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HEART AND COMPLETE PARTS OF (R, S)-HYPER BI-MODULE

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Abstract. In this article, we investigate several aspects of (R, S)-hyper bi-modules and describe some their properties. The concepts of fundamental relation, completes part and complete closure are studied regarding to (R, S)-hyper bi-modules. In particular, we show that any complete (R, S)-hyper bi-module has at least an identity and any element has an inverse. Finally, we obtain a few results related to the heart of (R, S)-hyper bi-modules.

1. Introduction and Preliminaries

Let R and S be rings and suppose that M be a left R-module and a right S-module. Then M is called a (R, S)-*bimodule* if for all $r \in R,\ s \in S$ and $m \in M$, $(rm)s = r(ms)$.

For positive integers n and m, the set $M_{n\times m}(T)$ of $n \times m$ matrices of real numbers is an (R, S)-bimodule, where R is the ring $M_n(T)$ of $n \times n$ matrices, and S is the ring $M_m(T)$ of $m \times m$ matrices. Addition and multiplication are carried out using the usual rules of matrix addition and matrix multiplication; the heights and widths of the matrices have been chosen so that multiplication is defined. Note that $M_{n\times m}(R)$ itself is not a ring (unless $n = m$). The crucial bimodule property, that $(rx)s = r(xs)$, is the statement that multiplication of matrices is associative.

A *hypergroupoid* $(H, \circ)$ is a non-empty set H together with a *hyperoperation* $\circ$ defined on H, that is, a mapping of $H \times H$ into $\wp^*(H)$, the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b.$$

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If $x \in H$, then $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *hypergroup* in the sense of Marty [15] if for all $x, y, z \in H$ the following two conditions hold: (i) $x \circ (y \circ z) = (x \circ y) \circ z$, (ii) $x \circ H = H \circ x = H$. The second condition is called the *reproduction axiom*. A *hyperring* [11, 17] is a multi-valued system $(R, +, \circ)$ which satisfies the ring-like axioms in the following way: (1) $(R, +)$ is a hypergroup in the sense of Marty, (2) $(R, \circ)$ is a semihypergroup, (3) the multiplication is distributive with respect to the hyperoperation $+$. Let $(M, +)$ be a hypergroup and $(R, +, \cdot)$ be a hyperring. According to [18] M is said to a left hypermodule over the hyperring R if there exists $\cdot : R \times M \to \wp^*(M)$, $(a, m) \mapsto a \cdot m$ such that for all $a, b \in R$ and $m_1, m_2, m \in M$, we have (1) $a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$, (2) $(a + b) \cdot m = (a \cdot m) + (b \cdot m)$, (3)$(a \cdot b) \cdot m = a \cdot (b \cdot m)$. Basic definitions and propositions about the hyperstructures are found in [6, 7, 9, 10, 18]. The notion of right hypermodules can be defined similarly.

Definition 1.1 ([16])**.** Let R, S be hyperrings and let M be a left R-hyper module and a right S-hypermodule. Then M is called an (R, S)*-hyperbimodule* if for all $r \in R,\ s \in S$ and $m \in M$, $(rm)s = r(ms)$.

Example 1. If R is a hyperring, then R itself is an (R, R)-hyperbimodule and so is R^n.

Example 2. Any two-sided hyperideal of a hyperring R is an (R, R)-hyperbimodule.

Example 3. If $R,\ S$ are hyperrings and $R \subseteq S$, then S is an (R, R)-hyperbimodule. It is also (R, S) and (S, R)-hyperbimodules.

Example 4. Let M be an (R, S)-hyper bi-module, N a left R-subhyper bi-module and T a right S-subhyper bi-module of M. If set $P := N \cap T$ $(P \neq \emptyset)$ then $(M/P,\ \oplus_P)$ with the following hyperoperation is an (R, S)-hyper bi-module.

$$\begin{array}{l} R \times M/P \times S \longrightarrow M/P \\ (r, m + P, s) \longmapsto r \cdot m \times s + P. \end{array}$$

We call this the *quotient hyperbimodule* M on P.

Example 5. Let $R,\ S$ be rings, M a left R-module and right S-module. Let $P,\ G$ be respectively subrings of $R,\ S$ which satisfy in the following condition:

$$\begin{cases} \forall \{a, b\} \subseteq R, & aGbG = abG \\ \forall \{a', b'\} \subseteq S, & a'Pb'P = a\prime b'P. \end{cases}$$

We define the relation ρ on M in the following way:

$$x\rho y \Leftrightarrow \exists t_1 \in G,\ t_2 \in P: \quad x = y + t_1 + t_2$$

also hyperoperation $\oplus$ on the set M/ρ in the following way:

$$\bar{x} \oplus \bar{y} := \{\bar{w} \in M/P \mid \bar{w} \subseteq \bar{x} + \bar{y}\}.$$

Now, we consider quotient hyperrings $R/G = \{\bar{a} = aG \mid a \in R\}$ and $S/P = \{\bar{b} = bP \mid b \in S\}$. Then, $(M/\rho, \oplus)$ with the following hyperoperation is an $(R/G, S/P)$-hyperbimodule

$$\begin{array}{l} R/G \times M/\rho \times S/P \longrightarrow M/\rho \\ (\bar{a}, \bar{x}, \bar{b}) \longmapsto \overline{a \cdot x \times b}. \end{array}$$

Example 6. Let M be a right A-hypermodule and N be a right A-subhypermodule of M. Also, suppose that M is a left B-hypermodule and T is a left B-subhypermodule of M. Set $P = N \cap T$ $(P \neq \emptyset)$ and define the relation ρ in the following way:

$$\forall (x, y) \in M^2,\ x\rho y \ \Leftrightarrow\ x + P = y + P.$$

Obviously, ρ is an equivalence relation on M. The set M/ρ with the following hyperoperations is an (A, B)-hyperbimodule:

$$\begin{array}{l} (x + P) \oplus (y + P) = \{z + P \mid z \in x + y\}, \\ b \otimes (x + P) \odot a = b \times x \cdot a + P, \end{array}$$

for all $a \in A$ and $b \in B$. Clearly, the condition $(bm)a = b(ma)$ holds for all $m \in M/\rho$.

The relation β was introduced by Koskas [14] and studied mainly by Corsini [6] and Freni [12, 13], and many others. Vougiouklis defined the relation γ on hyperrings.

Definition 1.2 ([17])**.** Let R be a hyperring. We define the relation γ as follows:

$x\gamma y$ if and only if there exist $n \in \mathbb{N}, (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$ such that

$$x, y \in \sum_{i=1}^{n} \Big(\prod_{j=1}^{k_i} x_{ij} \Big).$$

The relation γ is reflexive and symmetric. Let γ^* be the transitive closure of γ. Then the relation γ is the smallest strongly regular relation such that the quotient R/γ^* is a ring.

The following definition for the first time is introduced by Vougiouklis. We refer the readers to [18].

Definition 1.3 ([18])**.** Let R be a hyperring and M be a hypermodule over R. The relation ϵ is defined as follows:

$$x\epsilon y \;\Leftrightarrow\; \exists\, n\in\mathbb{N},\; \exists(m_1,\dots,m_n)\in M^n,\; \exists(k_1,k_2,\dots,k_n)\in\mathbb{N}^n,$$

$$\exists(x_{i1},x_{i2},\dots,x_{ik_i})\in R^{k_i},$$

such that

$$x,y\in\sum_{i=1}^{n}m'_i;\qquad m'_i=m_i\quad\text{or}\quad m'_i=\sum_{j=1}^{n_i}\Big(\prod_{k=1}^{k_{ij}}x_{ijk}\Big)m_i.$$

The relation ϵ is reflexive and symmetric. Let ϵ^* be the transitive closure of ϵ. Then ϵ^* is a strongly regular relation both on $(M,+)$ and M as an R-hyper module. Also, the (abelian group) M/ϵ^* is an R/γ^*- module, where R/γ^* is a ring and the relation ϵ^* is the smallest equivalence relation such that the quotient M/ϵ^* is an R/γ^*- module.

