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STABILITY OF s-VARIABLE ADDITIVE AND l-VARIABLE QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. In this paper we investigate the Hyers-Ulam stability of the s-variable additive and l-variable quadratic functional equations of the form

$$f\left(\sum_{i=1}^{s} x_i\right) + \sum_{j=1}^{s} f\left(-sx_j + \sum_{i=1, i\neq j}^{s} x_i\right) = 0$$

and

$$f\left(\sum_{i=1}^{l} x_i\right) + \sum_{j=1}^{l} f\left(-lx_j + \sum_{i=1, i\neq j}^{l} x_i\right) = (l+1)\sum_{i=1, i\neq j}^{l} f(x_i - x_j) + (l+1)\sum_{i=1}^{l} f(x_i)$$

$(s, l \in \mathbb{N}, s, l \geq 3)$ in quasi-Banach spaces.

1. Introduction and Preliminaries

In 1940, Ulam [18] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms. Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x)) \diamond h(y) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

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The Cauchy additive functional equation

$$f(x+y)=f(x)+f(y) \tag{1.1}$$

is called additive functional equation. In 1941, Hyers [8] considered the case of approximately additive mappings $f:E\to E_1$, where E and E_1 are Banach spaces and f satisfies Hyers inequality

$$\|f(x+y)-f(x)-f(y)\|\le\epsilon$$

for all $x,y\in E$. It was shown that the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x\in E$ and that $L:E\to E_1$ is the unique additive mapping satisfying

$$\|L(x)-f(x)\|\le\epsilon.$$

See [1, 2, 3, 5, 10, 11] for more information on functional equations and their stability. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping (see [6, 12, 13, 14, 15]). A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [17] for mappings $f:X\to Y$, where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [6] proved the Hyers-Ulam stability of the additive quadratic and cubic functional equation (see [7, 9, 16]).

In this paper we investigate the Hyers-Ulam stability of the s-variable and l-variable quadratic functional equations of the form

$$f\left(\sum_{i=1}^{s}x_i\right)+\sum_{j=1}^{s}f\left(-sx_j+\sum_{i=1,i\neq j}^{s}x_i\right)=0 \tag{1.3}$$

(1.4)

$$f\left(\sum_{i=1}^{l}x_i\right)+\sum_{j=1}^{l}f\left(-lx_j+\sum_{i=1,i\neq j}^{l}x_i\right)=(l+1)\sum_{i=1,i\neq j}^{l}f(x_i-x_j)+(l+1)\sum_{i=1}^{l}f(x_i)$$

$(s,l\in\mathbb{N}, s,l\ge 3)$ in quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([1, 13])**.** Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in R$ and all $x \in X$.

(iii) There is a constant $k \geq 1$ such that $\|x+y\| \leq k(\|x\|+\|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p-norm $0 < p \leq 1$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$.

In this case, a quasi-Banach space is called a p-Banach space. By the Aoki-Rolewicz theorem [13] (see also [1]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. Stability of s-variable Additive Functional Equation (1.3)

Assume that X is a quasi-normed space with quasi-norm $\|\cdot\|$ and that Y is a p-Banach space with p-norm $\|\cdot\|$. By using an idea of Gavruta [4], we prove the stability of the functional equation (1.1). For convenience, we use the following abbreviation for a given mapping $f : X \to Y$

$$Df(x_1 \ldots, x_s) = \left(f \sum_{i=1}^{s} x_i\right) + \sum_{j=1}^{s} f\left(-sx_j + \sum_{i=1, i\neq j}^{s} x_i\right)$$

for all $x_j \in X (1 \leq j \leq s)$.

