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APPROXIMATION BY MODIFIED POST-WIDDER OPERATORS

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Abstract. The current article manages with new generalization of Post-Widder operators preserving constant function and other test functions in Bohmann-Korovkin sense and studies the approximation properties via different estimation tools like modulus of continuity and approximation in weighted spaces. The viability of the recently modified operators as per classical Post-Widder operators is introduced in specific faculties also. Numerical examples are additionally introduced to verify our theortical results. In second last section we introduce Grüss-Voronovskaya results and in last section, we show the better approximation our new modified operators via graphical exmaples using Mathematica.

1. Introduction

The previous 70 years have seen progressively quick advancements in area of approximation theory. In the direction to response the question of how to reveal the best convergence to the given function, a lot of research has been printed. These examinations offer a number of ways to deal with the development of estimating functions. Meanwhile, P. P. Korovkin and autonomously by H. Bohman in the fifties presented one of the major noteworthy theorem in approximation theory, called by Bohman-Korovkin's approximation theorem, which give yardstick to look over that a given sequence $(L_{\mathrm{k}})_{\mathrm{k}} \geq 1$ of positive linear operators converge to the function as to the uniform norm of the space $C[a, b]$, which implies, whether it speaks to or not an estimation method. This result gives understanding into the investigations on linear positive operators furthermore, a few new developments of approximating operators have been found in literature. In addition, Post-Widder operators presented by May [16] are one of the most broadly utilized gatherings in estimation process and have been widely utilized for finding a better estimate to the selected function. The

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fundamental Post -Widder operators are presented as

$$(P_j\mathbf{h})(x) = \frac{1}{j!}\left(\frac{j}{x}\right)^{j+1}\int_0^\infty t^j e^{-jt/x}\mathbf{h}(t)dt, \tag{1.1}$$

At that point, an enormous measure of literature have showed up in tending to refine Post-Widder operators, which indicated that recently characterized operators have comparative same approximation features to old style operators (see for instance, [[8],[13],[17], [18], [19], [20]]). Very recently, Sofyalioğlu and Kanat [21] modified Post-Widder operators in following form

$$P^*_{j,a}(\mathbf{h};x) = \frac{(2a)^j}{(j-1)!(1-e^{\frac{-2ax}{j}})^j}\int_0^\infty t^{j-1}e^{\frac{-2at}{1-e^{-2ax/j}}}\mathbf{h}(t)dt, \tag{1.2}$$

where $a > 0$ and $j \in \mathbb{N}$ and examined the rate of convergence by utilizing various kinds of the modulus of continuity and manages with a quantitative Voronovskaya-type theorem. At the last, authors compared their new developed operators with Post-Widder operators constructed by Gupta and Tachev [14].

The fundamental point of present article is to offer a reasonable theoretical framework based on modification of Post-Widder operators that preserve the constant functions and we deal an adequate condition under which the modified operators perform superior to the classical ones. The present article is neatly categorized in following way:

- Section 2 presents the modified Post-Widder operator and determine the values of test function and central moments for the same.
- In Section 3, some estimated results are calculated via different tools of approximation process and compare the results with classical Post-Wider operators define by (1.1) theoretically as well as graphically.
- In Section 4, we examine the estimation error of the operators defined in Section 1 in weighted space via different weighted approximation tools and study the Grüss-Voronovskaya theorem.
- last section, we show some graphical examples of our constructed operators to verify the better approximation to function using mathematica.

2. A New Modification of Post-Widder Operators

In this segment we present a modification of Post-Widder operators and give some essential results which will be utilized in the remaining portion of this paper. All through this and the following sections, we mean by e_j as the polynomial function characterized by $e_j(t) = t^j$ and $\vartheta_{x,j}(t) = (t-x)^j$ for $x \in \mathbb{R}^+$ and $j \in \mathbb{N}$.

The recently referenced modification of our enthusiasm for this assessment is described by

$$\mathfrak{P}_j(\mathbf{h}; x) = \frac{(jx)^{j+2}}{\Gamma j+2} \int_0^\infty \exp^{-jx/v} v^{-j-3} \mathbf{h}(v) dv. \tag{2.1}$$

Or, we can redefine the above operators in kernel form as follows

$$\mathfrak{P}_j(\mathbf{h}; x) = \int_0^\infty \mathrm{G}_j(x,v) \mathbf{h}(v) dv,$$

where

$$\mathrm{G}_j(x,v) = \frac{(jx)^{j+2}}{\Gamma j+2} \exp^{-jx/v} v^{-j-3},$$

for bounded $\mathbf{h} \in \mathbb{R}^+,\ j \in \mathbb{N}$ and $x \in \mathbb{R}^+$. Obviously these new modified operators are linear and positive along these lines we will figure that this conditions satisfy starting now and into the foreseeable future.

