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A NOTE ON TWO KNOWN SUMS INVOLVING CENTRAL BINOMIAL COEFFICIENTS WITH AN APPLICATION

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Abstract. The aim of this note is to establish two known sums involving central binomial coefficients via a hypergeometric series approach. As an application, we discover two new closed-form evaluations of generalized hypergeometric function.

1. Introduction

The binomial coefficients are defined by [6]

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & ;\ n \geq k \\ 0 & ;\ n < k. \end{cases} \tag{1.1}$$

for non-negative integers n and k.

The central binomial coefficients are defined by [6]

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}, \qquad (n = 0, 1, 2, \ldots). \tag{1.2}$$

On June 29, 2021, Vuk Stejiljkovic of University of Novisad, Serbia asked on the Researchgate for finding the sum of an alternating series viz.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k\, 2^{2k}} \binom{2k}{k}$$

and its sum was obtained (by three methods) very recently by Li and Qi [2]. The result is asserted in the following theorem.

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Theorem 1.1. *The following identity holds true.*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k\,2^{2k}} \binom{2k}{k} = 2\ln\left(2(\sqrt{2}-1)\right). \tag{1.3}$$

Inspired by this result, very recently, Lim and Qi [3] obtained several interesting finite and infinite sums containing central binomial coefficients. however, here we would like to mention only one result asserted in the following theorem.

Theorem 1.2. *The following identity holds true.*

$$\sum_{k=1}^{\infty} \frac{1}{k\,2^{2k}} \binom{2k}{k} = 2\ln 2. \tag{1.4}$$

On the other hand, the generalized hypergeometric function [5, 7] with p numerator and q denominator parameters is defined by

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{matrix} \;;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1.5}$$

where $(a)_n$ is the well-known Pochhammer's symbol (or the shifted or raised factorial, since $(1)_n = n!$) defined for the complex number $a(\neq 0)$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\ldots(a+n-1) & , \ n \in \mathbb{N} \\ \\ 1 & , \ n = 0. \end{cases}$$

where $\Gamma(\cdot)$ is the well-known gamma function. Here, p and q are non-negative integers and the parameters $a_1, a_2, \cdots, a_p$ and $b_1, b_2, \cdots, b_q$ are arbitrary complex values with one exception that b_j $(1 \leq j \leq q)$ should not be zero or a negative integer.

Moreover, the generalized hypergeometric function ${}_pF_q$ converges in the following three cases:

(i) $|z| < \infty$ provided $p \geq q$,

(ii) $|z| < 1$ provided $p = q+1$,

(iii) $|z| = 1$ provided $p = q+1$ and $\Re(s) > 0$,

where s is the parametric excess given by

$$s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j.$$

For more detail about this function, we refer to the standard texts [5, 7].

The note is organized as follows. In Section 2, we shall derive the known results asserted in Theorems 1.1 and 1.2 via a hypergeometric series approach. In Section 3, as an application, we shall establish two new and interesting closed-form evaluations of the generalized hypergeometric function.

2. Derivations of the Results (1.3) and (1.4) via a Hypergeometric Series Approach

In order to establish the result (1.3) asserted in Theorem 1.1 via a hypergeometric series approach, we proceed as follows. Denoting the left-hand side of (1.3) by S, we have

$$S = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\, 2^{2k}} \binom{2k}{k}.$$

Replacing k by $k+1$, we have

$$S = -\frac{1}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)2^{2k}} \binom{2k+2}{k+1}.$$

Now expressing binomial coefficient in terms of Pochhammer's symbol using

$$\binom{2k+2}{k+1} = 2^{2k+1} \frac{\left(\frac{3}{2}\right)_k}{(2)_k}$$

and

$$k+1 = \frac{(2)_k}{(1)_k}.$$

We have

$$S = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (1)_k (1)_k \left(\frac{3}{2}\right)_k}{k!(2)_k (2)_k}.$$

Summing up the series in terms of the generalized hypergeometric function, we have

$$S = -\frac{1}{2}\, {}_3F_2 \left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; -1 \right].$$

Now, we observe that ${}_3F_2$ can be evaluated with the help of a known result [4, Equ. 365, p. 519 for $z = -1$] viz.

$${}_3F_2 \left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; -1 \right] = -4 \ln\left(2(\sqrt{2} - 1)\right)$$

and we at once arrive at the right-hand side of the result (1.3). This completes the proof of the result (1.3) asserted in the Theorem 1.1.

In exactly the same manner, the result (1.4) asserted in the Theorem 1.2 can be established with the help of the following known result [4, Equ. 365, p. 519 for $z = 1$] viz.

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; 1\right] = 4 \ln 2. $$

We left this as an exercise to the interested reader.