If M is an R-hyper module, then we set

$$\epsilon_0=\{(m,m)\mid m\in M\}$$

and for every integer $n\geq 1$, ϵ_n is the relation defined as follows:

$$x\epsilon_n y\Leftrightarrow x,y\in\sum_{i=1}^{n}m'_i.$$

Obviously, for every $n\geqslant 1$, the relation ϵ_n is symmetric, and the relation $\epsilon=\bigcup\limits_{n\geqslant 0}\epsilon_n$ is reflexive and symmetric. If M is a hypermodule over a hyperring R and $n\geq 1$ then $\epsilon_n\subseteq\epsilon_{n+1}$.

The fundamental relation ω^* on M can be defined as the smallest equivalence relation such that the quotient M/ω^* be a bimodule over the corresponding fundamental ring such that M/ω^* as a group is not abelian [16].

Definition 1.4 ([16])**.** Let R and S be hyperrings and suppose that M is an (R,S)-hyper bi-module. We define the relation ω as follows:

$x\omega y$ if and only if there exist $p\in\mathbb{N}$, $(m_1,\dots,m_p)\in M^p$, $(n_1,n_2,\dots,n_p)\in\mathbb{N}^p$, $(k_{i1},k_{i2},\dots,k_{in_i})\in\mathbb{N}^{n_i}$, $r_{ijk}\in R$, $(n'_1,n'_2,\dots,n'_p)\in\mathbb{N}^p$, $(k'_{i1},k'_{i2},\dots,k'_{in'_i})\in$

$\mathbb{N}^{n'_i}$, $s_{ijk} \in S$, such that

$$x, y \in \sum_{i=1}^{p} m'_i \text{ when } m'_i = \begin{cases} m_i \text{ or} \\ m_i\Big(\sum_{j=1}^{n'_i}\Big(\prod_{k=1}^{k'_{ij}} s_{ijk}\Big)\Big) \text{ or} \\ \Big(\sum_{j=1}^{n_i}\Big(\prod_{k=1}^{k_{ij}} r_{ijk}\Big)\Big)m_i \text{ or} \\ \Big(\sum_{j=1}^{n_i}\Big(\prod_{k=1}^{k_{ij}} r_{ijk}\Big)\Big)m_i\Big(\sum_{j=1}^{n'_i}\Big(\prod_{k=1}^{k'_{ij}} s_{ijk}\Big)\Big). \end{cases}$$

The relation ω is reflexive and symmetric. Let ω^* be transitive closure of ω.

Lemma 1.5 ([16])**.** *ω^* is a strongly regular relation on $(M,+)$ and M as an (R,S)-hyper bi-module too.*

Theorem 1.6 ([16])**.** *The relation ω^* is the smallest equivalence relation such that the quotient M/ω^* is an $(R/\gamma_R^*, S/\gamma_S^*)$-bi-module.*

Definition 1.7 ([16])**.** Let M be an (R,S)-hyper bi-module. Then we set $\omega_0 = \{(m,m) \mid m \in M\}$ and for every integer $n \geq 1$, ω_n is the relation defined as follows:

$$x\omega_n y \Leftrightarrow x \in \sum_{i=1}^{n} m'_i, \quad y \in \sum_{i=1}^{n} m'_i.$$

Obviously, for every $n \geqslant 1$, the relation ω_n are symmetric, and the relation $\omega = \bigcup_{n \geqslant 0} \omega_n$ is reflexive and symmetric.

2. Complete Closure of (R,S)-hyperbimodules

In this section we find some properties of complete parts of (R,S)-hyperbimodules which are valid in every (R,S)-hyperbimodule. In the following m'_i is the notation that defined in Definition 1.4

Definition 2.1 ([16])**.** Let M be an (R,S)-hyperbimodule and A be a non-empty subset of M. We say that A is a *complete part* of M if for every $n \in \mathbb{N}$, for every and for every $(m'_1, \ldots, m'_n)$

$$\sum_{i=1}^{n} m'_i \cap A \neq \emptyset \Rightarrow \sum_{i=1}^{n} m'_i \subseteq A.$$

We say an (R,S)-hyperbimodule M is *n-complete* if $\forall(m'_1,\dots,m'_n)$, we have

$$\omega\Big(\sum_{i=1}^{n} m'_i\Big)=\sum_{i=1}^{n} m'_i,$$

where $\omega\Big(\sum_{i=1}^{n} m'_i\Big)$ is the union of all ω-classes having a non-empty intersection with the set $\sum_{i=1}^{n} m'_i$.

Lemma 2.2 ([16]). *Let M be an (R,S)-hyperbimodule. For every $x,y,a\in M$, $r\in R$ and $s\in S$, if $x\omega_n y$ then*

$$\begin{aligned}
&\omega^*_n \subseteq \omega^*_{n+1},\\
&(x+a)\ \overline{\overline{\omega}}_{n+1}\ (y+a),\quad (x+a)\ \overline{\overline{\omega^*}}_{n+1}\ (y+a),\\
&(a+x)\ \overline{\overline{\omega}}_{n+1}\ (a+y),\quad (a+x)\ \overline{\overline{\omega^*}}_{n+1}\ (a+y),\\
&r\cdot a\ \overline{\overline{\omega}}_{n}\ r\cdot b,\qquad\qquad r\cdot a\ \overline{\overline{\omega^*}}_{n}\ r\cdot b,\\
&a\cdot s\ \overline{\overline{\omega}}_{n}\ b\cdot s,\qquad\qquad a\cdot s\ \overline{\overline{\omega^*}}_{n}\ b\cdot s,
\end{aligned}$$

Theorem 2.3. *Let M be an R-hyper bi-module and ρ be a strongly regular relation on M. Then $(M/\rho,\oplus)$ is an (R,S)-hyper bi-module if and only if for every $(x,y,z)\in M^3$;*

(1) $\rho(\rho(\rho(x)\oplus\rho(y))\oplus\rho(z))=\rho(\rho(x)\oplus\rho(\rho(y)\oplus\rho(z)))$,
(2) *for every* $r\in R$, $\quad r.\rho(x)=\rho(r.x)$.

Proof. Let $\overline{x}:=\rho(x)$. It is enough to observe that

$$\begin{aligned}
(\overline{x}\oplus\overline{y})\oplus\overline{z}\ &=\{\overline{u}\mid u\in\rho(x)+\rho(y)\}\oplus\overline{z}\\
&=\{\overline{v}\mid v\in\rho(u)+\rho(z),\ u\in\rho(x)+\rho(y)\}\\
&=\{\overline{v}\mid v\in(\rho(x)+\rho(y))+\rho(z)\}.
\end{aligned}$$

Analogously, we can write $\overline{x}\oplus(\overline{y}\oplus\overline{z})=\{\overline{w}\mid w\in\rho(x)+(\rho(y)+\rho(z))\}$.

