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GENERALIZED RELATIVE ORDER (α, β) BASED SOME GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper we wish to establish some results relating to the growths of composition of two entire functions with their corresponding left and right factors on the basis of their generalized relative order (α, β) and generalized relative lower order (α, β) where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1. Introduction, Definitions and Notations

For any entire function f defined in the open complex plane $\mathbb{C}$, the maximum modulus function $M_f(r)$ is defined as $M_f(r) = \max\limits_{|z|=r} |f(z)|$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim\limits_{s\rightarrow\infty} M_f^{-1}(s) = \infty$. The maximum term $\mu_f(r)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ can be defined in the following way:

$$\mu_f(r) = \max_{n\geq 0} (|a_n| r^n).$$

We use the standard notations and definitions of the theory of entire functions which are available in [11] and [12], and therefore we do not explain those in details.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \rightarrow +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \rightarrow +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0.

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Considering this, the value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

is called [8] generalized order (α, β) of an entire function f. For details about generalized order (α, β) one may see [8]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different directions. For the purposes of further applications, Biswas et al. [3] rewrote the definition of the generalized order (α, β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [8]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [6]:

Definition 1.1 ([3])**.** The generalized order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}$$

Since for $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R)\{cf.\ [10]\ \},$$

it is easy to see that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\mu_f(r))}{\beta(r)}$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progresses in the study of relative order, Biswas et al. [4] introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 1.2 ([4])**.** Let $\alpha,\ \beta \in L^0$. The generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function f with respect to an entire function g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

In terms of maximum terms of entire functions, Definition 1.2 can be reformulated as:

Definition 1.3 ([5])**.** Let α, $\beta \in L^0$. The growth indicators $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to another entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(\mu_g^{-1}(\mu_f(r)))}{\beta(r)}.$$

In fact, Definition 1.2 and Definition 1.3 are equivalent (e.g. see, [5]).

In the paper we wish to establish some newly developed results based on the comparative growth of composite entire functions on the basis of their generalized relative order (α, β) and generalized relative lower order (α, β).

2. Known Results

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([7])**.** *Let f and g are any two entire functions with $g(0) = 0$. Also let b satisfy $0 < b < 1$ and $c(b) = \frac{(1-b)^2}{4b}$. Then for all sufficiently large values of r,*

$$M_f(c(b)M_g(br)) \le M_{f\circ g}(r) \le M_f(M_g(r)).$$

In addition if $b = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \ge M_f\Big(\frac{1}{8}M_g\Big(\frac{r}{2}\Big)\Big).$$

Lemma 2.2 ([9])**.** *Let f and g be entire functions. Then for every $\delta > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\delta}{\delta - 1}\mu_f\Big(\frac{\delta R}{R-r}\mu_g(R)\Big).$$

Lemma 2.3 ([9])**.** *If f and g are any two entire functions. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\Big(\frac{1}{16}\mu_g\Big(\frac{r}{4}\Big)\Big).$$

Lemma 2.4 ([2])**.** *Suppose f is an entire function and $A > 1$, $0 < B < A$. Then for all sufficiently large r,*

$$M_f(Ar) \ge BM_f(r).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g, h and k be any four entire functions such that $\rho_{(\alpha_1,\beta_1)}[f \circ g]_h < \infty$ and $\lambda_{(\alpha_3,\beta_3)}[g]_k > 0$. Then*

$$\lim_{r\to+\infty} \frac{\{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r))))\}^2}{\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(\log r)))) \cdot \alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(r))))} = 0.$$

Proof. For arbitrary positive ε we have for all sufficiently large values of r that

$$\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r)))) \leq (\rho_{(\alpha_1,\beta_1)}[f \circ g]_h + \varepsilon) \log r. \tag{3.1}$$

Again for all sufficiently large values of r we get

$$\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(\log r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) \log r. \tag{3.2}$$

Similarly for all sufficiently large values of r we have

$$\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) r. \tag{3.3}$$

From (3.1) and (3.2) we have for all sufficiently large values of r that

$$\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(\log r))))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}[f \circ g]_h + \varepsilon) \log r}{(\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) \log r}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain from above that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(\log r))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f \circ g]_h}{\lambda_{(\alpha_3,\beta_3)}[g]_k}. \tag{3.4}$$

Again from (3.1) and (3.3) we get for all sufficiently large values of r that

$$\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(r))))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}[f \circ g]_h + \varepsilon) \log r}{(\lambda_{(\alpha_3,\beta_3)}[g]_k - \varepsilon) r}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows from above that

$$\lim_{r\to+\infty} \frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\mu_k^{-1}(\mu_g(\beta_3^{-1}(r))))} = 0. \tag{3.5}$$

