J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.2.109 ISSN(Online) 2287-6081
Volume 30, Number 2 (May 2023), Pages 109–129

INTRODUCTION OF T-HARMONIC MAPS

Mehran Aminian

Abstract. In this paper, we introduce a second order linear differential operator $\overset{T}{\square}: C^\infty(M) \to C^\infty(M)$ as a natural generalization of Cheng-Yau operator, [8], where T is a $(1,1)$–tensor on Riemannian manifold (M,h), and then we show on compact Riemannian manifolds, $\operatorname{div}T = \operatorname{div}T^t$, and if $\operatorname{div}T = 0$, and f be a smooth function on M, the condition $\overset{T}{\square} f = 0$ implies that f is constant. Hereafter, we introduce T-energy functionals and by deriving variations of these functionals, we define T-harmonic maps between Riemannian manifolds, which is a generalization of L_k-harmonic maps introduced in [3]. Also we have studied fT-harmonic maps for conformal immersions and as application of it, we consider fL_k-harmonic hypersurfaces in space forms, and after that we classify complete fL_1-harmonic surfaces, some fL_k-harmonic isoparametric hypersurfaces, fL_k-harmonic weakly convex hypersurfaces, and we show that there exists no compact fL_k-harmonic hypersurface either in the Euclidean space or in the hyperbolic space or in the Euclidean hemisphere. As well, some properties and examples of these definitions are given.

1. Introduction and Preliminaries

Harmonic maps are critical points of energy functionals, equivalently these maps are solutions of PDE systems when tension fields are zero, [9, 12]. In paper [3], the authors generalize energy functionals and the notions of tension fields to introduce L_k-harmonic maps. Following it, we introduce T-energy functionals and by computing the first variation of these functionals, Theorem 3.4, we define T-harmonic maps between two Riemannian manifolds. In the paper, we used technicks of [15] to get some of results and as in Proposition 3.10, Proposition 3.12 and Theorem 3.13.

In Section 2, we first introduce a second order linear differential operator $\overset{T}{\square}$: $C^\infty(M) \to C^\infty(M)$ as a natural generalization of Cheng-Yau operator, [8], where T is a $(1,1)$-tensor on a Riemannian manifold, and after studying some of its properties, in Theorem 2.7 and Theorem 2.9, we show that on compact Riemannian manifolds,

Received by the editors June 20, 2020. Accepted February 25, 2023.
2020 *Mathematics Subject Classification.* 58E20, 53C43.
Key words and phrases. L_k-operator, energy functional, harmonic map.

div$T$ = divT^t where T^t denote the transpose of tensor T with respect to Riemannian metric h on M,and constant functions are the only ones that $\overset{T}{\square} f = 0$. In a similar way, We introduce an operator and show its similar properties to $\overset{T}{\square}$ in Theorem 2.13 and Proposition 2.14.

In Section 3, we introduce T-energy functionals and by deriving variations of these functionals, we define T-harmonic maps between Riemannian manifolds, which is a generalization of L_k-harmonic maps introduced in [3] and then in Corollary 3.8, we show that a smooth map $\psi : M \to \overline{M}$ from a Riemannian manifold M to a Riemannian manifold $\overline{M}$ is fT-harmonic map if and only if

$$f\overset{\mathbf{T}}{\square}\psi + \frac{1}{2}d\psi\left((T+T^t)(\nabla f) + f div(T+T^t)\right) = 0\,,$$

where $\overset{\mathbf{T}}{\square}\psi$ is stated in Definition 3. After that in Theorem 3.13, we study fT-harmonic maps which are conformal immersions.