Since ρ is strongly regular, it follows that with the scalar hyperoperation $r\cdot\rho(x):=\rho(r\cdot x)$ we obtain a module, and the properties of M as an R-hyper bi-module, guarantee that the hypergroup M/ρ is an (R,S)-hyper bi-module. □

Theorem 2.4. *Let M be a hyper bi-module, $\phi_M: M\to M/K$ be the canonical projection. If N is a hyper bi-module and $f: M\to N$ is an (R,S)-homomorphism, then $g: M/K\to N$ exists such that $g\phi_M=f$.*

Proof. It is enough to check that for every $x \in M$, $g\phi_M(x) = f(x)$. First, g is well defined: in fact $\phi_M(x) = \phi_M(y)$ implies that xKy. Since N is a hyper bi-module, it follows that $f(x) = f(y)$. Moreover, g is an (R,S)-homomorphism because for every $x, y \in M$, and $u \in x + y$, we have

$$\begin{aligned} g(\phi_M(x) + \phi_M(y)) &= g\phi_M(x+y) = g\phi_M(u) = f(u) \\ &= f(x+y) = f(x) + f(y) = g\phi_M(x) + g\phi_M(y). \end{aligned}$$

Moreover, for every $r \in R$, and $v \in r \cdot x$ we have

$$g(\phi_M(r \cdot x)) = g(\phi_M(v)) = f(v) = f(r \cdot x) = r \cdot f(x) = r \cdot (g\phi_M(x)).$$

In the similar way, for every $s \in S$, $g(\phi_M(x) \cdot s) = (g\phi_M(x)) \cdot s$. □

Theorem 2.5. *If $f : M \to M'$ is an (R,S)-homomorphism, then*

(1) *for all $x \in M$, we have $f(C(x)) \subseteq C(f(x))$.*

(2) *f determines an (R,S)-homomorphism $f^* : M/K \to M'/K'$ defined*

$$f^*(\phi_M(x)) = \phi_{M'}(f(x)).$$

Proof. (1) It is easy to check that for every $n \in \mathbb{N}$, the following implication holds;

$$x \ \omega_n \ y \Rightarrow f(x) \ \omega_n \ f(x).$$

(2) f^* is well defined, in fact if $\phi_M(x) = \phi_M(y)$, then xKy. Then, we conclude that $f(x) \ K \ f(y)$, and so $f^*\phi_M(x) = f^*\phi_M(y)$. f^* is an (R,S)-homomorphism because for every $u \in x + y$,

$$\begin{aligned} f^*(\phi_M(x) + \phi_M(y)) &= f^*(\phi_M(u)) = \phi_{M'}(f(u)) = \phi_{M'}(f(u)) \\ &= \phi_{M'}(f(x) + f(y)) = \phi_{M'}(f(x)) + \phi_{M'}(f(y)) \\ &= f^*(\phi_M(x)) + f^*(\phi_M(y)), \end{aligned}$$

and for every $r \in R$ and $v \in r \cdot x$, we have

$$f^*(\phi_M(r \cdot x)) = \phi_{M'}(f(v)) = \phi_{M'}(f(r \cdot x)) = r \cdot \phi_M(f(x)) = r \cdot f^*(\phi_M(x)).$$

□

Theorem 2.6. *An (R,S)-hyperbimodule M is n-complete if and only if for every $(m'_1, \ldots, m'_n)$ and $z \in \sum_{i=1}^{n} m'_i$, we have*

$$\omega(z) = \sum_{i=1}^{n} m'_i.$$

Proof. Let M be n-complete, and suppose that $z \in \sum_{i=1}^{n} m'_i$. Then, we have

$$\omega(z) \subseteq \bigcup_{z \in \sum_{i=1}^{n} m'_i} \omega(z) = \omega\Big(\sum_{i=1}^{n} m'_i\Big) = \sum_{i=1}^{n} m'_i.$$

Hence, we obtain $\omega(z) \subseteq \sum_{i=1}^{n} m'_i$. Now, if $z \in \sum_{i=1}^{n} m'_i$, then $\omega(z) \subseteq \sum_{i=1}^{n} m'_i$. Consequently, if $y \in \sum_{i=1}^{n} m'_i$, then

$$z\omega_n y \Rightarrow z\omega y \Rightarrow y \in \omega(z).$$

Conversely, for every $(m'_1, \ldots, m'_n)$ and $z \in \sum_{i=1}^{n} m'_i$, we obtain $\omega(z) = \sum_{i=1}^{n} m'_i$. Therefore,

$$\omega\Big(\sum_{i=1}^{n} m'_i\Big) = \bigcup_{z \in \sum_{i=1}^{n} m'_i} \omega(z) = \sum_{i=1}^{n} m'_i$$

and hence M is n-complete. □

Theorem 2.7. *If M is an n-complete (R,S)-hyperbimodule then for all $(m'_1, \ldots, m'_n)$, $\sum_{i=1}^{n} m'_i$ is a complete part of M.*

Proof. For every $m \in \mathbb{N}$ and $(z'_1, \ldots, z'_m)$, if $\sum_{i=1}^{m} z'_i \cap \sum_{i=1}^{n} m'_i \neq \emptyset$, then there exists $a \in \sum_{i=1}^{m} z'_i \cap \sum_{i=1}^{n} m'_i$. Now, for every $y \in \sum_{i=1}^{m} z'_i$, we have $a\omega_m y$, and so $y \in \omega(a)$. Hence, we get $y \in \omega(a) = \sum_{i=1}^{n} m'_i$. Therefore, we conclude that $\sum_{i=1}^{m} z'_i \subseteq \sum_{i=1}^{n} m'_i$. □

Proposition 2.8. *If M is a n-complete (R,S)-hyperbimodule, then $\omega = \omega_n$.*

Proof. It is suffices to prove that $\omega \subseteq \omega_n$. Suppose that $x\omega y$. Then, there exists $m \in \mathbb{N}$, $x\omega_m y$. If $m \leq n$, then $\omega_m \subset \omega_n$. If $m > n$, then there exist $(m'_1, \ldots, m'_m)$ such that $x, y \in \sum_{i=1}^{m} m'_i$. Since $(M,+)$ is a hypergroup, it follows that there exist

$s \in M$ and $x \in \sum_{i=1}^{n-1} m'_i + s$ such that $y \in \omega(x) = \sum_{i=1}^{n} m'_i$. Therefore, we obtain $y \in \sum_{i=1}^{n} m'_i$, and so $x\omega_n y$. □

Definition 2.9. Let A be a non-empty subset of M. The intersection of the complete parts of M which contain A is called *complete closure* of A in M. It will be denoted by $C_M(A)$.

Theorem 2.10. *Let A be a non-empty subset of M. Assume that*

(1) $K_1(A) := A$,

(2) $K_{n+1}(A) := \{x \mid \exists\, p \in \mathbb{N},\ \exists(m'_1, \ldots, m'_p), x \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{p} m'_i \cap K_n(A) \neq \emptyset\}$,

(3) $K(A) := \bigcup_{n \geq 1} K_n(A)$.

Then $K(A) = C_M(A)$.

Proof. It is necessary to prove:

(1) $K(A)$ is a complete part of M,

(2) If $A \subseteq B$ and B is a complete part of M then $K(A) \subseteq B$.

Therefore,

(1) Let $\sum_{i=1}^{p} m'_i \cap K(A) \neq \emptyset$ then there exists $n \in \mathbb{N}$ such that $\sum_{i=1}^{p} m'_i \cap K_n(A) \neq \emptyset$. For every $y \in \sum_{i=1}^{n} m'_i$ we have $y \in K_{n+1}(A)$ and $\sum_{i=1}^{n} m'_i \subseteq K(A)$, and so $K(A)$ is a complete part of M.

