also, a variety is called factorial if every Weil divisor on it is Cartier. From now on, we shall denote by $\mathbb{NCIT}(n, m)$ a nodal complete intersection threefold of two hypersurfaces G_n and G_m in $\mathbb{P}^5$ of degree n and m, $n \geq m$, respectively, such that G_m is smooth. In the present article, we study the factoriality of $\mathbb{NCIT}(n, m)$.

The factoriality depends both on local types of singularities and on their global position. Note that a smooth threefold is factorial. Cheltsov [2] obtained a sharp bound on the number of nodes on a factorial nodal hypersurface in $\mathbb{P}^4$.

Theorem 1. *If* $\#|\operatorname{Sing}(\mathbb{NCIT}(n, 1))| < (n-1)^2$, *then* $\mathbb{NCIT}(n, 1)$ *is factorial.*

For $m \geq 2$, Kosta [9] proved the following result.

Theorem 2. *If* $\#|\operatorname{Sing}(\mathbb{NCIT}(n, m))| < (n+m-2)^2 - (n+m-2)(m-1)$, *then* $\mathbb{NCIT}(n, m)$ *is factorial.*

Thereafter, Cynk and Rams [5], Kloosterman [8] consider the case of a nodal complete intersection in projective space of dimension ≥ 5. Let N be a nodal complete intersection threefold in $\mathbb{P}^{3+c}$ defined by homogeneous equations $f_1, \ldots, f_c$

Received by the editors April 12, 2022. Accepted April 26, 2022.
2010 *Mathematics Subject Classification.* 14C20, 14J17, 14J30.
Key words and phrases. complete intersection 3-fold, nodal variety, factoriality.

of multidegree $d_1, \ldots, d_c$ with $d_1 \leq \cdots \leq d_c$. Suppose that the set $V(f_1, \ldots, f_i)$ is smooth in codimension 3 for $i \leq c-1$. Cynk and Rams [5] gave a sharp bound on the minimal number of nodes of N which contains a smooth complete intersection surface that is not a Cartier divisor. On the other hand, Kloosterman [8] gave a sharp bound on the minimal number of nodes of N which has a slightly different non-degeneracy condition than Cynk and Rams, and assume that either $c = 2$ or $d_1 + \ldots + d_{c-1} < d_c$.

The aim of this article is to give some examples of a factorial $\mathbb{NCIT}(n, m)$ which has many singular points greater than the bound of Kosta [9]. Two papers [1, 9], Example 7, Lemma 8 enable us to propose the conjecture below.

Conjecture 3. *$\mathbb{NCIT}(n, m)$ is factorial if it has at most $(n+m-2)^2 - (n-1)(m-1)$ nodes and contains no planes.*

Remark 4. Because, for $m = 1$, a complete proof of Conjecture 3 was given in the paper [3], we may assume that $m \geq 2$.

Our main result is the following:

Theorem 5. *Conjecture 3 holds for $\mathbb{NCIT}(3, 2)$ and $\mathbb{NCIT}(4, 2)$.*

2. Preliminaries

To check the factoriality of $\mathbb{NCIT}(n, m)$, we use the following theorem.

Theorem 6. *$\mathbb{NCIT}(n, m)$ is factorial if the points of $\operatorname{Sing}(\mathbb{NCIT}(n, m))$ impose independent linear conditions on sections in $H^0(\mathcal{O}_{\mathbb{P}^5}(2n + m - 6)|_{G_m})$.*

Proof. See [4, Theorem 2] and [1, Corollary 14]. □

Now, we present a non-factorial $\mathbb{NCIT}(n, m)$, which motivates our study.

Example 7 ([9, Example 1.2])**.** Let X be a complete intersection of two smooth hypersurfaces

$$\begin{cases} G_n := xf(x, y, z, w, s, t) + yg(x, y, z, w, s, t) + zh(x, y, z, w, s, t) = 0, \\ G_m := x\tilde{f}(x, y, z, w, s, t) + y\tilde{g}(x, y, z, w, s, t) + z\tilde{h}(x, y, z, w, s, t) = 0, \end{cases}$$

in $\mathbb{P}^5 \cong \operatorname{Proj}\left(\mathbb{C}[x, y, z, w, s, t]\right)$ of degree n and m, $n \geq m$, respectively. Then X has exactly $(n + m - 2)^2 - (n - 1)(m - 1)$ nodes and contains the plane π defined by $\{x = y = z = 0\}$. In this case, X is not factorial and the set $\operatorname{Sing}(X)$ lies on the plane π.