Lemma 1. *The operator $\mathfrak{P}_j(.; x)$ satisfies underneath equalities:*

(i) $\mathfrak{P}_j(1; x) = 1$,

(ii) $\mathfrak{P}_j(t; x) = \dfrac{jx}{j+1}$,

(iii) $\mathfrak{P}_j(t^2; x) = \dfrac{jx^2}{(j+1)}$,

(iv) $\mathfrak{P}_j(t^3; x) = \dfrac{j^3x^3}{j(j-1)(j+1)}$,

(v) $\mathfrak{P}_j(t^4; x) = \dfrac{j^4x^4}{j(j-1)(j-2)(j+1)}$,

(vi) $\mathfrak{P}_j(t^5; x) = \dfrac{j^5x^5}{j(j-1)(j-2)(j-3)(j+1)}$,

(vii) $\mathfrak{P}_j(t^6; x) = \dfrac{j^6x^6}{j(j-1)(j-2)(j-3)(j-4)(j+1)}$.

Proof. Using definition of operators $\mathfrak{P}_j(.; x)$ and simple calculations, the proof of identities (i)-(vii) is immediate. Hence we skip the details. □

It is obvious from the above lemma that operators $\mathfrak{P}_j(.; x)$ conserve following in limit case as

$$\mathfrak{P}_j(e_1; x) \to x$$

$$\mathfrak{P}_j(e_2; x) \to x^2$$

$$\mathfrak{P}_j(e_3; x) \to x^3$$

$$\mathfrak{P}_j(e_4; x) \to x^4$$

$$\mathfrak{P}_j(e_5; x) \to x^5$$

$$\mathfrak{P}_j(e_6; x) \to x^6$$

Above results directly indicate that sequence of operators $\{\mathfrak{P}_j\}_{j\geq 1}$ do not fulfill Korovkin test function criteria except constant. Although as indicated by Bohman-Korovkin Theorem $\{\mathfrak{P}_j\}_{j\geq 1}$, is an estimation procedure on any compact subset $D \subset \mathbb{R}^+$ since the test functions conserve Korovkin polynomial functions in limiting case.

In general, for $x \in \mathbb{R}^+$ and $j \in \mathbb{N}$, following equality holds for $\mathfrak{P}_j$ as

$$\mathfrak{P}_j(e_k; x) = \frac{j^k \Gamma(j-k+2)}{\Gamma(j+2)} e^k(x), \; k = 0, 1, 2, \cdots$$

As an outcome of Lemma 1, we obtain:

Lemma 2. *The operator $\mathfrak{P}_j$ verifies the following central moments values:*

(i) $\mathfrak{P}_j(\varphi_{x,1}; x) = \dfrac{-x}{j+1}$,
(ii) $\mathfrak{P}_j(\varphi_{x,2}; x) = \dfrac{(x^2}{(j+1)}$,
(iii) $\mathfrak{P}_j(\varphi_{x,4}; x) = \dfrac{(3j+2)x^4}{(j-1)(j-2)(j+1)}$,
(iv) $\mathfrak{P}_j(\varphi_{x,6}; x) = \dfrac{(15j^2+190j+24)x^6}{(j-1)(j-2)(j-3)(j-4)(j+1)}$.

Lemma 3. *For every $x \in [0, \infty)$, we have*

(i) $\lim_{j\to\infty} j\, \mathfrak{P}_j((t-x); x) = x$;
(ii) $\lim_{j\to\infty} j\, \mathfrak{P}_j((t-x)^2; x) = x^2$;
(iii) $\lim_{j\to\infty} j^2\, \mathfrak{P}_j((t-x)^4; x) = 3x^4$;
(iv) $\lim_{j\to\infty} j^3\, \mathfrak{P}_j((t-x)^6; x) = 15x^6$.

Proof. The confirmation of Lemma 3 follows effectively from Lemma 2, so the subtleties are precluded. □

3. Main Results

Next, we define $C_B[0,\infty)$ as the normed linear space of all the function having property of boundedness and uniformly continuity on $[0,\infty)$ favored the norm defined as

$$||\mathbf{h}|| = \sup_{x\in[0,\infty)} |\mathbf{h}(x)|.$$

In the going with theorem we show that the operators $\mathfrak{P}_j$ is an estimation process for functions in $C_B(\mathbb{R}^+)$.

Theorem 1. *Let $\mathcal{D}$ be a compact subset of $[0,\infty)$. Then, for $\mathbf{h}\in C_B(\mathbb{R}^+)$,*

$$\lim_{j\to\infty} \mathfrak{P}_j(\mathbf{h};x) \to \mathbf{h}(x),$$

uniformly on $\mathcal{D}$.