(2) We have $A = K_1(A) \subseteq B$. Suppose that B is a complete part of M and $K_n(A) \subseteq B$. We prove that this implies $K_{n+1}(A) \subseteq B$. For every $z \in K_{n+1}$ there exist $p \in \mathbb{N}$, $(m'_1, \ldots, m'_p)$ such that $z \in \sum_{i=1}^{p} m'_i$, $\sum_{i=1}^{p} m'_i \cap K_n(A) \neq \emptyset$. Thus $\sum_{i=1}^{p} m'_i \cap B \neq \emptyset$, hence $z \in \sum_{i=1}^{p} m'_i \subseteq B$ and so $K_{n+1}(A) \subseteq B$. □ □

Lemma 2.11. *The following statements hold:*

(1) *For all $n \geq 2$ and $m \in M$, we have $K_n(K_2(m)) = K_{n+1}(m)$.*

(2) *If $m \in K_n(z)$, then $z \in K_n(m)$.*

Proof. (1) We can write $K_2(K_2(m)) :=$

$$\{z \mid \exists p \in \mathbb{N},\ \exists (m'_1, \dots, m'_p) :\ z \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{p} m'_i \cap K_2(m) \neq \emptyset\} = K_3(m).$$

We now proceed by induction: If $K_{n-1}(K_2(m)) = K_n(m)$, then

$K_n(K_2(m)) :=$

$$\{z \mid \exists p \in \mathbb{N},\ \exists (m'_1, \dots, m'_p),\ \exists \sigma \in \mathbb{S}_p :\ z \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{p} m'_i \cap K_{n-1}(K_2(m)) \neq \emptyset\} =$$

$$\{z \mid \exists p \in \mathbb{N},\ \exists (m'_1, \dots, m'_p),\ \exists \sigma \in \mathbb{S}_p :\ z \in \sum_{i=1}^{p} m'_i,\ \sum_{i=1}^{p} m'_i \cap K_n(m) \neq \emptyset\} = K_{n+1}(m).$$

(2)] We do the proof by mathematical induction. It is clear that $x \in K_2(y) \Leftrightarrow y \in K_2(x)$. Suppose that $x \in K_{n-1}(y) \Leftrightarrow y \in K_{n-1}(x)$. Let $x \in K_n(y)$, then there exist $q \in \mathbb{N},\ (m'_1, \dots, m'_q)$ and $\sigma \in \mathbb{S}_q$ such that

$$x \in \sum_{i=1}^{q} m'_i \text{ and } \sum_{i=1}^{q} m'_i \cap K_{n-1}(y) \neq \emptyset,$$

by this it follows that there exists $v \in \sum_{i=1}^{n} m'_i \cap K_{n-1}(y)$. Therefore by choosing $\sigma = 1$, $v \in K_2(x)$ is obtained. From $v \in K_{n-1}(y)$ we have $y \in K_{n-1}(K_2(x)) = K_n(x)$. □

Theorem 2.12. *The relation $xKy \Leftrightarrow x \in K(\{y\})$ is an equivalence relation.*

Proof. We write $C_M(x)$ instead of $C_M(\{x\})$. Clearly, K is reflexive. Now, let xKy and yKz,. If P is a complete part of M and $z \in P$, then $C_M(z) \subseteq P$, $y \in P$ and consequently $x \in C_M(y) \subseteq P$. For this reason $x \in C_M(z)$ that is xKz. The symmetrically of K follows in a direct way from the preceding lemma. □

Theorem 2.13. *For each (R, S)-hyperbimodule M, if $R \cdot m = M = m \cdot S$, for every $(r, s) \in R \times S$ and $m \in M$, then $K = \omega^*$.*

Proof. Suppose that $x\omega y$. Then

$$\exists n \in \mathbb{N} : x\omega y \Rightarrow \exists (m'_1, \dots, m'_n),\ x, y \in \sum_{i=1}^{n} m'_i.$$

Now, we have $\sum_{i=1}^{n} m'_i \cap \{x\} \neq \emptyset$, and so

$$x \in K_2(y) \Rightarrow x \in C_M(y) \Rightarrow xKy \Rightarrow \omega \subseteq K.$$

For every $(r,s) \in R \times S$ and $m \in M$, we conclude that $\omega^* \subseteq K$.

Conversely, if xKy, then there exists $n \in \mathbb{N}$ such that $x \in K_{n+1}(y)$. This implies that there exist $m \in \mathbb{N}$, $(m_1'^1, \dots, m_m'^1)$ such that

$$x \in \sum_{i=1}^{m} m_i'^1 \text{ and } \sum_{i=1}^{m} m_i'^1 \cap K_n(y) \neq \emptyset.$$

Thus, there exists $x_1 \in \sum_{i=1}^{m} m_i'^1 \cap K_n(y)$. Consequently, we obtain $x\omega x_1$ and $x_1 \in K_n(y)$, and so there exists $(m_1'^2, \dots, m_l'^2)$ such that

$$x_1 \in \sum_{i=1}^{l} m_i'^2,\ \sum_{i=1}^{l} m_i'^2 \cap K_{n-1}(y) \neq \emptyset \Rightarrow \exists x_2 \in \sum_{i=1}^{l} m_i'^2 \cap K_{n-1}(y) \Rightarrow x_1 \omega x_2.$$

So as a consequence one obtains:

$$\exists x_n \in \sum_{i=1}^{s} m_i'^n \cap K_{n-(n-1)}(y) \Rightarrow x_n \in K_1(y) = \{y\} \Rightarrow x_n = y.$$

Therefore, $x\omega x_1 \dots \omega x_n = y$. This implies that $K \subseteq \omega$. Since $\omega \subseteq \omega^*$, it follows that $K \subseteq \omega^*$. □

Theorem 2.14. *If B is a non-empty subset of M, then $C_M(B) = \bigcup_{b \in B} C_M(b)$.*

Proof. It is clear for every $b \in B$, $C_M(b) \subseteq C_M(B)$, because every complete part containing B contains $\{b\}$. Therefore, $\bigcup_{b \in B} C_M(b) \subseteq C_M(B)$. In order to prove the converse remember that $C_M(B) = \bigcup_{n \geq 1} K_n(B)$, by Theorem 2.10, one clearly has

$$K_1(B) = B = \bigcup_{b \in B} \{b\} = \bigcup_{b \in B} K_1(b).$$

We demonstrate the theorem by induction. Suppose that it is true for n, that is, $K_n(B) \subseteq \bigcup_{b \in B} K_n(b)$ and we prove that $K_{n+1}(B) \subseteq \bigcup_{b \in B} K_{n+1}(b)$. If $z \in K_{n+1}(B)$, then there exist $q \in \mathbb{N}$, $(m_1', \dots, m_q')$, $\sigma \in \mathbb{S}_q$ such that

$$z \in \sum_{i=1}^{q} m_i' \text{ and } \sum_{i=1}^{q} m_i' \cap K_n(B) \neq \emptyset,$$

by the hypothesis induction $\sum_{i=1}^{q} m'_i \cap (\bigcup_{b\in B} K_n(b)) \neq \emptyset$, hence there exists $b' \in B$ such that $\sum_{i=1}^{q} m'_i \cap K_n(b') \neq \emptyset$. Since $z \in \sum_{i=1}^{q} m'_i$ one gets $z \in K_{n+1}(b')$ and so one has prove $K_{n+1}(B) \subseteq \bigcup_{b\in B} K_{n+1}(b)$. Therefore, $C_M(B) \subseteq \bigcup_{b\in B} C_M(b)$. □

Corollary 2.15. *If A is a complete part of M, then for every $B \in P^*(M)$, $A + B, B + A$ are complete parts of M.*

Proof. We have: $C_M(A+B) = A + B + H(M) = A + H(M) + B = C_M(A) + B = A + B$. □

Corollary 2.16. *Let $A \in P^*(M)$. Then, A is a complete part of M if and only if $A + H(M) = A$.*

Proof. We have $C_M(A) = A + H(M) = A$. □

Corollary 2.17. *If $A \in P^*(M)$, then $H(M) + A = A + H(M) = C_M(A)$.*

3. ω_n^*-Complete (R, S)-hyper Bi-modules

In [3], Davvaz and Anvariyeh studied θ-part and θ-closure of hypermodules. Also, see [4, 5].