From the above example, if a plane is contained in $\mathbb{NCIT}(n,m)$, then $\mathbb{NCIT}(n,m)$ is not factorial. More precisely, we have the following result.

Lemma 8. *If* $\mathbb{NCIT}(n,m)$ *contains a plane, then* $\mathbb{NCIT}(n,m)$ *has at least* $(n+m-2)^2-(n-1)(m-1)$ *nodes and is not factorial.*

Proof. Assume that a plane π is given by $\{x=y=z=0\}$ such that $\pi \subset \mathbb{NCIT}(n,m) \subset \mathbb{P}^5 \cong \mathrm{Proj}(\mathbb{C}[x,y,z,w,s,t])$. Then $\mathbb{NCIT}(n,m)$ can be written as a complete intersection of two hypersurfaces

$$\begin{cases} G_n := xf(x,y,z,w,s,t)+yg(x,y,z,w,s,t)+zh(x,y,z,w,s,t)=0, \\ G_m := x\tilde{f}(x,y,z,w,s,t)+y\tilde{g}(x,y,z,w,s,t)+z\tilde{h}(x,y,z,w,s,t)=0 \end{cases}$$

of degree n and m, $n \geq m$, respectively, where G_m is smooth. Because $\mathbb{NCIT}(n,m)$ has only ordinary double points as singularities, the set $\mathrm{Sing}(\mathbb{NCIT}(n,m))$ is contained in the set given by the system of five equations

$$\{x=y=z=f\tilde{g}-g\tilde{f}=f\tilde{h}-h\tilde{f}=0\},$$

for $q \in \mathrm{Sing}(\mathbb{NCIT}(n,m))$, the plane π, $\{f\tilde{g}-g\tilde{f}=0\}, \{f\tilde{h}-h\tilde{f}=0\}$ meet transversally at the point q. Note that if $s \in \{x=y=z=f=\tilde{f}=0\} \subset \{x=y=z=f\tilde{g}-g\tilde{f}=f\tilde{h}-h\tilde{f}=0\}$, then $s \notin \mathrm{Sing}(\mathbb{NCIT}(n,m))$. Therefore, $\mathbb{NCIT}(n,m)$ has at at least $(n+m-2)^2-(n-1)(m-1)$ nodes and is not factorial. □

3. Useful Tools

The following result is originally due to the paper [6]. It help us to make our proofs simpler.

Theorem 9. *Let* Σ *be a sets in* $\mathbb{P}^N$ *and let* $d \geq 2$ *be an integer. If no* $dk+2$ *points of* Σ *lie in a projective* k*-plane for all* $k \geq 1$*, then* Σ *imposes linearly independent conditions on forms of degree* d *in* $\mathbb{P}^N$*.*

Proof. See [7]. □

Let $V_{n,m}$ be a nodal complete intersection of two hypersurfaces F_n and F_m in $\mathbb{P}^5$ of degree n and m, $n \geq m$, respectively. Then the set of $\mathbb{NCIT}(n,m)$ is contained in the set of $V_{n,m}$.

Remark 10. Lemma 8 holds for $V_{n,m}$.

Lemma 11. *There exists a hypersurface $\tilde{F_n}$ in $\mathbb{P}^5$ of degree n such that $V_{n,m} = \tilde{F_n} \cap F_m$ and* $\mathrm{Sing}(\tilde{F_n}) \subseteq \mathrm{Sing}(V_{n,m})$.

Proof. [1, Lemma 33]. □

Moreover, we have two lemmas about the position of $\mathrm{Sing}(V_{n,m})$.

Lemma 12. *The following assertions hold:*

(1) A curve of degree l in $\mathbb{P}^5$ contains at most $l(n+m-2)$ nodes of $V_{n,m}$;
(2) If a plane contains $\lfloor \frac{n(n+m-2)}{2} \rfloor + 1$ nodes of $V_{n,m}$, then the plane is contained in $V_{n,m}$.