Proof. Considering Lemma 1, we have

$$\mathfrak{P}_j(e_k;x) \to e_k, \text{ as } j\to\infty, \text{ uniformly on } \mathcal{D}, \text{ for } k=0,1,2.$$

Accordingly, the vital outcome goes to by seeking the Bohman-Korovkin criterion. □

In our next result, we examine the rate of convergence of the operators $\mathfrak{P}_j(.;x)$ and then compare outcome with the result of classical Post-Widder operators given in (1.1).

Theorem 2. *Let $\omega_{y_0+1}(\mathbf{h},\delta)$ be the modulus of continuity on the finite interval $[0,y_0+1]\subset[0,\infty)$ for $y_0>0$ and $\mathbf{h}\in C_B[0,\infty)$. Then the following inequality holds*

$$|\mathfrak{P}_j(\mathbf{h};x)-\mathbf{h}| \leq 3\mathcal{M}_{\mathbf{h}}\frac{1}{j+1}y_0^2(1+y_0)^2 + 2\omega_{y_0+1}\left(\mathbf{h}, \sqrt{\frac{1}{j+1}y_0}\right),$$

where $\mathcal{M}_{\mathbf{h}}$ is a constant just depending on $\mathbf{h}$.

Proof. By considering $0\leq x\leq y_0$, $t>y_0+1$ and $\mathbf{h}\in C_B[0,\infty)$, we can write for $t-x>1$

$$\begin{aligned}|\mathbf{h}(t)-\mathbf{h}(x)| &\leq \mathcal{M}_{\mathbf{h}}(2+t^2+x^2)\\ &\leq \mathcal{M}_{\mathbf{h}}(t-x)^2(2x^2+2x+3)\\ &\leq 3\mathcal{M}_{\mathbf{h}}(t-x)^2(1+y_0).\end{aligned}$$

Again, for $\mathbf{h} \in C_B[0,\infty), 0 \leq x \leq y_0$ and $t < y_0 + 1$ we have following inequalities

$$\begin{aligned} |\mathbf{h}(t) - \mathbf{h}(x)| &\leq \omega_{y_0+1}(\mathbf{h}, |t-x|) \\ &\leq \omega_{y_0+1}(\mathbf{h}, \delta)\bigg(1 + \frac{1}{\delta}(|t-x|)\bigg), \end{aligned}$$

now combining above inequalities, for $0 \leq x \leq y_0,\ 0 \leq t < \infty$ we conclude that

$$|\mathbf{h}(t) - \mathbf{h}(x)| \leq 3\mathcal{M}_{\mathbf{h}}(t-x)^2(1+y_0) + \omega_{y_0+1}(\mathbf{h}, \delta)\bigg(1 + \frac{1}{\delta}(|t-x|)\bigg).$$

Now operating $\mathfrak{P}_j$ and Cauchy-Schwarz inequality to above expression, we obtain

$$\begin{aligned} |\mathfrak{P}_j(\mathbf{h}; x) - \mathbf{h}(x)| &\leq 3\mathcal{M}_{\mathbf{h}}\mathfrak{P}_j((t-x)^2; x)(1+y_0)^2 \\ &\quad + \omega_{y_0+1}(\mathbf{h}, \delta)\bigg(1 + \frac{1}{\delta}\sqrt{(\mathfrak{P}_j((t-x)^2); x)}\bigg) \\ &\leq 3\mathcal{M}_{\mathbf{h}}\frac{1}{j+1}y_0^2(1+y_0)^2 + \omega_{y_0+1}\bigg(\mathbf{h}, \sqrt{\frac{1}{(j+1)}y_0^2}\bigg) \end{aligned}$$

Letting $\delta = \sqrt{\frac{1}{(j+1)}y_0^2}$ we finish the proof. □

Remark 1. Let $(P_j f)(x)$ be the classical Post-Widder operators given in (1.1). It can be easily verified that

$$|(P_j\mathbf{h})(x) - \mathbf{h}(x)| \leq 3\mathcal{M}_{\mathbf{h}}\frac{1}{j}y_0^2(1+y_0)^2 + \omega_{y_0+1}\bigg(\mathbf{h}, \sqrt{\frac{1}{j}y_0^2}\bigg)$$

where modulus of continuity is given by $\omega_{y_0+1}(\mathbf{h}, \delta)$ on finite interval $[0, y_0+1],\ y_0 > 0$ and $\mathcal{M}_{\mathbf{h}}$ is a constant depends only on $\mathbf{h}$ and $\mathbf{h} \in C_B[0,\infty)$.