In Section 4, by use of Theorem 3.13, in Theorem 4.1, we prove that the oriented immersed hypersurfaces in simply connected space forms are fL_k-harmonic if and only if $H_{k+1} = 0$ and $P_k\nabla f = 0$, where H_{k+1} is $(k+1)$-th mean curvature and P_k's are Newton transformations. In Theorem 4.4 and Theorem 4.5, we show that an immersion from a connected oriented surface into a simply connected space form is fL_1-harmonic if and only if the principal curvatures are zero and $2H_1$, and $H_1\nabla_v f = 0$ for every vector v in the distribution of space of principal vectors of zero's principal curvature, and if the surface in Euclidean space $\mathbb{R}^3$ or in unit Euclidean sphere $\mathbb{S}^3$ is complete, then it is a cylinder over planar curve in Euclidean space and $H_1\nabla_v f = 0$ for every vector v in the distribution of space of principal vectors of zero's principal curvature, or it is totally geodesic sphere $\mathbb{S}^2(1)$ and f is arbitrary smooth positive function on the surface. As a result of Theorem 4.1, in Corollary 4.6, we study some isoparametric hypersurfaces in space forms which are fL_k-harmonic. In Theorem 4.7, by property of weakly convex hypersurfaces, we show that if these hypersurfaces of space forms be fL_k-harmonic, then they are totally geodesic and f is arbitrary smooth positive function on the hypersurface if $k \neq 0$, and f is constant positive function if $k = 0$. Finally in Corollary 4.8, we get that there exists no compact orientable fL_k-harmonic hypersurface either in the Euclidean space or in the hyperbolic space or in the Euclidean hemisphere.

We recall the prerequisites from [1, 5, 6, 7, 13, 16]. Let $R^{n+1}(c)$ be the simply connected Riemannian space form of constant sectional curvature c which is the

Euclidean space $\mathbb{R}^{n+1}$, for $c = 0$, and the Hyperbolic space $\mathbb{H}^{n+1}$, for $c = -1$, and the Euclidean sphere $\mathbb{S}^{n+1}$, for $c = +1$. Let $\varphi : M^n \to R^{n+1}(c)$ be a connected oriented hypersurface isometrically immersed into $R^{n+1}(c)$ with N as a unit normal vector field, ∇ and $\overline{\nabla}$ the Levi-Civita connections on M and $R^{n+1}(c)$, respectively. For simplicity we also denote the induced connection on the pullback bundle $\varphi^*TR^{n+1}(c)$ by $\overline{\nabla}$. Let X, Y be vector fields on M. We have the following formula for the shape operator of M,

$$\begin{aligned}&\overline{\nabla}_X d\varphi(Y) = d\varphi(\nabla_X Y) + \langle SX, Y\rangle\, N\,,\\ &d\varphi(SX) = -\overline{\nabla}_X N\,.\end{aligned}$$

As it is known, the shape operator is a self-adjoint linear operator. Let $k_1, \ldots, k_n$ be its eigenvalues which are called principal curvatures of M . Define $s_0 = 1$ and

$$s_k = \sum_{1\leq i_1<\cdots<i_k\leq n} k_{i_1}\cdots k_{i_k}\,.$$

The k-th mean curvature of M is defined by

$$\binom{n}{k} H_k = s_k\,.$$

The Newton transformations $P_k : \mathcal{X}(M) \to \mathcal{X}(M)$ are defined inductively by $P_0 = I$ and

$$P_k = s_k I - S\circ P_{k-1}\,,\ 1\leq k\leq n.$$

From the Cayley-Hamilton theorem, one gets that $P_n = 0$. Each P_k is a self adjoint linear operator which commutes with S. For $0 \leq k \leq n-1$, the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ as the natural generalization of the Laplace operator for Euclidean hypersurfaces M, is defined by

$$L_k f = tr(P_k \circ \nabla^2 f)\,,$$

where $\nabla^2 f$ is metrically equivalent to the Hessian of f and is defined by $\big\langle(\nabla^2 f)X,\, Y\big\rangle = \langle\nabla_X(\nabla f),\, Y\rangle$ for all vector fields X, Y on M, and ∇f is the gradient vector field of f.

We recall the definition of harmonic maps, [9]. Let $\psi : M \to \overline{M}$ be a smooth map between Riemannian manifolds (M, h) and $(\overline{M}, l)$ with Levi-Civita connections ∇ and $\overline{\nabla}$, respectively. We denote the induced connection on the pullback bundle $\psi^*T\overline{M}$ by $\overline{\nabla}$ as well.