We will use the following lemma.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies (1.3) if and only if if the mapping $f : X \to Y$ is additive.*

Proof. We first assume that the mapping $f : X \to Y$ satisfies (1.1). Setting $x = y = 0$ in (1.1), we get $f(0) = 0$. Letting $y = -x$ in (1.1), we get $f(-x) = -f(x)$. Thus f is odd. If we replace y by x and x and y by $2x$ and x in (1.1), we get

$$f(2x) = 2f(x), f(3x) = 3f(x).$$

In general for any positive integer s, we obtain

$$f(sx) = sf(x)$$

for all $x \in X$. Letting $x = x_1 + x_2, y = x_3$ in (1.1), we get

$$f(x_1 + x_2 + x_3) = f(x_1) + f(x_2) + f(x_3) \tag{2.1}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_3 by $-3x_3$ in (2.1), we get

$$f(x_1 + x_2 - 3x_3) = f(x_1) + f(x_2) - 3f(x_3) \tag{2.2}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_2 by $-3x_2$ in (2.1), we have

$$f(x_1 - 3x_2 + x_3) = f(x_1) - 3f(x_2) + f(x_3) \tag{2.3}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_1 by $-3x_1$ in (2.1), we get

$$f(-3x_1 + x_2 + x_3) = -3f(x_1) + f(x_2) + f(x_3) \tag{2.4}$$

for all $x_1, x_2, x_3 \in X$. It follows from (2.1), (2.2), (2.3) and (2.4) that

$$f(-3x_1+x_2+x_3)+f(x_1-3x_2+x_3)+f(x_1+x_2-3x_3)+f(x_1+x_2+x_3) = 0 \tag{2.5}$$

for all $x_1, x_2, x_3 \in X$.

Continuing in this same way by using (2.5) up to s times, we get (1.3).

Letting $x_1 = x_2 = \cdots = x_s = 0$ in (1.3), we get $f(0) = 0$. Setting $x_1 = x_2 = , \ldots, = x_s = x$ and by using (2), we get $f(-x) = -f(x)$. Thus f is odd. Letting $x_1 = \frac{3x+y}{8}, x_2 = \frac{3y+x}{8}, x_3 = x_4 = \cdots = x_s = 0$ in (1.3), we get

$$f(-y) + f(-x) + 2f\left(\frac{4x + 4y}{8}\right) = 0 \tag{2.6}$$

for all $x, y \in X$. It follows from (2.6), (2) and the oddness of f that (1.1) holds for all $x \in X$. So the mapping $f : X \to Y$ is additive.

The converse follows from the additivity. □

Theorem 2.2. *Let $\varphi : X \times X \times \cdots \times X \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{s^n}\varphi(s^n x_1, \ldots, s^n x_s) = 0 \tag{2.7}$$

for all $x_1, x_2, \cdots, x_s \in X$ and

$$\tilde{\varphi} := \sum_{i=0}^{\infty} \frac{1}{s^{ip}} \left(\varphi(s^i x, \ldots, s^i x)\right)^p < \infty \tag{2.8}$$

for all $x \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \ldots, x_s)\| \le \varphi(x_1, \ldots, x_s) \tag{2.9}$$

for all $x_1, x_2, \cdots, x_s \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} \frac{1}{s^n} f(s^n x) \tag{2.10}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying

$$\|f(x) - A(x)\| \le \frac{1}{s}[\tilde{\varphi}(x)]^{\bar{p}} \tag{2.11}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = \cdots = x_s = x$ in (1.3), we get

$$\|f(sx) - sf(x)\| \le \varphi(\underbrace{x, x, \ldots, x}_{s-times}) \tag{2.12}$$

for all $x \in X$. Replacing x by $s^n x$ in (2.12) and dividing both sides of (2.12) by s^{n+1}, we get

$$\left\| \frac{1}{s^{n+1}} f(s^{n+1}x) - \frac{1}{s^n} f(s^n x) \right\| \le \frac{1}{s^{n+1}} \varphi(\underbrace{s^n x, \ldots, s^n x}_{s-times}) \tag{2.13}$$

for all $x \in X$ and all nonnegative integers n. Since Y is p-Banach space, we have

$$\begin{aligned} \left\| \frac{1}{s^{n+1}} f(s^{n+1}x) - \frac{1}{s^r} f(s^r x) \right\|^p &\le \sum_{i=r}^{n} \frac{1}{s^{n+1}} \left\| \frac{1}{s^{i+1}} f(s^{i+1}x) - \frac{1}{s^i} f(s^i x) \right\|^p \varphi(\underbrace{s^n x, \ldots, s^n x}_{s-times}) \\ &\le \frac{1}{s^{ip}} (\varphi(s^i x, \ldots, s^i x))^p \end{aligned} \tag{2.14}$$