Definition 3.1. An (R, S)-hyper bi-module M is said to be *ω_n^*-complete (R, S)-hyper bi-module* if there exists $n \in \mathbb{N} \cup \{0\}$, and n is the smallest integer such that $\omega_n^* = \omega^*$ and $\omega_n^* \neq \omega_{n-1}^*$.

Lemma 3.2. *An (R, S)-hyper bi-module M is ω_0^*-complete if and only if M is an (R, S)-bi-module.*

Proof. Suppose that M is an ω_0^*-complete (R, S)-hyper bi-module. Then $\omega_0^* = \omega^*$, and hence $\omega_2 \subseteq \omega_0$ and $\omega_1 \subseteq \omega_0$. Now, for every $x \in m_1 + m_2$ and $y \in m_2 + m_1$, we have $x\omega_2 y$, so $x = y$. Also, for every $x, y \in r \cdot m$, or $x, y \in m \cdot s$, we have $x\omega_1 y$, so $x = y$. Thus, we conclude that $m_1 + m_2 = m_2 + m_1$, $r \cdot m$ and $m \cdot s$ are singleton. Therefore, we conclude that M is an (R, S)-bi-module.

Conversely, if M is an (R,S)-bi-module, then $\sum_{i=1}^{n} m'_i$ is singleton and $|\sum_{i=1}^{n} m'_i| = 1$. By the definition, $x\omega_n y$ if and only if $x = \sum_{i=1}^{n} m'_i$, $y = \sum_{i=1}^{n} m'_i$, thus $x = y$ and $x\omega_0 y$. □

Corollary 3.3. *If M is an ω_n^*-complete (R,S)-hyper bi-module, then M/ω_n^* is an $(R/\Gamma_R^*, S/\Gamma_S^*)$-bi-module.*

Proposition 3.4. *Every finite (R,S)-hyper bi-module is ω_n^*-complete, for some n.*

Proof. Since M is finite, it follows that the succession $\omega_1^* \subseteq \omega_2^* \subseteq \dots$ is stationary. Thus, there exists $n \in \mathbb{N}$ such that $\omega_n^* = \omega^*$ and $\omega_n^* \neq \omega_{n-1}^*$. □

Let M be an (R,S)-hyper bi-module and $\pi : M \to M/\omega^*$ be the canonical projection. We set $H(M) := \pi^{-1}(0_{M/\omega^*})$.

Theorem 3.5. *For every non-empty subset A of an (R,S)-hyper bi-module M, we have*

(1) $\pi^{-1}(\pi(A)) = H(M) + A = A + H(M)$.
(2) *If A is a complete part of M, then $\pi^{-1}(\pi(A)) = A$.*

Proof. (1) For every $x \in H(M) + A$, there exists a pair $(a,b) \in H(M) \times A$ such that $x \in a + b$, so $\pi(x) = \pi(a) \otimes \pi(b) = 0_{M/\omega^*} \otimes \pi(b) = \pi(b)$. Therefore $x \in \pi^{-1}(\pi(b)) \subseteq \pi^{-1}(\pi(A))$.

Conversely, for every $x \in \pi^{-1}(\pi(A))$, an element $b \in H$ exists such that $\pi(x) = \pi(b)$. By the reproducibility, there is $a \in M$ such that $x \in a + b$, and so $\pi(b) = \pi(x) = \pi(a) \otimes \pi(b)$. This implies that $\pi(a) = 0_{M/\omega^*}$ and $a \in \pi^{-1}(0_{M/\omega^*}) = H(M)$. Therefore, we have $x \in a+b \subseteq H(M)+A$. This shows that $\pi^{-1}(\pi(A)) = H(M)+A$. In the same way, we can prove that $\pi^{-1}(\pi(A)) = A + H(M)$.

(2) It is obvious that $A \subseteq \pi^{-1}(\pi(A))$. Moreover, if $x \in \pi^{-1}(\pi(A))$, then there exists an element $b \in A$ such that $\pi(x) = \pi(b)$. Since A is a complete part, it follows that $x \in \omega^*(x) = \omega^*(b) \subseteq A$ and therefore $\pi^{-1}(\pi(A)) \subseteq A$. □

Theorem 3.6. *We have*

(1) *If for every $(v,w) \in H(M)^2$, $v\omega_n w$, then $\omega = \omega_{n+1}$*
(2) *If for every $(v,w) \in H(M)^2$, $v\omega_n^* w$, then $\omega = \omega_{n+1}^*$.*

Proof. (1) If $x\omega y$, since $H(M)+M=M+H(M)=M$, then there exists $(v,w)\in H(M)^2$ such that $y\in x+v$ and $y\in x+w$. By hypothesis $v\omega_n w$. Now, using Lemma 2.2, we have $(x+v)\ \overline{\overline{\omega}}_{n+1}\ (x+w)$, whence $x\omega_{n+1}y$, and so $\omega\subseteq\omega_{n+1}$.

(2) The result follows from (1) and Lemma 2.2. □

Corollary 3.7. *If $v\omega_n^* w$, for every $(v,w)\in H(M)^2$, and there exists $(u',w')\in H(M)^2$ such that $(u',w')\notin\omega_{n-1}^*$, then M is ω_n^*-complete or ω_{n+1}^*-complete.*

4. Complete (R,S)-hyper Bi-modules

In this section, we present an important class of (R,S)-hyper bi-module: complete (R,S)-hyper bi-modules. We investigate some interesting properties of this class of (R,S)-hyper bi-module, for instance we show that any complete (R,S)-hyper bi-module has at least an identity and any element has an inverse.

If M is an (R,S)-hyper bi-module and A is a non-empty subset of M, then we recall the complete closure of A by $C(A)$.

Theorem 4.1. *Let M be an (R,S)-hyper bi-module. The following conditions are equivalent*

(1) *for all $n\geq 1$, $m'_1,\ldots,m'_n$ and for all $a\in\sum_{i=1}^n m'_i$, $C(a)=\sum_{i=1}^n m'_i$,*

(2) *for all $m'_1,\ldots,m'_n$, $C\Big(\sum_{i=1}^n m'_i\Big)=\sum_{i=1}^n m'_i$,*

Proof. $(1\Rightarrow 2)$: We have $C\Big(\sum_{i=1}^n m'_i\Big)=\bigcup_{a\in\sum_{i=1}^n m'_i} C(a)=\sum_{i=1}^n m'_i$.

$(2\Rightarrow 1)$: From $a\in\sum_{i=1}^n m'_i$, we obtain $C(a)\subseteq C(\sum_{i=1}^n m'_i)=\sum_{i=1}^n m'_i$. This means that $C(a)\cap\sum_{i=1}^n m'_i\neq\emptyset$, whence $\sum_{i=1}^n m'_i\subseteq C(a)$. Therefore, $C(a)=\sum_{i=1}^n m'_i$. □

Definition 4.2. An (R,S)-hyper bi-module is complete if it satisfies one of the above equivalent conditions.

Example 7. Suppose that $R = \{x, y\}$. Then $(R, +, \cdot)$ is a hyperring, where

$+$	x	y
x	x	y
y	y	x

$\cdot$	x	y
x	R	R
y	R	R

If we consider R as a (R, R)-hyper bi-module, then it is easy to check that the condition (2) of Theorem 4.1 is satisfied. Therefore, R is complete.

Corollary 4.3. *If M is a complete (R, S)-hyper bi-module, then either there exist $m'_1, \ldots, m'_n$ such that $\omega^*(x) = \sum_{i=1}^{n} m'_i$.*

Theorem 4.4. *If M is a complete (R, S)-hyper bi-module, then*

(1) $H(M) = \{e \in M : \forall x \in M, x \in x + e \cap e + x\}$, *which means that H is the set of two-sided identities of H.*

(2) $(M, +)$ *has at least an identity and any element has an inverse and reversible*

Proof. (1) If $u \in H(M)$, then for all $m \in M$, we have $m \in C(m) = m + H(M) = m + u$. Similarly we have $mu + m$, which means that u is a two-sided identity of M.