Our assertion is that error approximation of modified Post-Widder operators gives us better error estimation in comparison with their classical Post-Widder operators for $\mathbf{h} \in C_B[0,\infty)$ and $y_0 > 0$ given in (3.1). Therefore as to demonstrate this case we just need to show $\frac{y_0^2}{j(j+1)} \geq 0$ for $y_0 \geq 0$. In reality

$$\frac{y_0^2}{j} - \frac{y_0^2}{j+1} = \frac{y_0^2}{j(j+1)} \geq 0 \text{ for } y_0 \geq 0\ j \geq 0.$$

This outcome ensures that the error term $\sqrt{\frac{1}{(j+1)}y_0^2}$ for our modified operator is smaller than $\sqrt{\frac{1}{j}y_0^2}$ in (3.1), which guarantees the proof of our claim. In addition, we will introduce some numerical examples to address our case by numerically.

Table 1. Error table for the operators $\mathfrak{P}_j(.;x)$ and $(P_j.)(x)$

j	1	5	10	15	50	100
error $\mathfrak{P}_j(.;x)$	0.707107x	0.408248x	0.301511x	0.25x	0.140028x	0.995037x
error $(P_j.)(x)$	x	0.447214x	0.316228x	0.258199x	0.141421	0.1

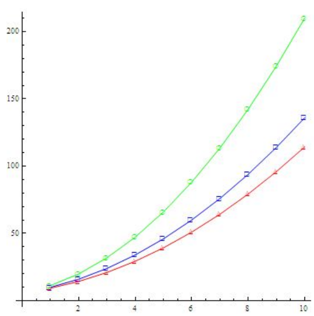

Figure 1

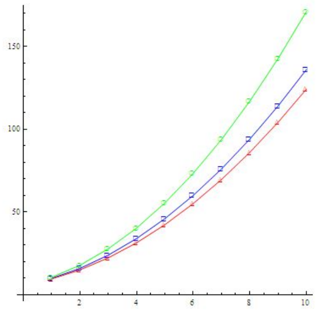

Figure 2

Figure 1 shows convergence of operators $\mathfrak{P}_j(.;x)$ and $(P_j.)(x)$ to the function $\mathbf{h}(x) = x^2 + 3x + 6$ for $j = 5$ and $x \in [0, 10]$.
Figure 2 shows convergence of operators $\mathfrak{P}_j(.;x)$ and $(P_j.)(x)$to the function $\mathbf{h}(x) = x^2 + 3x + 6$ for $j = 10$ and $x \in [0, 10]$.

Blue, green and red curves represent the function $h(x)$, the Post-Widder operators $(P_j\mathfrak{h})(x)$ and modified Post-Widder operators $\mathfrak{P}_j(\mathfrak{h};x)$ respectively.

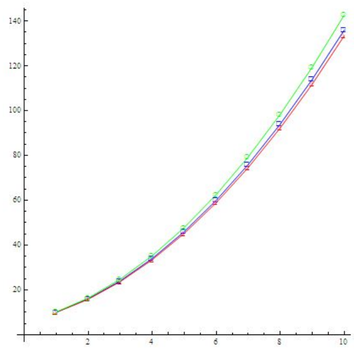

Figure 3

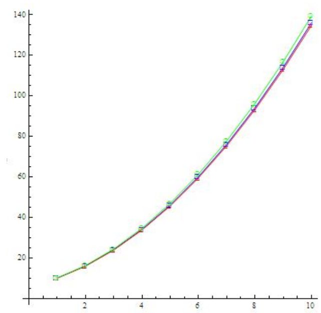

Figure 4

Figure 3 shows convergence of operators $\mathfrak{P}_j(.;x)$ and $(P_j.)(x)$ to the function $\mathbf{h}(x) = x^2 + 3x + 6$ for $j = 5$ and $x \in [0, 10]$.
Figure 4 shows convergence of operators $\mathfrak{P}_j(.;x)$ and $(P_j.)(x)$to the function $\mathbf{h}(x) = x^2 + 3x + 6$ for $j = 10$ and $x \in [0, 10]$.

Blue, green and red curves represent the function $h(x)$, the Post-Widder operators $(P_j\mathfrak{h})(x)$ and modified Post-Widder operators curves $\mathfrak{P}_j(\mathfrak{h};x)$ respectively.
In light of above table and graphs, we conclude that the approximation given by recently consider modified Post-Widder operators $\mathfrak{P}_j(.;x)$ is much better than the classical Post-Widder operators $(P_j.)(x)$.