Proof. Let $V_{n,m} \subset \mathbb{P}^5 \cong \mathrm{Proj}(\mathbb{C}[x,y,z,w,s,t])$. Suppose that $V_{n,m}$ is given by a system of equations

$$\begin{cases} F_n := f(x,y,z,w,s,t) = 0, \\ F_m := g(x,y,z,w,s,t) = 0. \end{cases}$$

Then the singular locus of $V_{n,m}$ is contained in the hypersurface

$$V'_{n,m} := \alpha_1\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \alpha_2\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + \ldots + \alpha_5\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial x}\right) = 0 \tag{13}$$

of degree $n+m-2$, where $\alpha_i \in \mathbb{C}$. Let $C \subset \mathbb{P}^5$ be a curve of degree l. Since $V_{n,m}$ has only nodes as singularities, $C \cap V'_{n,m}$ is zero-dimensional. Thus C contains at most $l(n+m-2)$ singular points of $V_{n,m}$.

If $V_{n,m}$ contains no a plane π, then $\pi \not\subset (F_n \cup F_m)$. We may assume that $\pi \not\subset F_n$, since $n \geq m$. Then the curve $\pi \cap F_n$ is singular where F_n is singular. By Lemma 11, we assume that $\mathrm{Sing}(F_n) \subseteq \mathrm{Sing}(V_{n,m})$. Then the curve $\pi \cap F_n$ can pass through at most $\lfloor \frac{n(n+m-2)}{2} \rfloor$ points of $\mathrm{Sing}(V_{n,m})$. □

Lemma 14. *Let $\Xi_{n,m,r} = \mathrm{Sing}(V_{n,m}) \cap \mathrm{Sing}(S_r)$, where S_r is a surface of degree $r \geq 2$, and let $\#|\Xi_{n,m,r}|$ be a number of $\Xi_{n,m,r}$.*

(1) If a plane $\pi \subset S_r$ contains $\lfloor \frac{n(n+m-2)}{2} \rfloor - \#|\Xi_{n,m,r}| + 1$ nodes of $V_{n,m}$, then $\pi \subset V_{n,m}$;
(2) If an irreducible component S_i of S_r contains $\lfloor \frac{in(n+m-2)}{2} \rfloor - \#|\Xi_{n,m,i}| + 1$ nodes of $V_{n,m}$, then $S_i \subset V_{n,m}$.

Proof. Suppose that a plane $\pi \subset S_r$. Using the notation in (13), let $S_r \cap V'_{n,m}|_\pi$ be the restriction of $S_r \cap V'_{n,m}$ to π. Because $V_{n,m}$ has only isolated singularities, $S_r \cap V'_{n,m}$ is a curve of degree $r(n+m-2)$. Moreover, the curve $S_r \cap V'_{n,m}|_\pi$ of degree

$n+m-2$ is singular where $S_r \cap V'_{n,m}$ is singular. If $V_{n,m}$ contains no planes, then $S_r \cap V'_{n,m}|_\pi$ can pass through at most $\lfloor \frac{n(n+m-2)}{2} \rfloor - \#|\Xi_{n,m,r}|$ points of $\mathrm{Sing}(V_{n,m})$.

Note that $i \le r$. Assume that $S_i \not\subset V_{n,m}$. Then $S_i \cap V'_{n,m}$ is a curve of degree $i(n+m-2)$ not contained in $V_{n,m}$. Therefore, $S_i \cap V'_{n,m}$ cannot meet $V_{n,m}$ at more than $\lfloor \frac{in(n+m-2)}{2} \rfloor - \#|\Xi_{n,m,i}|$ points of $\mathrm{Sing}(V_{n,m})$. □