Thus the theorem follows from (3.4) and (3.5). $\square$

Theorem 3.2. *Let f, g, h, k, l and m be six entire functions such that $\rho_{(\alpha_1,\beta_1)}[f]_l < +\infty$, $\lambda_{(\alpha_3,\beta_3)}[h]_m > 0$, $\lambda_{(\alpha_4,\beta_4)}[k] > 0$ and $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$. Also let C and D be any two positive constants.*

(i) Any one of the following four conditions are assumed to be satisfied:

(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*
(b) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*;*
(c) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*
(d) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*, then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))} = \infty.$$

(ii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*;*
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))} = \infty.$$

(iii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_3(r) = D\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L^0$*;*
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L^0$*, then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))} = \infty.$$

(iv) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L^0$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))} = \infty.$$

Proof. Taking $R = 2r$ in Lemma 2.2 we obtain for all sufficiently large values of r that

$$\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant$$

$$(1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)\beta_1(\mu_g(2\beta_2^{-1}(\log r))). \tag{3.6}$$

Case I. Let $\beta_1(r) = C(\exp(\alpha_2(r)))$. Then we have from (3.6) for all sufficiently large values of r that

$$\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.7}$$

Case II. Let $\exp(\alpha_2(r)) > \beta_1(r)$. Then we have from (3.6) for all sufficiently large values of r that

$$\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.8}$$

CASE III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then we get from(3.6) for all sufficiently large values of r that

$$\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))))) \leqslant r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.9}$$

Further in view of the inequalities $\mu_g(r) \leq M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ [10]$\}$, for $0 \leq r < R$, it follows from Lemma 2.3 and Lemma 2.4 for all sufficiently large values r that

$$\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))) \geq \alpha_3\Big(\mu_m^{-1}\Big(\mu_h\Big(\frac{1}{80}\mu_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\Big)$$

i.e., $\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))) \geq$

$$(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)\beta_3\Big(\mu_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big). \tag{3.10}$$

CASE IV. Let $\beta_3(r) = D\exp(\alpha_4(r))$ Then from (3.10) it follows for all sufficiently large values of r that

$\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))) \geq$

$$D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.11}$$

CASE V. Let $\beta_3(r) > \exp(\alpha_4(r))$. Now from (3.10) it follows for all sufficiently large values of r that

$\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))) >$

$$(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.12}$$

CASE VI. Let $\alpha_4(\beta_3^{-1}(r)) \in L^0$. Then from (3.10) we obtain for all sufficiently large values of r that

$$\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))))) \geqslant r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{3.13}$$

Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$ we can choose $\varepsilon(>0)$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_4,\beta_4)}[k] - \varepsilon. \tag{3.14}$$

Now combining (3.7) of Case I and (3.11) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))} \geq \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]_m - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_l + \varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (3.14) and above we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}=\infty. \tag{3.15}$$

Similarly combining (3.7) of Case I and (3.12) of Case V we get that

$$\liminf_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}=\infty. \tag{3.16}$$

Analogously combining (3.8) of Case II and (3.11) of Case IV, we obtain that

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}=\infty. \tag{3.17}$$

Likewise combining (3.8) of Case II and (3.12) of Case V it follows that

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}=\infty. \tag{3.18}$$

Hence the first part of the theorem follows from (3.15), (3.16), (3.17) and (3.18).

Again combining (3.7) of Case I and (3.13) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}\geq \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_l+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (3.14) and above we obtain that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}=\infty.$$

Similarly combining (3.8) of Case II and (3.13) of Case VI we also get same conclusion. Therefore the second part of the theorem is established.

Again combining (3.9) of Case III and (3.11) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}\geq \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]_m-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}. \tag{3.19}$$

Now in view of (3.14) we obtain from (3.19) that

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}=\infty. \tag{3.20}$$

Similarly combining (3.9) of Case III and (3.12) of Case V we get that

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}=\infty. \tag{3.21}$$

Hence the third part of the theorem follows from (3.20) and (3.21).

Further combining (3.9) of Case III and (3.13) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}\geq\frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Now in view of (3.14) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\mu_m^{-1}(\mu_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}=\infty.$$

This proves the fourth part of the theorem.

Thus the theorem follows. □

Theorem 3.3. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\rho_{(\alpha_2,\beta_2)}[g]>0$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\geq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definition of $\rho_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))\leq(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r. \tag{3.22}$$

Further in view of the inequalities $\mu_g(r)\leq M_g(r)\leq\frac{R}{R-r}\mu_g(R)$ $\{cf.$ [10]$\}$, for $0\leq r<R$, it follows from Lemma 2.3 and Lemma 2.4 for all sufficiently large values r that

$$\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))\geq(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\beta_1\Big(\mu_g\Big(\frac{\beta_2^{-1}(r)}{4}\Big)\Big).$$

Since $\alpha_2(\beta_1^{-1}(r))\in L^0$, we obtain from above for a sequence of values of r tending to infinity that

$$\begin{aligned}\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))&\geq(1+o(1))\alpha_2\Big(\mu_g\Big(\frac{\beta_2^{-1}(r)}{4}\Big)\Big)\\ \text{i.e., }\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))&\geq(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]-\varepsilon)r.\end{aligned}$$

Now combining (3.22) and above we get that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\geq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Hence the theorem follows. □

Theorem 3.4. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g]>0$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

Theorem 3.5. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g]>0$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

The proofs of Theorem 3.4 and Theorem 3.5 would run parallel to that of Theorem 3.3. We omit the details.