for all $x \in X$ and all nonnegative integers n and r with $n \ge r$. Therefore, we conclude from (2.9) and (2.14) that the sequence $\left\{\frac{1}{s^n} f(s^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{s^n} f(s^n x)\right\}$ converges in Y for all $x \in X$. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{s^n} f(s^n x) \tag{2.15}$$

for all $x \in X$. Letting $r = 0$ and passing the limit $n \to \infty$ in (2.14), we get (2.11). Now, we show that A is an additive mapping. It follows from (2.8), (2.10) and (2.15) that

$$\|DA(x_1, \ldots, x_s)\| = \lim_{n\to\infty} \frac{1}{s^n} \|Df(s^n x_1, \ldots, s^n x_s)\| \le \lim_{n\to\infty} \frac{1}{s^n} \varphi(s^n x_1, \ldots, s^n x_s) = 0$$

for all $x_s \in X$. Hence the mapping A satisfies (1.3). So by Lemma 2.1, the mapping $x \to A(x)$ is additive.

To prove the uniqueness of A, let $B : X \to Y$ be another additive mapping satisfying (2.11). It follows from (2.11) and (2.15) that

$$\|A(x) - B(x)\|^p = \lim_{n\to\infty} \frac{1}{s^{np}} \|f(s^n x) - B(s^n x)\|^p \le \frac{1}{s^p} \lim_{n\to\infty} \tilde{\varphi}(s^n x) = 0$$

for all $x \in X$. So $A = B$. □

Theorem 2.3. *Let $\varphi : X \times X \times \cdots \times X \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} s^n \varphi\left(\frac{x_1}{s^n}, \ldots, \frac{x_s}{s^n}\right) = 0 \tag{2.16}$$

for all $x_1, x_2, \cdots, x_s \in X$ and

$$\tilde{\varphi} := \sum_{i=0}^{\infty} \frac{1}{s^{ip}} \left(\varphi\left(\frac{x}{s^i}, \ldots, \frac{x}{s^i}\right)\right)^p < \infty \tag{2.17}$$

for all $x \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \ldots, x_s)\| \le \varphi(x_1, \ldots, x_s) \tag{2.18}$$

for all $x_1, x_2, \cdots, x_s \in X$. Then the limit

$$A(x) = \lim_{n\to\infty} s^n f\left(\frac{x}{s^n}\right) \tag{2.19}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying

$$\|f(x) - A(x)\| \le \frac{1}{s} [\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{2.20}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.4. *Let $\beta, r_j (1 \le j \le s)$ be nonnegative real numbers such that $r_j > 1$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \ldots, x_s)\| \le \beta \sum_{i=1}^{s} \|x_i\|^{r_i}$$

for all $x_j \in X (1 \le j \le s)$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \frac{n\beta}{\|s^p - s^{pr_1}\|} \|x\|^{r_1}$$

for all $x \in X$.

Proof. The result follows from Theorems 2.2 and 2.3. □

Corollary 2.5. *Let $\beta, r_j (1 \le j \le s)$ be nonnegative real numbers such that $r_j > 1$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \dots, x_s)\| \le \beta \prod_{i=1}^{s} \|x_i\|^{r_i}$$

for all $x_j \in X (1 \le j \le s)$. Then the mapping $f : X \to Y$ is additive.

Proof. The result follows from Theorems 2.2 and 2.3. □

3. Stability of l-variable Quadratic Functional Equation (1.4)

In this section, we investigated the general solution and the Hyers-Ulamstability of the functional equation (1.4).