The smooth map ψ is called harmonic if it is a critical point of the energy functional:

$$E(\psi) = \frac{1}{2}\int_\Omega |d\psi|^2 \mathrm{d}\Omega$$

for any compact domain Ω in M where $|d\psi|^2 = \sum_i \langle d\psi(e_i), d\psi(e_i)\rangle_h$ for a local orthonormal frame field $\{e_i\}_{i=1}^n$ on M. One can prove that ψ is harmonic if and only if $\tau(\psi) = 0$, [9], where the tension field $\tau(\psi)$ is defined as

$$\tau(\psi) = \sum_i \left(\overline{\nabla}_{e_i} d\psi(e_i) - d\psi(\nabla_{e_i} e_i)\right) .$$

We recall the Divergence Theorem (cf. [7]), to be used later.

Theorem 1.1 (Divergence Theorem). *Let M be a compact Riemannian manifold and X be a vector field on it. Then*

$$\int_M \mathrm{div} X \, \mathrm{d}M = 0 .$$

2. Second Order Linear Differential Operator $\overset{T}{\Box}$

Definition 2.1. Let $T : \mathcal{X}(M) \to \mathcal{X}(M)$ be a tensor on Riemannian manifold (M, h). We define a second order linear differential operator $\overset{T}{\Box}: C^\infty(M) \to C^\infty(M)$ as the following:

$$\overset{T}{\Box} f = \sum_{i,j} T_{ij} H^f(e_i, e_j) = \sum_i H^f(Te_i, e_i) , \tag{2.1}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M and $T_{ij} = \langle Te_j, e_i\rangle_h$.

It is easily seen that the equation (2.1) is independent of choice of frames and so it is well defined. When the tensor T is symmetric, operator $\overset{T}{\Box}$ is Cheng-Yau operator $\Box$ introduced in [8].

In local coordinates $\{x^i\}$ for M and $h = [h_{ij}]$, we have

$$\begin{aligned}\overset{T}{\Box} f =& h^{l_1 l_3} h^{l_2 l_4} \left\langle T(\frac{\partial}{\partial x^{l_2}}), \frac{\partial}{\partial x^{l_1}} \right\rangle_h H^f(\frac{\partial}{\partial x^{l_3}}, \frac{\partial}{\partial x^{l_4}}) \\ =& h^{l_1 l_3} h^{l_2 l_4} \left\langle T(\frac{\partial}{\partial x^{l_2}}), \frac{\partial}{\partial x^{l_1}} \right\rangle_h \left(\frac{\partial^2 f}{\partial x^{l_3} \partial x^{l_4}} - \Gamma^{l_i}_{l_3 l_4} \frac{\partial f}{\partial x^{l_i}} \right) ,\end{aligned}$$

where Γ^k_{ij}'s are Christoffel symbols of the Levi-Civita connection ∇ on M.

Remark 2.2. Let T^t denote the transpose of tensor T with respect to Riemannian metric h. Since $\Gamma^{l_i}_{l_3 l_4} = \Gamma^{l_i}_{l_4 l_3}$ and $\dfrac{\partial^2 f}{\partial x^{l_3}\partial x^{l_4}} = \dfrac{\partial^2 f}{\partial x^{l_4}\partial x^{l_3}}$ then $\overset{T}{\Box} f = \overset{T^t}{\Box} f$.

Remark 2.3. One can see that $\overset{T}{\Box} f = \operatorname{div}(T\nabla f) - \left\langle \nabla f, \operatorname{div}T^t \right\rangle$. In fact, let $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M, then

$$\begin{aligned}\operatorname{div}(T\nabla f) &= \sum_i \langle (\nabla_{e_i} T)\nabla f + T\nabla_{e_i}\nabla f, e_i\rangle \\ &= \sum_i \left\langle \nabla f, (\nabla_{e_i} T^t) e_i \right\rangle + \sum_i \langle T\nabla_{e_i}\nabla f, e_i\rangle = \left\langle \nabla f, \operatorname{div}T^t \right\rangle + \overset{T}{\Box} f\,.\end{aligned}$$

Example 2.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(x_1, \ldots, x_n) = x_1$, T be an arbitrary $(1,1)$ tensor on $\mathbb{R}^n$, and $\{\partial_i\}_{i=1}^n$ be the canonical orthonormal frame on $\mathbb{R}^n$. We have $df(\partial_i) = \delta_{i1}$ and $\nabla_{\partial_i}\partial_j = 0$ where ∇ is the canonical Levi-Civita connection on $\mathbb{R}^n$. Then we get $\overset{T}{\Box} f = 0$.