4. WEIGHTED APPROXIMATION PROPERTIES

Around there, we give Korovkin type results for weighted estimation of modified Post Widder operators, $(\mathfrak{P}_j)_{j\geq 1}$. As per this reason, we define some weighted spaces on $[0,\infty)$ as follows

- $B_\sigma(\mathbb{R}^+)$ denotes space of all bounded functions in weighted space i.e it contains all the functions $\mathbf{h}$ such that $|\mathbf{h}(x)| \leq \mathcal{M}_{\mathbf{h}}\sigma(x)$.
- $C_\sigma(\mathbb{R}^+)$ is space of all continuous and bounded functions in weighted space, it means $C_\sigma(\mathbb{R}^+)$ contains all the functions of type $\mathbf{h} : \mathbf{h} \in B_\sigma(\mathbb{R}^+) \cap C[0,\infty))$.
- Lastly $C_\sigma^k(\mathbb{R}^+)$ denotes the class of function of type $\mathbf{f} \in C_\sigma(\mathbb{R}^+)$ and $\lim_{x\to\infty} \frac{\mathbf{f}(x)}{\sigma(x)} = k$ (some constant)) in weighted space.

Here, the weight function is denoted by $\sigma(x) = 1+x^2$ and the constant $\mathcal{M}_{\mathbf{h}}$ depends only on the function.

The proof of existence as normed linear space of $C_\sigma(\mathbb{R}^+)$is shown in [4] favored the norm $||\mathbf{h}||_\sigma := \sup\limits_{x\geq 0} \dfrac{|\mathbf{h}(x)|}{\sigma(x)}$.

It is remarkable that if for continuous function $\mathbf{h}$ on an infinite interval, the classical modulus of continuity $\omega(\mathbf{h};\delta) \nrightarrow o.$. In this manner, so as to examine the estimation of functions in thee of space of weight function i.e $C_\sigma^k(\mathbb{R}^+)$, Ispir and Atakut [4] presented the accompanying modulus of continuity for weighted space

$$\Omega(\mathbf{h};\delta) = \sup_{x\in[0,\infty),|g|\leq\delta} \frac{|\mathbf{h}(x+g)-\mathbf{h}(x)|}{(1+g^2)(1+x^2)}, \tag{4.1}$$

and indicated that

$$\lim_{\delta\to 0}\Omega(\mathbf{h};\delta) = 0, \omega(\mathbf{h};\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(\mathbf{h};\delta), \lambda > 0$$

and

$$
\begin{aligned}
&|\mathbf{h}(u)-\mathbf{h}(x)\\
&\le 2\left(1+\frac{|u-x|}{\delta}\right)(1+\delta^2)(1+x^2)(1+(u-x)^2)\Omega(\mathbf{f};\delta),\ u,x\in[0,\infty).
\end{aligned}\tag{4.2}
$$

In the following theorem we show that the operator $\mathfrak{P}_j$ is an estimation method for functions belonging to the weighted space $C^k_\sigma(\mathbb{R}^+)$:

Theorem 3. *The following equality is satisfied by the the sequence of linear positive operators $\mathfrak{P}_j$, for each $\mathbf{h}\in C^k_\sigma(\mathbb{R}^+)$,*

$$\lim_{j\to\infty}||\mathfrak{P}_j(\mathbf{h};x)-\mathbf{h}(x)||_\sigma=0.$$

Proof. By Lemma 1, clearly $\lim_{j\to\infty}||\mathfrak{P}_j(1;x)-1||_\sigma=0$.
Now,

$$
\begin{aligned}
\sup_{x\ge0}\frac{|\mathfrak{P}_j(t;x)-x|}{1+x^2} &\le \left|\frac{-1}{j+1}\right|\sup_{x\ge0}\frac{x}{1+x^2}\\
&\le \left|\frac{-1}{j+1}\right|,
\end{aligned}
$$

which affirms that $\lim_{\alpha\to\infty}||\mathfrak{P}_j(t;x)-x||_\sigma=0$. Again,

$$
\begin{aligned}
\sup_{x\ge0}\frac{|\mathfrak{P}_j(u^2;x)-x^2|}{1+x^2} &\le \left|\frac{-1}{(j+1)}\right|\sup_{x\ge0}\frac{x}{1+x^2}\\
&\le \left|\frac{-1}{(j+1)}\right|,
\end{aligned}
$$

which implies that $\lim_{\alpha\to\infty}||\mathfrak{P}_j(t^2;x)-x^2||_\sigma=0$. Hence the weighted Korovkin-type theorem presented in [9] gives the confirmation of appropriate result. □

In the accompanying theorem, the rate of convergence is gotten by methods for the weighted modulus of continuity.