Conversely, any two-sided identity u of M is an element of $H(M)$, since $\pi(u) = 0$.

(2) Let a, b, c be elements of M and e be a two-sided identity, such that $e \in b + a \cap a + c$. Then, $b + a = H(M) = a + c$ and $a + b \subseteq a + b + c \subseteq a + H(M) + c = H(M) + a + c = H(M)$, hence $a + b = H(M)$, so b is an inverse of a. Moreover, if $a \in u + v$, then $H(M) = b + a \subseteq b + u + v$, so for any inverse v' of v, we have $v' \in H(M) + v' \subseteq b + uvv' = b + u + H(M) = b + u$. Similarly, from here we obtain $u' \in v + b$, and so $u' + a \subseteq v + b + a = C(v)$, whence $v \subseteq C(v) = u' + a$. In a similar way, we obtain uav'. □

Definition 4.5. An (R, S)-hyper bi-module is called *flat* if for all subhyper bi-module K of M, we have $H(K) = H(M) \cap K$.

Example 8. Let $R = \{0, 1, 2, 3\}$ be a set together with the hyperoperation $+$ and the binary operation $\cdot$ defined as follows:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	$\{0, 2, 3\}$	$\{2, 3\}$
3	3	3	$\{2, 3\}$	$\{0, 1, 2\}$

and $a \cdot b = 0$ for all $a, b \in R$. Then $(R, +, \cdot)$ is a hyperring. According to Example 1, R is an (R, R)-hyper bi-module. Clearly, $\{0\}$, $\{0, 1\}$ and R are subhyper bi-modules of R. Since $H(\{0\}) = \{0\}$, $H(\{0, 1\}) = \{0\}$ and $H(R) = \{0\}$, we conclude that

$$\begin{aligned} &H(\{0\}) = H(R) \cap \{0\}, \\ &H(\{0,1\}) = H(R) \cap \{0,1\}, \\ &H(R) = H(R) \cap R. \end{aligned}$$

This means that R is a flat (R, R)-hyper bi-module.

Theorem 4.6. *Any complete (R, S)-hyper bi-module is flat.*

Proof. Let M be a complete (R, S)-hyper bi-module and suppose that K is a subhyper bi-module M. We have

$$\begin{aligned} H(M) \cap K &= \{e \in M : \forall x \in M, x \in x + e \cap e + x\} \cap K \\ &= \{e \in K : \forall x \in M, x \in x + e \cap e + x\} \subseteq H(K). \end{aligned}$$

Moreover, we have

$$y \in C_K(x) \Rightarrow y\omega_K^* x \Rightarrow y\omega_M^* x \Rightarrow y \in C_M(x),$$

which means that $C_K(x) \subseteq C_M(x)$. Clearly, $H(M) \cap K \neq \emptyset$. If $x \in H(M) \cap K \subseteq H(K)$, then $C_K(x) = H(K)$, $C_M(x) = H(M)$. Hence $H(K) \subseteq H(M)$ whence $H(K) \subseteq H(M) \cap K$. Therefore, we have $H(K) = H(M) \cap K$. □

Corollary 4.7. *If K is a subhyper bi-module of a complete (R, S)-hyper bi-module M, then $H(K) = H(M)$.*

Proof. Set $x \in H(M) \cap K$. We have $H(M) = C(x+x) = x+x \subseteq H(M) \cap K$, whence $H(M) \subseteq H(M) \cap K$, then we apply the above theorem. Hence, $H(K) = H(M)$. □

Theorem 4.8. *Let M and N be two complete (R, S)-hyper bi-modules and $f : M \to N$ be a good homomorphism. Then we have $f(H(M)) = H(N)$.*

Proof. Let $x \in H(M)$. Then $x + x = H(M)$, whence $f(x) + f(x) = f(H(M))$. On the other hand, $f(x)$ is an identity of N, since x is an identity of M, which means that $f(x) \in H(N)$. Therefore, $H(N) = f(x) + f(x) = f(H(M))$. □

5. Heart of (R, S)-hyperbimodules

In [8], Corsini and Leoreanu investigated the heart of hypergroups. In [1] and [2], Anvariyeh and Davvaz studied the characterizations of hearts of hypermodules, and established a few results concerning the sequence of heart. In this section, we examine and study the heart of (R,S)-hyper modules.

Theorem 5.1. *Let M be an (R,S)-hyper bi-module and B the union of summations of finite numbers of $\sum_{i=1}^{n} m'_i$, containing at least one right and at least one left identity and be scalar multiplicatively closed. Then $B = H(M)$.*

Proof. We set $E_l(E_r)$ the set of left (right) identities and $T = \{P \in B \mid P \cap E_l \neq \emptyset,\ P \cap E_r \neq \emptyset\}$. Furthermore, for every $x \in M$, we denote with $i_l(x)(i_r(x))$ the set of left (right) inverses of x. The first, we prove that for every $a \in B,\ i_l(a) \subseteq B \supseteq i_r(a)$. Let $a \in M$, then a $\sum_{i=1}^{n} m'_i = P \in T$ exists such that $a \in P$. If $a' \in i_l(a), e' \in i_l$ exists such that $e' \in a' + a$; if $a'' \in i_l(a), e'' \in E_r$ exists such that $e'' \in a + a''$. We now consider the $P_1 = a' + \sum_{i=1}^{n} m'_i + a + a''$, we have $P_1 \subseteq T$, in fact $\{e', e''\} \subseteq e' + e'' \subseteq a' + a + a + a'' \subseteq P_1$. Furthermore, $\{a', a''\} \subseteq P_1$; in fact $a' + a + a'' \subseteq P_1$ and $a' \in a' + e'' \subseteq a' + a + a''$, also $a'' \in e' + a'' \subseteq a' + a + a''$.

Now, we prove that B is a complete part of M. Let $a \in \sum_{i=1}^{n} m'_i \cap B \neq \emptyset$, hence there exists $\sum_{i=1}^{t} z'_i = P \in T$ such that $a \in P$. Now let e', e'' be the left and right identities, respectively. We have $a', a'' \in M$ such that $e' \in a' + a,\ e'' \in a + a''$. Then $\sum_{i=1}^{n} m'_i \subseteq e' + \sum_{i=1}^{n} m'_i + e'' \subseteq a' + a + \sum_{i=1}^{n} m'_i + a + a'' \subseteq a' + P + \sum_{i=1}^{n} m'_i + P + a'' \supseteq a' + a + a + a'' \supseteq \{e', e''\}$, thus $a' + P + \sum_{i=1}^{n} m'_i + P + a'' = P_1$. Therefore $\sum_{i=1}^{n} m'_i \subseteq P_1 \in T$ and for this reason $\sum_{i=1}^{n} m'_i \subseteq B$.

Let $a, b \in M$, such that $a \in P, b \in Q$ where $P, Q \in T$. Then $a + b \in B$. Also, for every $(r,s) \in R \times S$, $r \cdot a \subseteq B$ and $a \cdot s \subseteq B$.

Furthermore, B satisfies the conditions of reproducibility. Since M is an (R,S)-hyper bi-module, the properties of M as an $(R,S)-$hyperbimodule, guarantee that the hypergroup B is an (R,S)-hyper bi-module. It is clear that $B \subseteq H(M)$. As seen from the above, it turns out that B is a complete part subhyper bi-module, thus $H(M) \subseteq B$. $\square$

We denote $\sum_{C}(A)$ the set hypersums A of elements of M such that $C(A) = A$.