Lemma 2.5. *Let (M, h) be a Riemannian manifold and T be a tensor on it, and f and g be smooth functions on M. Then*

$$\overset{T}{\Box}(fg) = g\overset{T}{\Box} f + f\overset{T}{\Box} g + \langle \nabla f, T\nabla g\rangle + \langle \nabla g, T\nabla f\rangle\ .$$

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. Then at p, by equation (2.1) we have

$$\begin{aligned}\overset{T}{\Box}(fg) &= \sum_i \nabla_{Te_i}\nabla_{e_i}(fg) = \sum_i \nabla_{Te_i}(g\nabla_{e_i} f + f\nabla_{e_i} g) \\ &= \sum_i (g\nabla_{Te_i}\nabla_{e_i} f + f\nabla_{Te_i}\nabla_{e_i} g + (\nabla_{Te_i} g)(\nabla_{e_i} f) + (\nabla_{Te_i} f)(\nabla_{e_i} g)) \\ &= g\overset{T}{\Box} f + f\overset{T}{\Box} g + \langle \nabla f, T\nabla g\rangle + \langle \nabla g, T\nabla f\rangle\ .\end{aligned}$$

□

Lemma 2.6. *Let (M, h) be a compact Riemannian manifold and T be a tensor on it, and f and g be smooth functions on M. Then*

$$\int_M f\overset{T}{\Box} g\,\mathrm{d}M = \int_M (g\overset{T}{\Box} f + \langle \nabla f, T\nabla g\rangle_h - \langle \nabla g, T\nabla f\rangle_h + \langle g\nabla f - f\nabla g, \operatorname{div}T\rangle_h)\mathrm{d}M\,.$$

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. We define the following well defined vector fields on M:

$$X = f\sum_{i,j} T_{ji}\left\langle \nabla g, e_j\right\rangle_h e_i \ , Y = g\sum_{i,j} T_{ji}\left\langle \nabla f, e_j\right\rangle_h e_i \, .$$

Therefore at p, we have

$$\begin{aligned} divX = \sum_k \left\langle \nabla_{e_k} X, e_k\right\rangle_h = \sum_{i,j}(T_{ji}(\nabla_{e_i} f)\left\langle \nabla g, e_j\right\rangle_h + f(\nabla_{e_i} T_{ji})\left\langle \nabla g, e_j\right\rangle_h \\ + fT_{ji}\left\langle \nabla_{e_i}\nabla g, e_j\right\rangle_h) = \left\langle \nabla g, T\nabla f\right\rangle_h + f\left\langle \nabla g, \mathrm{div}T\right\rangle_h + f\overset{T}{\Box}\, g \, , \end{aligned} \tag{2.2}$$

and similarly

$$divY = \left\langle \nabla f, T\nabla g\right\rangle_h + g\left\langle \nabla f, \mathrm{div}T\right\rangle_h + g\overset{T}{\Box}\, f \, . \tag{2.3}$$

So by equations (2.2) and (2.3), and Divergence Theorem we get the result. □

Theorem 2.7. *Let (M, h) be a compact Riemannian manifold and T be a tensor on it. Then* $\mathrm{div}T = \mathrm{div}T^t$.

Proof. By use of Remark 2.2 and Lemma 2.6, for tensors T^t and $\frac{T+T^t}{2}$, we get

$$\int_M (\left\langle \nabla f, T^t\nabla g\right\rangle_h - \left\langle \nabla g, T^t\nabla f\right\rangle_h + \frac{1}{2}\left\langle g\nabla f - f\nabla g, \mathrm{div}(T^t - T)\right\rangle_h)\mathrm{d}M = 0 \, . \tag{2.4}$$

Since f and g are arbitrary, equation (2.4) implies that $\mathrm{div}(T^t - T) = 0$, and so $\mathrm{div}T = \mathrm{div}T^t$. □

Remark 2.8. Compactness of Theorem 2.7 is necessary. For instance, by considering $T(x_1, x_2) = \begin{bmatrix} 0 & 0 \\ x_2 & 0 \end{bmatrix}$ on $\mathbb{R}^2$, we have $\mathrm{div}T = 0$ whilst $\mathrm{div}T^t = \partial_1$.