Lemma 3.1. *A mapping $f : X \to Y$ satisfies (1.4) if and only if the mapping $f : X \to Y$ is quadratic.*

Proof. A mapping $f : X \to Y$ satisfies the functional equation (1.4). Letting $x_1 = x_2 = \cdots = x_l = 0$ in (1.4), we get $f(0) = 0$. Letting $x_1 = x, x_2 = \cdots = x_l = 0$ in (1.4), we get $f(-x) = f(x)$ and so f is an even mapping. Letting $x_1 = 0, x_2 = x, x_3 = \cdots = x_l = 0$ in (1.4), we get

$$lf(x) + f(lx) = l(l+1)f(x) \tag{3.1}$$

for all $x \in X$. It follows from (3.1) that

$$f(lx) = l^2 f(x)$$

for all $x \in X$. Letting $x_1 = x, x_2 = y, x_3 = \cdots = x_l = 0$ in (1.4), we have

$$2f(x+y) + f(-3x+y) + f(x-3y) = 4(f(x-y)) + 8f(x) + 8f(y) \tag{3.2}$$

for all $x, y \in X$. It follows from (3.2) and (1.2) that

$$f(x+y) + f(-x+y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Therefore the mapping $f : X \to Y$ is quadratic.

The converse is similar to the proof of Lemma 2.1. □

Theorem 3.2. *Let $\varphi : X \times X \times \cdots \times X \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{l^{2n}} \varphi\left(\underbrace{l^n x_1, \cdots, l^n x_l}_{l-times}\right) = 0 \tag{3.3}$$

for all $x_1, x_2, \cdots, x_l \in X$ *and*

$$\tilde{\varphi(x)} := \sum_{i=0}^{\infty} \frac{1}{l^{2ip}} \left(\varphi(l^i x, \ldots, l^i x)\right)^p < \infty \tag{3.4}$$

for all $x \in X$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, \ldots, x_l)\| \leq \varphi(x_1, \ldots, x_l) \tag{3.5}$$

for all $x_1, x_2, \cdots, x_l \in X$. *Then the limit*

$$Q(x) = \lim_{n\to\infty} l^{2n} f\left(l^n x\right) \tag{3.6}$$

exists for all $x \in X$ *and the mapping* $Q : X \to Y$ *is a unique quadratic mapping satisfying*

$$\|f(x) - Q(x)\| \leq \frac{1}{l^2} [\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{3.7}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let* $\varphi : \underbrace{X \times X \times \cdots \times X}_{l-times} \to [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty} l^{2n} \varphi\left(\frac{x_1}{l^n}, \ldots, \frac{x_l}{l^n}\right) = 0 \tag{3.8}$$

for all $x_1, x_2, \cdots, x_l \in X$ *and*

$$\tilde{\varphi(x)} := \sum_{i=0}^{\infty} l^{2ip} \left(\varphi\left(\frac{x}{l^i}, \ldots, \frac{x}{l^i}\right)\right)^p < \infty \tag{3.9}$$

for all $x \in X$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x_1, \ldots, x_l)\| \leq \varphi(x_1, \ldots, x_l) \tag{3.10}$$

for all $x_1, x_2, \cdots, x_l \in X$. *Then the limit*

$$Q(x) := \lim_{n\to\infty} l^{2n} f\left(\frac{x}{l^n}\right)$$

exists for all $x \in X$ *and the mapping* $Q : X \to Y$ *is a unique quadratic mapping satisfying*

$$\|f(x) - Q(x)\| \leq \frac{1}{l^2} [\tilde{\varphi}(x)]^{\frac{1}{p}}$$

for all $x \in X$.

Corollary 3.4. *Let $\beta, r_j (1 \leq j \leq l)$ be nonnegative real numbers such that $r_j > 1$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \ldots, x_s)\| \leq \beta \sum_{i=1}^{l} \|x_i\|^{r_i}$$

for all $x_j \in X (1 \leq j \leq s)$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \frac{n\beta}{\|l^{2p} - l^{2pr_1}\|^{\frac{1}{p}}} \|x\|^{r_1}$$

for all $x \in X$.

Proof. The result follows from Theorems 3.2 and 3.3. □

Corollary 3.5. *Let $\beta, r_j (1 \leq j \leq s)$ be nonnegative real numbers such that $r_j > 1$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \ldots, x_l)\| \leq \beta \prod_{i=1}^{l} \|x_i\|^{r_i}$$

for all $x_j \in X (1 \leq j \leq l)$. Then the mapping $f : X \to Y$ is quadratic.

Proof. The result follows from Theorems 3.2 and 3.3. □

Competing Interests

The authors declare that they have no competing interests.

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