Theorem 5.2. *If M is an (R, S)-hyper bi-module and $(x'_1, \ldots, x'_n)$ such that $\sum_{i=1}^{n} x'_i \in \sum_{C}(M)$, then there exists $(y'_1, \ldots, y'_n)$ such that $\sum_{i=1}^{n} x'_i + \sum_{i=1}^{n} y'_i = H(M)$.*

Proof. We set $x'_i = \sum_{j=1}^{n_i} \Big(\prod_{k=1}^{k_{ij}} r_{ijk}\Big) x_i$. For $1 \leq t \leq n$, let a_t be an element of $H(M)$. Then, there exists $y_t \in M$ such that $a_t \in x_t + y_t$, and hence

$$\sum_{j=1}^{n_t} \prod_{k=1}^{k_{tj}} r_{tjk} a_t \subseteq \sum_{j=1}^{n_t} \prod_{k=1}^{k_{tj}} r_{tjk} x_t + \sum_{j=1}^{n_t} \prod_{k=1}^{k_{tj}} r_{tjk} y_t = x'_t + y'_t.$$

Since $H(M)$ is a complete part, it follows that $x'_t + y'_t \subseteq H(M)$. Therefore

$$\begin{aligned} \sum_{i=1}^{n} x'_i + y'_n &= H(M) + \sum_{i=1}^{n} x'_i + y'_n = \sum_{i=1}^{n-1} x'_i + H(M) + x'_n + y'_n \\ &= \sum_{i=1}^{n-1} x'_i + H(M) = H(M) + \sum_{i=1}^{n-1} x'_i \end{aligned}$$

and so

$$\sum_{i=1}^{n} x'_i + y'_n + y'_{n-1} = H(M) + \sum_{i=1}^{n-2} x'_i + x'_{n-1} + y'_{n-1} = H(M) + \sum_{i=1}^{n-2} x'_i.$$

Going on the same way one arrives to

$$\sum_{i=1}^{n} x'_i + \sum_{i=1}^{n} y'_i = H(M) + x'_1 + y'_1 = H(M).$$

□

Lemma 5.3. *Let $(M, +)$ be an (R, S)-hyper bi-module, then*

(1) *$M - H(M)$ is a complete part of M.*

(2) *If $M - H(M)$ is a hypersum, then $H(M)$ is also a hypersum.*

Proof. (1) It is straightforward.

(2) For (1), $M - H(M)$ is a complete part. Now by using Theorem 3.2, the proof is completed.

□

Remark 1. Let M be an (R,S)-hyper bi-module endowed with a complete hyper-sum. The following implication is satisfied for every $A \in P^*(M)$:

$$A \cap \sum_{i=1}^{n} m_i' = \emptyset \Rightarrow C(A) \cap \sum_{i=1}^{n} m_i' = \emptyset.$$

Assume that $z \in C(A) \cap \sum_{i=1}^{n} m_i'$, then $a \in A$ exists such that $z \in C(a)$, hence $C(a) = C(z)$. The hypothesis $\sum_{i=1}^{n} m_i' = C\Big(\sum_{i=1}^{n} m_i'\Big)$ implies

$$C(z) \subseteq \bigcup_{y \in \sum_{i=1}^{n} m_i'} C(y) = C\Big(\sum_{i=1}^{n} m_i'\Big) = \sum_{i=1}^{n} m_i'.$$

Therefore $a \in A$, $a \in C(z) \subseteq \sum_{i=1}^{n} m_i'$, where $\sum_{i=1}^{n} m_i' \cap A \neq \emptyset$ which absurd.

Let $(M,+)$ be an (R,S)-hyper bi-module. Let's consider the sequence

$$(*) \qquad M \supseteq H(M) = H_1 \supseteq H(H(M))) = H_2 \supseteq \ldots \supseteq H_k \supseteq H_{k+1} \supseteq \ldots \supseteq H_n \supseteq \ldots$$

Proposition 5.4. *Let M be an (R,S)-hyper bi-module. Then the following conditions are equivalent:*

(1) *The sequence $(*)$ is finite;*
(2) *there is $(n,k) \in \mathbb{N}^2$, where $n > k+1$, such that H_n is a complete part of H_k;*
3) *there is $(n,k) \in \mathbb{N}^2$ where $n > k+1$, such that for any $(x,y) \in (H_k - H_n) \times (H_k - H_n)$; $(x+y) \cap (H_k - H_n) \neq \emptyset$ implies $x + y \subseteq H_k - H_n$;*
(4) *there is $(n,k) \in \mathbb{N}^2$, where $n > k+1$, such that for any H_n is an ω_n-conjugable.*

Proof. $(1 \Rightarrow 2)$ If the sequence $(*)$ is finite, then there is $n \in \mathbb{N}$ such that $H_n = H_{n-1}$, hence H_{n-2} is a complete part of H_n.

$(2 \Rightarrow 3)$ If H_n is a complete part of H_k, then $H_k - H_n$ is a complete part of H_k.

$(3 \Rightarrow 4)$ Ones proves easily that for any $s \in \mathbb{N}$, H_s is a closed subhyperbimodule of M. Moreover, for all $a, b \in H_k$, if $\{a,b\} \subseteq H_k - H_n$, we have $a + b \subseteq H_n$, if $a \neq b$ and $|\{a,b\} \cap H_n| = 1$, we have $a + b \subseteq H_k - H_n$. Then, we obtain that H_n is H_k-conjugable.

($4 \Rightarrow 1$) We know H_n is a complete part subhyperbimodule of H_k. Hence $H_{k+1} = H(H_k) \subseteq H_n \subseteq H_{k+1}$ from which $H_n = H_{k+1}$. So, we have: $H_{n+1} = H(H_n) = H(H_{k+1}) = H_{k+2} \supseteq H_n = H_{k+1} \supseteq H_{k+2}$. Therefore, $H_n = H_{k+2} = H_{n+1}$. Let $H_{n+s} = H_{k+1}$. It follows $H_{n+s+1} = H(H_{n+s}) = H_{k+1} = H_{k+2} = H_{k+1}$. Then, for any m such that $m \geq n$, we have $H(M) = H_n$. □

Theorem 5.5. *Let $(M, +)$ be an (R, S)-hyper bi-module such that the sequence $(*)$ is finite, and let N be a complete part subhyper bi-module of M. Then there is $p \in \mathbb{N}$ such that $H_{p+1}(N) = H_{p+1}(M)$.*

Proof. Let's remark that $H(N)$ is a subhyper bi-module of $H(M)$. Indeed, for any $a \in H(K)$, there is $e \in N$ such that $a \in a + e$, it's clear that $a \in \omega_K(e) \subseteq \omega_M(e) = H(M)$. Moreover, since N is a complete part subhyper bi-module of M, we have $H(M) \subseteq N$. Then $H_1(N) \subseteq H_1(M) \subseteq N$. For any $s \geq 1$, from $H_s(N) \subseteq H_s(M) \subseteq H_{s-1}(N)$, one obtains $H_{s+1}(N) \subseteq H_{s+1}(M) \subseteq H_s(N)$, and hence $N \supseteq H_1(M) \supseteq H_1(N) \supseteq H_2(M) \supseteq H_2(N) \supseteq \ldots$.