Theorem 3.6. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\rho_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\leq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))\geq(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)r. \tag{3.23}$$

Further taking $R=2r$ in Lemma 2.2 we obtain for all sufficiently large values of r that

$$\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(r)))\leqslant(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\mu_g(2r)). \tag{3.24}$$

Since $\alpha_2(\beta_1^{-1}(r))\in L^0$, we obtain from above for all sufficiently large values of r that

$$\begin{aligned}\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))&\leq(1+o(1))\alpha_2(\mu_g(2\beta_2^{-1}(r)))\\ \textit{i.e.,}\ \alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))&\leq(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)r.\end{aligned}$$

Now combining (3.23) and above we get that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\leq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Hence the theorem follows. □

Theorem 3.7. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\lambda_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\leq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Theorem 3.8. *Let f, g and h be any three entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$ and $\rho_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(r))))))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(r))))}\leq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\rho_{(\alpha_1,\beta_1)}[f]_h}.$$

The proofs of Theorem 3.7 and Theorem 3.8 would run parallel to that of Theorem 3.6. We omit the details.

Remark 3.9. Theorem 3.1 to Theorem 3.8 can also be deduced in terms of maximum modulus of entire functions with the help of Lemma 2.1.

Theorem 3.10. *Let f, g and h be any three entire functions such that $\rho_{(\alpha_2,\beta_2)}[g]<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h$. Also let C be any positive constant and $\beta_1\in L^0$.*
(i) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r)=C(\exp(\alpha_2(r)))$;
(b) $\exp(\alpha_2(r))>\beta_1(r)$; then

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

Proof. From the definition of generalized relative lower order (α_1,β_1) of f with respect to h, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\geq r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}.\tag{3.25}$$

As $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h$ we can choose $\varepsilon(>0)$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon. \tag{3.26}$$

Now in view of (3.7) of Case I and (3.25) we have for all large positive numbers of r,

$$\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} \leq \frac{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}.$$

In view of (3.26) we get from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} = 0. \tag{3.27}$$

Again in view of (3.8) of Case II and (3.25) it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} \leq \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}.$$

Now in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} = 0. \tag{3.28}$$

Further in view of (3.9) of Case III and (3.25) it follows for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} \leq \frac{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}.$$

So in view of (3.26) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} = 0. \tag{3.29}$$

Now in view of (3.6) we get that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))} \leq \rho_{(\alpha_1,\beta_1)}[f]_h. \tag{3.30}$$

From (3.27) and (3.30) we obtain for all sufficiently large values of r that

$$\begin{aligned}
&\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r)))) \cdot \beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\\
=\ &\lim_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))} \cdot \limsup_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}\\
\leq\ &0\cdot\rho_{(\alpha_1,\beta_1)}[f]_h = 0.
\end{aligned} \tag{3.31}$$

Similarly from (3.28) and (3.30) we obtain that

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

Therefore the first part of the theorem follows from (3.31) and above.

Again from (3.29) and (3.30) we get for all large values of r that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}$$
$$=\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))}\cdot\limsup_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}$$
$$\leq 0\cdot\rho_{(\alpha_1,\beta_1)}[f]_h=0.$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\mu_l^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\log r))))\cdot\beta_1(\mu_g(2\beta_2^{-1}(\log r)))}=0.$$

Thus the second part of the theorem is established. □

In the line of Theorem 3.10 and with the help of Lemma 2.1, one can easily proof the following theorem and therefore its proof is omitted:

Theorem 3.11. *Let f, g and h be any three entire functions such that $\rho_{(\alpha_2,\beta_2)}[g]<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h$. Also let C be any positive constant and $\beta_1\in L^0$.*
(i) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r)=C(\exp(\alpha_2(r)))$;
(b) $\exp(\alpha_2(r))>\beta_1(r)$, then

$$\limsup_{r\to+\infty}\frac{\{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))\}^2}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))\cdot\beta_1(M_g(\beta_2^{-1}(\log r)))}=0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r))\in L^0$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(M_h^{-1}(M_f(\beta_1^{-1}(\log r)))))\cdot\beta_1(M_g(\beta_2^{-1}(\log r)))}=0.$$