4. A Proof of Theorem 5

We assume that $\#|\mathrm{Sing}(\mathbb{NCIT}(n,m))| \le (n+m-2)^2-(n-1)(m-1)$. To prove the factoriality of $\mathbb{NCIT}(n,m)$, by Theorem 6, for $p \in \mathrm{Sing}(\mathbb{NCIT}(n,m))$ we will construct a hypersurface in $\mathbb{P}^5$ of degree $2n+m-6$ that contains all the points of $\mathrm{Sing}(\mathbb{NCIT}(n,m)) \setminus \{p\}$ but not the point p, in other word, the set $\mathrm{Sing}(\mathbb{NCIT}(n,m))$ is $(2n+m-6)$-normal in $\mathbb{P}^5$. By Lemma 12(2), we assume that a plane contains at most $\lfloor \frac{n(n+m-2)}{2} \rfloor$ points of $\mathrm{Sing}(\mathbb{NCIT}(n,m))$, otherwise, $\mathbb{NCIT}(n,m)$ contains a plane and not factorial by Lemma 8.

4.1. A sextic threefold $\mathbb{NCIT}(3,2)$ in $\mathbb{P}^5$ Suppose that $\#|\mathrm{Sing}(\mathbb{NCIT}(3,2))| \le 7$ and no 5 points of $\mathrm{Sing}(\mathbb{NCIT}(3,2))$ lie on a single plane. By Lemma 12(1), a line contains at most 3 singular points of $\mathbb{NCIT}(3,2)$. Then the set $\mathrm{Sing}(\mathbb{NCIT}(3,2))$ satisfies the condition of Theorem 9, since $\#|\mathrm{Sing}(\mathbb{NCIT}(3,2))| \le 7$. Thus, for $p \in \mathrm{Sing}(\mathbb{NCIT}(3,2))$ we can find a quadratic hypersurface in $\mathbb{P}^5$ that contains all the points of $\mathrm{Sing}(\mathbb{NCIT}(3,2)) \setminus \{p\}$ but not the point p, and $\mathbb{NCIT}(3,2)$ is factorial by Theorem 6.

4.2. A octic threefold $\mathbb{NCIT}(4,2)$ in $\mathbb{P}^5$ Assume that $\#|\mathrm{Sing}(\mathbb{NCIT}(4,2))| \le 13$ and a 2-plane contains at most 8 nodes of $\mathbb{NCIT}(4,2)$. By Lemma 12(1), a line passes through at most 4 nodes of $\mathbb{NCIT}(4,2)$. Hence, by Theorem 9, the points of $\mathrm{Sing}(\mathbb{NCIT}(4,2))$ impose independent linear conditions on forms of degree 4 in $\mathbb{P}^5$, i,e., the set $\mathrm{Sing}(\mathbb{NCIT}(4,2))$ is 4-normal in $\mathbb{P}^5$, and $\mathbb{NCIT}(4,2)$ is factorial by Theorem 6.

References

1. Cheltsov, I.: Factoriality of nodal three-dimensional varieties and connectedness of the locus of log canonical singularities. Mat. Sb. **197** (2006), 387-414.
2. Cheltsov, I.: Factorial threefold hypersurfaces. J. Algebraic Geom. **19** (2010), 781-791.
3. Cheltsov, I.: On a conjecture of Ciliberto. Sb. Math. **201** (2010), 1069-1090.
4. Cynk, S.: Defect of a nodal hypersurface. Manuscripta Math. **104** (2001), 325-331.

5. Cynk, S. & Rams, S.: Non-factorial nodal complete intersection threefolds. Commun. Contemp. Math. **15**, no. 5, 1250064, 14 pp (2013)
6. Edmonds, J.: Minimum partition of a matroid into independent subsets. J. Res. Nat. Bur. Standards Sect. B **69 B** (1965), 67-72.
7. Eisenbud, D. & Koh, J.-H.: Remarks on points in a projective space. in: Commutative Algebra, Berkeley, CA, in: Mathematical Science Research Institute Publications, vol. **15**, Springer, New York, 1987, pp. 157-172.
8. Kloosterman, R.: Nodal complete intersection threefold with defect. arXiv preprint arXiv:1503.05420. (2015), 1-21.
9. Kosta, D.: Factoriality of complete intersections in $\mathbb{P}^5$. Proc. Steklov Inst. Math. **264** (2009), 102-109.

DEPARTMENT OF MATHEMATICS EDUCATION, JEONJU UNIVERSITY, JEONJU 55069, REPUBLIC OF KOREA
Email address: kszooj@jj.ac.kr