As a generalization result of Maximum Principle for operators we give the following theorem.

Theorem 2.9. *Let (M, h) be a compact Riemannian manifold, T be a $(1,1)$ tensor on M which is definite and* $\mathrm{div}T = 0$*, and f be a smooth function on M. If $\overset{T}{\Box}\, f = 0$ then f is constant.*

Proof. By Lemma 2.5, we have

$$\overset{T}{\Box}\, f^2 = 2\left\langle T(\nabla f), \nabla f\right\rangle_h \, . \tag{2.5}$$

Now using Lemma 2.6, we get

$$\int_M \overset{T}{\Box}\, f^2 \mathrm{d}M = 0 \, . \tag{2.6}$$

So equations (2.5) and (2.6) result in

$$\int_M \langle T(\nabla f), \nabla f \rangle_h \, \mathrm{d}M = 0 \,.$$

Since T is definite, we get $\langle T(\nabla f), \nabla f \rangle_h = 0$ and so $\nabla f = 0$. Therefore f is constant. $\square$

Definition 2.10. Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$ and $V \in \mathcal{X}(\psi)$ be a smooth vector field. We define the operator $\overset{T}{\square}: \mathcal{X}(\psi) \to \mathcal{X}(\psi)$ as follow:

$$\overset{T}{\square} V = \sum_{i,j} T_{ij} (\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} V - \overline{\nabla}_{\nabla_{e_i} e_j} V), \tag{2.7}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M and $T_{ij} = \langle Te_j, e_i \rangle_h$.

It is easily seen that the equation (2.7) are independent of choice of frames and so it is well defined.

Remark 2.11. Let $\bar{R}$ be the curvature tensor of the induced connection on the pullback bundle $\psi^* T\overline{M}$. One can see that $\overset{T^t}{\square} V = \overset{T}{\square} V + \sum_i \bar{R}(e_i, Te_i)V$. When $T = I$, $\overset{I}{\square} V$ is the rough Laplacian.

Lemma 2.12. *Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$ and X, Y be smooth vector fields on M. Then*

$$\begin{aligned} \overset{T}{\square} \langle X, Y \rangle_{\psi^* l} = & \left\langle \overset{T}{\square} d\psi(X), d\psi(Y) \right\rangle_l + \left\langle d\psi(X), \overset{T}{\square} d\psi(Y) \right\rangle_l \\ & + \sum_i \left(\left\langle \overline{\nabla}_{T(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y) \right\rangle_l + \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{T(e_i)} d\psi(Y) \right\rangle_l \right) , \end{aligned}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame on M.

Proof. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. Then at p, by equations (2.1) and (2.7), we have

$$\begin{aligned} \overset{T}{\square} \langle X, Y \rangle_{\psi^* l} &= \sum_{i,j} T_{ij} \left(\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} \langle d\psi(X), d\psi(Y) \rangle_l \right) \\ &= \sum_{i,j} T_{ij} \left(\overline{\nabla}_{e_i} \left(\left\langle \overline{\nabla}_{e_j} d\psi(X), d\psi(Y) \right\rangle_l + \left\langle d\psi(X), \overline{\nabla}_{e_j} d\psi(Y) \right\rangle_l \right) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} T_{ij} \Big(\left\langle \overline{\nabla}_{e_j} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y) \right\rangle_l + \left\langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} d\psi(X), d\psi(Y) \right\rangle_l \\
&\quad + \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{e_j} d\psi(Y) \right\rangle_l + \left\langle d\psi(X), \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} d\psi(Y) \right\rangle_l \Big) \\
&= \left\langle \overset{T}{\square}\, d\psi(X), d\psi(Y) \right\rangle_l + \left\langle d\psi(X), \overset{T}{\square}\, d\psi(Y) \right\rangle_l \\
&\quad + \sum_{i,j} T_{ij} \left\langle \overline{\nabla}_{e_j} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y) \right\rangle_l + \sum_{i,j} T_{ij} \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{e_j} d\psi(Y) \right\rangle_l \\
&= \left\langle \overset{T}{\square}\, d\psi(X), d\psi(Y) \right\rangle_l + \left\langle d\psi(X), \overset{T}{\square}\, d\psi(Y) \right\rangle_l \\
&\quad + \sum_{i} \Big(\left\langle \overline{\nabla}_{T(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y) \right\rangle_l + \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{T(e_i)} d\psi(Y) \right\rangle_l \Big) .
\end{aligned}
$$