Theorem 4. *Let $\mathbf{h}\in C^k_\sigma(\mathbb{R}^+)$. Then for sufficiently large j and a constant $\mathcal{K}$ (liberated from $\mathfrak{h}$ and j) the following inequality is verified*

$$\sup_{x\in[0,\infty)}\frac{|\mathfrak{P}_j(\mathfrak{h};x)-\mathbf{h}(x)|}{(1+x^2)^{5/2}}\le\mathcal{K}\Omega\left(\mathbf{h};\frac{1}{\sqrt{j}}\right).$$

Proof. Bearing in the mind the weighted modulus of continuity, Lemma 3 and Cauchy-Schwarz inequality, one without a very remarkable stretch sees that

$$\begin{aligned}
|\mathfrak{P}_j(\mathbf{h};x)-\mathbf{h}(x)| &\leq \mathfrak{P}_j(|\mathbf{h}(u)-\mathbf{h}(x)|;x)\\
&\leq 2(1+\delta^2)(1+x^2)\Omega(\mathbf{h};\delta)\mathfrak{P}_j\left(\left(1+\frac{|\vartheta_{x,1}(u)|}{\delta}\right)(1+\vartheta_{x,2}(u));x\right)\\
&\leq 2(1+\delta^2)(1+x^2)\Omega(\mathbf{h};\delta)\Big(\mathfrak{P}_j(e_0(x);x)+\mathfrak{P}_j(\vartheta_{x,2}(u);x)\\
&\quad+\frac{1}{\delta}(\mathfrak{P}_j(\vartheta_{x,2}(u);x))^{1/2}\\
&\quad+\frac{1}{\delta}(\mathfrak{P}_j(\vartheta_{x,2}(u);x))^{1/2}\times(\mathfrak{P}_j(\vartheta_{x,4}(u);x))^{1/2}\Big)
\end{aligned}$$

Now, picking $\delta=\frac{1}{\sqrt{j}}$, we appear to end immediately. □

In the accompanying outcome, using the weighted modulus of continuity we demonstrate a quantitative Voronovskaja type theorem.

Theorem 5. *Let* $\mathbf{h}\in C_\sigma^k(\mathbb{R}^+)$ *such that* $\mathbf{h}',\mathbf{h}''\in C_\sigma^k(\mathbb{R}^+)$.

Then, for any $x\in[0,\infty)$, *we have following equality*

$$\frac{j}{j+1}\left|(j+1)\mathfrak{P}_j(\mathbf{h};x)-(j+1)\mathbf{h}(x)+h'(x)x-h''(x)x^2\right|=8(1+x^2)\Omega(\mathbf{h}'';\delta)O(1).$$

Proof. For each $\mathbf{h},\ \mathbf{h}''\in C_\sigma^k[0,\infty)$ and $u<\varsigma<x$, by Taylor's expansion, we have

$$\begin{aligned}
\mathbf{h}(u) &= \mathbf{h}(x)+\mathbf{h}'(x)(u-x)+\frac{\mathbf{h}''(\varsigma)}{2!}(u-x)^2\\
&= \mathbf{h}(x)+\mathbf{h}'(x)(u-x)+\frac{\mathbf{h}''(x)}{2!}(u-x)^2+\Upsilon(u,x), \qquad (4.3)
\end{aligned}$$

where $\Upsilon(u,x)$ is given by

$$\Upsilon(u,x)=\frac{\mathbf{h}''(\zeta)-\mathbf{h}''(x)}{2!}(u-x)^2.$$

Applying operator $\mathfrak{P}_j$ on equation (4.3), we obtain

$$\begin{aligned}
&\left|\mathfrak{P}_j(\mathbf{h};x)-\mathbf{h}(x)-\mathbf{h}'(x)\mathfrak{P}_j(u-x;x)-\frac{\mathbf{h}''}{2!}\mathfrak{P}_j((u-x)^2;x)\right|\\
&\leq|\mathfrak{P}_j(\Upsilon(u,x);x)|. \qquad (4.4)
\end{aligned}$$

By the definition (4.1) of weighted modulus of continuity,

$$
\begin{aligned}
\Big|\Upsilon(u,x)\Big| &\leq \frac{1}{2!}\Omega(\mathbf{h}^{''};|\Upsilon-x|)(1+(\Upsilon-x)^2)(1+x^2)(u-x)^2 \\
&\leq \frac{1}{2!}\Omega(\mathbf{h}^{''};|\Upsilon-x|)(1+(\Upsilon-x)^2)(1+x^2)(u-x)^2 \\
&\leq \left(1+\frac{|u-x|}{\delta}\right)(1+\delta^2)\Omega(\mathbf{h}^{''};\delta)(1+(u-x)^2)(1+x^2)(u-x)^2,\ \delta>0 \\
&\leq \begin{cases} 2(1+\delta^2)^2(1+x^2)\Omega(\mathbf{h}^{''};\delta)(u-x)^2, & |u-x|<\delta; \\ 2(1+\delta^2)^2(1+x^2)\dfrac{(u-x)^4}{\delta^4}\Omega(\mathbf{h}^{''};\delta)(u-x)^2, & |u-x|\geq\delta \end{cases} \\
&\leq 2(1+\delta^2)^2(1+x^2)\Omega(\mathbf{h}^{''};\delta)\left(1+\frac{(u-x)^4}{\delta^4}\right)(u-x)^2. \qquad (4.5)
\end{aligned}
$$