Finally, it is not difficult to see that the results (1.3) and (1.4) can be written in terms of generalized hypergeometric functions as follows:

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; -1\right] = -4 \ln\left(2(\sqrt{2} - 1)\right) \tag{2.1} $$

and

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ 2, 2 \end{matrix} ; 1\right] = 4 \ln 2. \tag{2.2} $$

We conclude this section by remarking that the application of the results (2.1) and (2.2) will be given in the next section.

3. Two New Closed-form Evaluations of the Generalized Hypergeometric Function

In this section, as an application of the results (2.1) and (2.2), we shall establish the following two new and interesting closed-form evaluations of the generalized hypergeometric function asserted in the following theorem.

Theorem 3.1. *The following results hold true.*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{3}{4}, \frac{5}{4} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} ; 1\right] = 2 \ln\left(\sqrt{2} - 1\right), \tag{3.1} $$

$$ {}_4F_3\left[\begin{matrix} 1, 1, \frac{5}{4}, \frac{7}{4} \\ \frac{3}{2}, 2, 2 \end{matrix} ; 1\right] = \frac{16}{3} \ln\left(4(\sqrt{2} - 1)\right). \tag{3.2} $$

Proof. In order to establish the results (3.1) and (3.2) asserted in Theorem 3.1, we shall make use of the following two results recorded in Exton [1] viz.

$$\begin{aligned}&{}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1}\\ \\ b_1, b_2, \ldots, b_q\end{matrix}\;;1\right] + {}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1}\\ \\ b_1, b_2, \ldots, b_q\end{matrix}\;;-1\right]\\ &= 2\; {}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{1}{2}a_1, \frac{1}{2}a_1+\frac{1}{2}, \ldots, \frac{1}{2}a_{q+1}, \frac{1}{2}a_{q+1}+\frac{1}{2}\\ \\ \frac{1}{2}, \frac{1}{2}b_1, \frac{1}{2}b_1+\frac{1}{2}, \ldots, \frac{1}{2}b_q, \frac{1}{2}b_q+\frac{1}{2}\end{matrix}\;;1\right],\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&{}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1}\\ \\ b_1, b_2, \ldots, b_q\end{matrix}\;;1\right] - {}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1}\\ \\ b_1, b_2, \ldots, b_q\end{matrix}\;;-1\right]\\ &= \frac{2a_1a_2\cdots a_{q+1}}{b_1b_2\cdots b_q}\; {}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{1}{2}a_1+\frac{1}{2}, \frac{1}{2}a_1+1, \ldots, \frac{1}{2}a_{q+1}+\frac{1}{2}, \frac{1}{2}a_{q+1}+1\\ \\ \frac{3}{2}, \frac{1}{2}b_1+\frac{1}{2}, \frac{1}{2}b_1+1, \ldots, \frac{1}{2}b_q+\frac{1}{2}, \frac{1}{2}b_q+1\end{matrix}\;;1\right].\end{aligned} \tag{3.4}$$

□

It is not out of place to mention here that the results (3.3) and (3.4) can be easily established by resolving generalized hypergeometric functions

$${}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1}\\ \\ b_1, b_2, \ldots, b_q\end{matrix}\;;\pm 1\right]$$

into even and odd components and making use of the following elementary identities of Pochhammer symbols viz.

$$(a)_{2n} = 2^{2n}\left(\frac{1}{2}a\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n$$

and

$$(a)_{2n+1} = a2^{2n}\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(\frac{1}{2}a+2\right)_n.$$

Now, we are ready to establish the results (3.1) and (3.2). For this, in (3.3) and (3.4) if we set $q = 2$, $a_1 = a_2 = 1, a_3 = \frac{3}{2}, b_1 = b_2 = 2$ and making use of the results (2.1) and (2.2) on the left-hand side of (3.3) and (3.4), we easily arrive at the results (3.1) and (3.2) simultaneously. This completes the proof of the results (3.1) and (3.2) asserted in Theorem 3.1.

We conclude this section by remarking that the results (3.1) and (3.2) can be written in series involving binomial coefficients as follows:

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)\,2^{4k}} \binom{4k+2}{2k+1} = 4\ln(\sqrt{2}-1) \tag{3.5}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k\,2^{4k}} \binom{4k}{2k} = 2\ln\left(\frac{4}{\sqrt{2}+1}\right). \tag{3.6}$$

Remark 3.2. The results (3.1) and (3.2) have been verified using MAPLE software.

4. Concluding Remark

In this note, we established the known sums

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k\,2^{2k}} \binom{2k}{k}$$

via a hypergeometric series approach. As an application, we discover two new and interesting closed-form evaluations of the generalized hypergeometric functions. We conclude this note by remarking that following the same technique, by employing the known sums

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{(k+m)\,2^{2k}} \binom{2k}{k}, \quad (0 \le m \le 16)$$

obtained very recently by Lim and Qi [3] several interesting sums of the form

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^k}{2^{2k}\,k(k+1)\cdots(k+j+1)}, \quad (0 \le j \le 16)$$

are under investigations and will form a part of subsequent paper in this direction.

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