By Theorem 5.4, there is $(n, p) \in \mathbb{N} \times \mathbb{N}$, where $n > p + 1$, such that $H_n(M) = H_{p+1}(M)$, therefore $H_{p+1}(M) = H_{p+1}(N)$. □

Remark 2. If N_1, N_2 are subhyperbimodules of M, then

$$H(N_1 \cap N_2) \leq H(N_1) \cap H(N_2).$$

Proposition 5.6. *If $N_1, N_2 \leq M$, where M has a finite sequence $(*)$, then there exists $p \in \mathbb{N}$, such that $H_{p+1}(N_1 \cap N_2) = H_{p+1}(H(N_1) \cap H(N_2))$.*

Proof. Let's consider $\overline{M} := N_1 \cap N_2$ and $\overline{N} := H(N_1) \cap H(N_2)$. Then $\overline{N}$ is a subhyperbimodule, complete part of $\overline{M}$. (We can verify this using the definition of a complete part.) Now, we can use the proof of Theorem 5.5. □

Also, we can give a relation for (R, S)-subhyper bi-module of M :

$$\exists p \in \mathbb{N}, \quad H_{p+1}(N_1 \cap N_2 \cap \ldots \cap N_m) = H_{p+1}(H(N_1) \cap H(N_2) \cap \ldots \cap H(N_m)).$$

Remark 3. If $N_1, N_2 \leq M$, then

$$H(N_1) \subseteq N_1 \cap \ H(\langle N_1 \cup N_2 \rangle).$$

Generally, we have not equality. Let M_1 and M_2 be two (R, S)-hyper bi-modules. Let m_1, n_1 arbitrary in M_1 and m_2, n_2 arbitrary in M_2. Let's define on $M = M_1 \cup M_2 \cup \{a\}$

$(a \notin M_1 \cup M_2)$ with the following hyperoperations:

$+'$	m_1	a	m_2
n_1	$n_1 + m_1$	a	M
a	a	M_1	M
n_2	M	M	$n_2 + m_2$

and for every $(r, s) \in R \times S, x \in M_1$ and $y \in M_2$ scalar multiplication

$$r \cdot' x = r \cdot_1 x, \quad r \cdot' y = r \cdot_2 y \quad x \cdot' s = x \cdot_1 s, \quad y \cdot' s = y \cdot_2 s$$

and $r \cdot' a = a \cdot' s = a$. We can easily verify $(M, +')$ with scalar multiplication $\cdot'$ is an (R, S)-hyper bi-module. We consider subhyper bi-modules $N_1 = M \cup \{a\},\ N_2 = M_2,\ \ N_1 \cup N_2 = M,\ \langle N_1 \cup N_2 \rangle = M$, then $H(\langle N_1 \cup N_2 \rangle) = M$. So

$$H(N_1) = M_1 \subsetneqq N_1 \cap H(\langle N_1 \cup N_2 \rangle) = N_1 = M_1 \cup \{a\}.$$

Theorem 5.7. *Let M be an (R, S)-hyper bi-module and N_1, N_2 be two subhyper bi-module of M. If for every $a \in \langle N_1 \cup N_2 \rangle -\ (N_1 \cup N_2)$, there exists $(n_1, n_2) \in N_1 \times N_2$, such that $a \in n_1 + n_2$ and if $\langle H(N_1) \cup H(N_2) \rangle$ is a closed subhyper bi-module of $H(\langle N_1 \cup N_2 \rangle)$ then $\langle H(N_1) \cup H(N_2) \rangle = H(\langle N_1 \cup N_2 \rangle)$.*

Proof. We shall prove that $\langle H(N_1) \cup H(N_2) \rangle$ is conjugable in $\langle N_1 \cup N_2 \rangle$ as hyper bi-module. $\langle H(N_1) \cup H(N_2) \rangle$ is closed in $\langle N_1 \cup N_2 \rangle$ because, from $a \in b + x$, where $(a, b) \in \langle H(N_1) \cup H(N_2) \rangle^2$ and $x \in \langle N_1 \cup N_2 \rangle$, it results $(a, b) \in (H^2 \langle N_1 \cup N_2 \rangle)$ and so $x \in H(\langle N_1 \cup N_2 \rangle)$. Using now the condition given in the proposition, $x \in \langle H(N_1) \cup H(N_2) \rangle$.

As regards an arbitrary element $a \in \langle N_1 \cup N_2 \rangle$, we have three situation:

$$\begin{aligned} &a \in N_1 \Rightarrow \exists a' \in N_1,\ a + a' \subseteq H(N_1) \subseteq \langle H(N_1) \cup H(N_2) \rangle; \\ &a \in N_2 \Rightarrow \exists a' \in N_2,\ a + a' \subseteq H(N_2) \subseteq \langle H(N_1) \cup H(N_2) \rangle; \\ &a \in \langle N_1 \cup N_2 \rangle - (N_1 \cup N_2) \Rightarrow \exists n_1 \in N_1, \exists n_2 \in N_2, a \in n_1 + n_2. \end{aligned}$$

For n_i there exists $n_i' \in N_i$, such that $n_i + n_i' \in H_{n_i},\ i = 1, 2$.

So, $a + n_1' + n_2' \subseteq (n_1' + n_2') + (n_2 + n_2') \subseteq H(N_1) \oplus H(N_2) \subseteq \langle H(N_1) \cup H(N_2) \rangle$, whence for every $t \in n_1' + n_2'$, $a + t \subseteq \langle H(N_1) \cup H(N_2) \rangle$. □

An (R, S)-hyper bi-module M is called $1-(R, S)$-hyper bi-module if $H(M)$ is a singleton.

Lemma 5.8. *If M is a 1-(R, S)-hyper bi-module, then M is an ω_2^*-complete (R, S)-hyper bi-module.*

Proof. Suppose that $H(M) = \{e\}$. Then for all $m \in M$, we have $m + e = e + m$ and so the classes module ω are $\{e, m\}$. It follows that $\omega = \omega_2 = \omega_2^*$. □

Theorem 5.9. *Let M be a $1-(R,S)$-hyper bi-module and $H(M) = \{e\}$. Then*

(1) *The ω^*-classes are the summations $e + a$, where $a \in M$.*
(2) *Every (R,S)-subhyper bi-module of M is complete part.*
(3) *If $\{M_i\}_{i\in I}$ is a family of (R,S)-subhyper bi-module of M, then $\bigcap_{i\in I} M_i$ is an (R,S)-subhyper bi-module of M.*
(4) *The direct product of $1-(R,S)$-hyper bi-modules is a $1-(R,S)$-hyper bi-module.*

Proof. (1) It is straightforward.

(2) If N is a subhyper bi-module of M, we have $N \cap W(M) \neq \emptyset$, for this reason $H(M) \subseteq N$, hence $N = N + H(M)$ and therefore N is a complete part.

(3) For (2), for every $i \in I$, $e \in M_i$. We set $M = \bigcap_{i\in I} M_i$, hence $M \neq \emptyset$. Then for every $x, y \in M$, $b \in M$ exist such that $y \in b + x$, but for every $i \in I$, for (2), M_i is a closed submodule, thus $b \in M_i$. Also, for every $r \in R, m \in M$, we have $r.m \subseteq M$.

(4) Set $N = \prod_{i\in I} N_i$, $m' = (m'_i)_{i\in I} \in N$, $e = (e_i)_{i\in I}$. We have $x\omega_n e$ if and only if $z'^1 = (z_i'^1)_{i\in I}$, $z'^2 = (z_i'^2)_{i\in I}, \ldots, z'^n = (z_i'^n)_{i\in I}$, exists such that $x, e \in \sum_{i=1}^{n} z'^k$, that is if and only if for each $i \in I$, $z'_i, e_i \in \sum_{k=1}^{n} z_i'^k$. Then $z'_i = \sum_{k=1}^{n} z_i'^k = e_i$, from $x = e$, for this reason $H(M) = \{e\}$. □

6. Conclusion

The notion of (R,S)-hyper bi-modules is a generalization of hypermodules and bimodules. The heart of an (R,S)-hyper bi-module is the neutral element of the quotient fundamental bi-module. We studied the properties of the heart and complete parts of (R,S)-hyper bi-modules. In particular, we proved that any compact (R,S)-hyper bi-module has at least one identity element.

For future research, we will study the properties of exact sequences of (R,S)-hyper bi-modules.

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