Now, selecting $\delta<1$, from (4.5), we obtain

$$
|\Upsilon(u,x)| \leq 8(1+x^2)\Omega(\mathbf{h}^{''};\delta)\left((u-x)^2+\frac{(u-x)^2(u-x)^4}{\delta^4}\right) \tag{4.6}
$$

Applying operator $\mathfrak{P}_j$ on above inequality and considering Lemma 3, we obtain

$$
\begin{aligned}
|\mathfrak{P}_j(\Upsilon(u,x);x)| &\leq 8j(1+x^2)\Omega(\mathbf{h}^{''};\delta)\left(\mathfrak{P}_j(\vartheta_{x,2}(u);x)+\frac{1}{\delta^4}\mathfrak{P}_j(\vartheta_{x,6}(u);x)\right) \\
&= 8(1+x^2)\Omega(\mathbf{h}^{''};\delta)\left(O\left(\frac{1}{j}\right)+\frac{1}{\delta^4}O\left(\frac{1}{j^3}\right)\right),\ \text{as } j\to\infty. \qquad (4.7)
\end{aligned}
$$

Now, choosing $\delta=\dfrac{1}{\sqrt{j}}$, we obtain

$$
|\mathfrak{P}_j(\Upsilon(u,x);x)| \leq 8(1+x^2)\Omega\left(\mathbf{h}^{''};\frac{1}{\sqrt{j}}\right)O(1). \tag{4.8}
$$

On gathering (4.4), (4.8) and using Lemma 2, we show up to required outcome. □

In our next outcome we talk about Grüss-Voronovskaja type theorem for the operator defined by (2.1). The difference of integral of two functions with the product of integral of the two functions is measured by Grüss inequality [12]. The utilization of Grüss inequality was used by Acu et al. [3] very initially, in approximation theory. In [11], Gonska and Tachev introduced Grüss-type inequality using second order modulus of smoothness. Gal and Gonska [10], showed Grüss-Voronovskaya approximation for the first time using Grüss inequality for Bernstein operators and for a class of Bernstein-Durrmeyer polynomials of real and complex variables. Tariboon and Ntouyas [22] introduced Grüss inequality in q-calculus. After that [[1],

[2], [6], [5],[15] and [23]] investigated approximation results for many different linear positive operators via Grüss-Voronovskaja theorem.

Theorem 6. *The sequence of operators* $\{\mathfrak{P}_j\}_{j\geq 1}$ *assures the resulting equality in sense of Grüss-inequality as*

$$\lim_{j\to\infty} j\{\mathfrak{P}_j(\mathbf{h}g;x) - \mathfrak{P}_j(\mathbf{h};x)\mathfrak{P}_j(g;x)\} = x^2\mathbf{h}'(x)g'(x),$$

where $\mathbf{h}, \mathbf{h}', \mathbf{h}'', g, g', , g'', (\mathbf{h}g)', (\mathbf{h}g)'' \in C_\sigma^k(\mathbb{R}^+)$.

Proof. By a straightforward calculation, we may write
$\mathrm{J}\{\mathfrak{P}_j(\mathbf{h}g;x) - \mathfrak{P}_j(\mathbf{h};x)\mathfrak{P}_j(g;x)\}$

$$\begin{aligned}
= \quad & j\Bigg\{\mathfrak{P}_j(\mathbf{h}g;x) - \mathbf{h}(x)g(x) - \mathfrak{P}_j(\vartheta_{x,1}(u);x)(\mathbf{h}g)'(x) - \frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}(\mathbf{h}g)''(x) \\
& -g(x)\left[\mathfrak{P}_j(\mathbf{h};x) - \mathbf{h}(x) - \mathfrak{P}_j(\vartheta_{x,1}(u);x)\mathbf{h}'(x) - \frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}\mathbf{h}''(x)\right] \\
& -\mathfrak{P}_j(\mathbf{h};x)\left[\mathfrak{P}_j(g;x) - g(x) - \mathfrak{P}_j(\vartheta_{x,1}(u);x)g'(x) - \frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}g''(x)\right] \\
& +2\frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}\mathbf{h}'(x)g'(x) + g''(x)\frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}[\mathbf{h}(x) - \mathfrak{P}_j(\mathbf{h};x)] \\
& g'(x)\mathfrak{P}_j(\vartheta_{x,1}(u);x)[\mathbf{h}(x) - \mathfrak{P}_j(\mathbf{h};x)]\Bigg\}.
\end{aligned}$$