Theorem 3.12. *Let f, g, h, l and k be any five entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f]_h<\infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k>0$ and $\rho_{(\alpha_3,\beta_3)}[f\circ g]_l<\infty$ where $\alpha_2,\beta_1\in L^0$. Then*

$$\limsup_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r))))\cdot\alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r))))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))\cdot\alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r))))}\leq\frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]_l\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

Proof. For all sufficiently large values of r we have

$$\alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r)))) \leq (\rho_{(\alpha_3,\beta_3)}[f\circ g]_l + \varepsilon)r. \tag{3.32}$$

Again for all sufficiently large values of r it follows that

$$\alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r)))) \geq (\lambda_{(\alpha_2,\beta_2)}[g]_k - \varepsilon)r. \tag{3.33}$$

Now combining (3.32) and (3.33) we have for all sufficiently large values of r that

$$\frac{\alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r))))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]_l + \varepsilon}{\lambda_{(\alpha_2,\beta_2)}[g]_k - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r))))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]_l}{\lambda_{(\alpha_2,\beta_2)}[g]_k}. \tag{3.34}$$

Now from(3.30) and (3.34) we obtain that

$$\begin{aligned}
&\limsup_{r\to+\infty}\frac{\alpha_1(\mu_h^{-1}(\mu_{f\circ g}(\beta_2^{-1}(\log r)))) \cdot \alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r))))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r))) \cdot \alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r))))}\\
\leq\;& \limsup_{r\to+\infty}\frac{\alpha_1(\mu_{f\circ g}(\beta_2^{-1}(\log r)))}{\beta_1(\mu_g(2\beta_2^{-1}(\log r)))} \cdot \limsup_{r\to+\infty}\frac{\alpha_3(\mu_l^{-1}(\mu_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\mu_k^{-1}(\mu_g(\beta_2^{-1}(r))))}\\
\leq\;& \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]_l \cdot \rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.
\end{aligned}$$

Hence the theorem follows. □

In the line of Theorem 3.12 and with the help of Lemma 2.1, one can easily proof the following theorem and therefore its proof is omitted:

Theorem 3.13. *Let f, g, h, l and k be any five entire functions such that $\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\rho_{(\alpha_3,\beta_3)}[f\circ g]_l < \infty$ where $\alpha_2, \beta_1 \in L^0$. Then*

$$\begin{aligned}
&\limsup_{r\to+\infty}\frac{\alpha_1(M_h^{-1}(M_{f\circ g}(\beta_2^{-1}(\log r)))) \cdot \alpha_3(M_l^{-1}(M_{f\circ g}(\beta_3^{-1}(r))))}{\beta_1(M_g(2\beta_2^{-1}(\log r))) \cdot \alpha_2(M_k^{-1}(M_g(\beta_2^{-1}(r))))}\\
&\leq \frac{\rho_{(\alpha_3,\beta_3)}[f\circ g]_l \cdot \rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.
\end{aligned}$$

Theorem 3.14. *Let f, g, h and k be any four entire functions such that $\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\lambda_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r)))}{\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r))) \leq \Delta \cdot \alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\beta_3(r))))). \tag{3.35}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}[f]_h$, it follows for all sufficiently large values of r that

$$\alpha_1(\mu_h^{-1}(\mu_f(\beta_1^{-1}(\beta_3(r))))) \leq (\rho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r). \tag{3.36}$$

Thus from (3.35) and (3.36), we have for a sequence of values of r tending to infinity that

$$\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r))) \leq \Delta(\rho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r)$$

$$i.e., \ \frac{\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r)))}{\beta_3(r)} \leq \frac{\Delta(\rho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r)}{\beta_3(r)}$$

$$i.e., \ \liminf_{r+\infty} \frac{\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r)))}{\beta_3(r)} = \lambda_{(\alpha_3,\beta_3)}[f\circ g]_k < \infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 3.15. Theorem 3.14 is also valid with "limit superior" instead of "limit" if $\lambda_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$ is replaced by $\rho_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$ and the other conditions remain the same.

Analogously one may also state the following theorem without its proof as it may be carried out in the line of Theorem 3.14.

Theorem 3.16. *Let f, g, h and k be any four entire functions such that $\rho_{(\alpha_1,\beta_1)}[g]_h < \infty$ and $\rho_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha_3(\mu_k^{-1}(\mu_{f\circ g}(r)))}{\alpha_1(\mu_h^{-1}(\mu_g(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Remark 3.17. Theorem 3.16 is also valid with "limit" instead of "limit superior" if $\rho_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$ is replaced by $\lambda_{(\alpha_3,\beta_3)}[f\circ g]_k = \infty$ and the other conditions remain the same.

Remark 3.18. Theorem 3.14, Theorem 3.16, Remark 3.15 and Remark 3.17 can also be deduced in terms of maximum modulus of entire functions.

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