Now, in view of Theorem 3, it follows that $\mathfrak{P}_j(\mathbf{h};x) \to \mathbf{h}(x)$, as $j \to \infty$ and using Theorem 5, we have

$$j\{\mathfrak{P}_j(\mathbf{h};x) - \mathbf{h}(x) - \mathfrak{P}_j(\vartheta_{x,1}(u);x)\mathbf{h}'(x) - \frac{\mathfrak{P}_j(\vartheta_{x,2}(u);x)}{2!}\mathbf{h}''(x)\} \to 0 \text{ as } j \to \infty,$$

since $\mathbf{h}', \mathbf{h}'' \in C_\sigma^k(\mathbb{R}^+)$.
Thus, using Theorems 3, 5 and Lemma 3, we obtain the required result

$$\lim_{j\to\infty} j\{\mathfrak{P}_j(\mathbf{h}g;x) - \mathfrak{P}_j(\mathbf{h};x)\mathfrak{P}_j(g;x)\} = x^2\mathbf{h}'(x)g'(x).$$

□

5. Graphical Examples

In this module, we constructed some graphical examples in favour to show better approximation of our modified Post-Widder operator by increasing the value of j. To support our claim, we also presented an error table by considering the error value

$\sqrt{\frac{1}{j+1}x^2}$ for operators $\mathfrak{P}_j(.;x)$.

Table 2: Error table for the operators $\mathfrak{P}_j(.;x)$

j	1	5	10	50	100	1000	5000
error	0.707107x	0.408248x	0.301511x	0.140028x	0.0995037x	0.031607x	0.0141407x

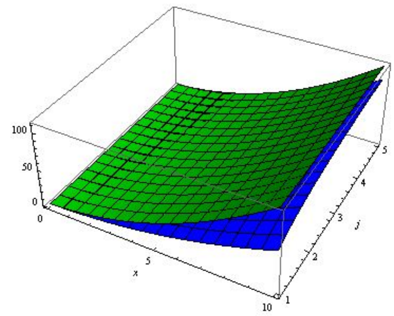

Figure 5

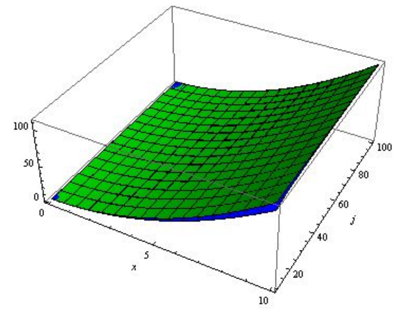

Figure 6

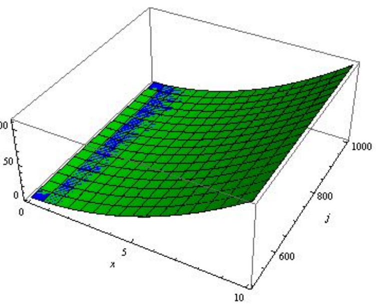

Figure 7

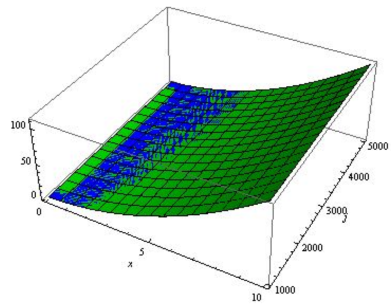

Figure 8

The operators $\mathfrak{P}_j(.;x)$ represented by blue curve and function $\mathbf{h}(x)$ represented by green curve.

Figure 5 shows convergence of operators $\mathfrak{P}_j(.;x)$ to the function $\mathbf{h}(x) = x^2 + x + 1$ for $1 \leq j \leq 5$ and $x \in [0, 10]$.

Figure 6 shows convergence of operators $\mathfrak{P}_j(.;x)$ to the function $\mathbf{h}(x) = x^2 + x + 1$ for $10 \leq j \leq 100$ and $x \in [0, 10]$.

Figure 7 shows convergence of operators $\mathfrak{P}_j(.;x)$ to the function $\mathbf{h}(x) = x^2 + x + 1$ for $500 \leq j \leq 1000$ and $x \in [0, 10]$.

Figure 8 shows convergence of operators $\mathfrak{P}_j(.;x)$ to the function $\mathbf{h}(x) = x^2 + x + 1$ for $1000 \leq j \leq 5000$ and $x \in [0, 10]$.

In view of above table and graphics we conclude that the new constructed modified Post-Widder operators $\mathfrak{P}_j(.;x)$ show better approximation as the value of j increases.

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