□

Theorem 2.13. *Let (M, h) be a compact Riemannian manifold, T be a $(1,1)$ tensor on M which is definite and $\mathrm{div}T = 0$, X be a smooth vector field on M, and $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$. If $\overset{T}{\square}\, d\psi(X) = 0$ then $d\psi(X)$ is parallel.*

Proof. By Lemma 2.6 and Lemma 2.12, we have

$$\overset{T}{\square} \langle X, X \rangle_{\psi^* l} = 2 \sum_i \left\langle \overline{\nabla}_{T(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(X) \right\rangle_l \text{ and } \int_M \overset{T}{\square} \langle X, X \rangle_{\psi^* l} \, \mathrm{d}M = 0.$$

Thus $\int_M \sum_i \left\langle \overline{\nabla}_{\left(\frac{T+T^t}{2}\right)(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(X) \right\rangle_l \mathrm{d}M = 0$. Since T and so $\frac{T+T^t}{2}$ is definite, there is a local orthonormal frame $\{e_i\}_{i=1}^n$ on M which diagonalize $\frac{T+T^t}{2}$, and let $\{\lambda_i\}_{i=1}^n$ be its corresponding eigenvalues. Therefore

$$\int_M \sum_i \lambda_i \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{e_i} d\psi(X) \right\rangle_l \mathrm{d}M = 0.$$

Definiteness implies that the integrand is zero, and so for every i, $\overline{\nabla}_{e_i} d\psi(X) = 0$. Thus $\overline{\nabla} d\psi(X) = 0$ on M. □

As an extra property of $\overset{T}{\square}$, we state the following proposition.

Proposition 2.14. *Let T be a tensor on a compact Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$, and X, Y be smooth vector fields on M. Then*

$$\int_M \left\langle d\psi(X), \overset{T}{\Box}\, d\psi(Y)\right\rangle_l \mathrm{d}M = \int_M \left\langle d\psi(Y), \overset{T}{\Box}\, d\psi(X)\right\rangle_l \mathrm{d}M$$
$$+ \int_M \left(\left\langle \overline{\nabla}_{T(e_i)} d\psi(Y), \overline{\nabla}_{e_i} d\psi(X)\right\rangle_l - \left\langle \overline{\nabla}_{T(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y)\right\rangle_l\right) \mathrm{d}M$$
$$+ \int_M \left(\left\langle d\psi(Y), \overline{\nabla}_{\mathrm{div}T} d\psi(X)\right\rangle_l - \left\langle d\psi(X), \overline{\nabla}_{\mathrm{div}T} d\psi(Y)\right\rangle_l\right) \mathrm{d}M\,.$$

where $\{e_i\}_{i=1}^n$ *is a local orthonormal frame on* M*.*

Proof. Assume a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. Let's define a well-defined vector fields Z_1 and Z_2 on M as

$$Z_1 := \sum_{i,j} T_{ij} \left\langle d\psi(X), \overline{\nabla}_{e_j} d\psi(Y)\right\rangle_l e_i\ , Z_2 := \sum_{i,j} T_{ij} \left\langle d\psi(Y), \overline{\nabla}_{e_j} d\psi(X)\right\rangle_l e_i.$$

So at p, we have

$$\begin{aligned} divZ_1 =& (\nabla_{e_i} T_{ij}) \left\langle d\psi(X), \overline{\nabla}_{e_j} d\psi(Y)\right\rangle_l \\ & + T_{ij} \left\langle \overline{\nabla}_{e_i} d\psi(X), \overline{\nabla}_{e_j} d\psi(Y)\right\rangle_l + T_{ij} \left\langle d\psi(X), \overline{\nabla}_{e_i}\overline{\nabla}_{e_j} d\psi(Y)\right\rangle_l \\ =& \left\langle d\psi(X), \overline{\nabla}_{\mathrm{div}T^t} d\psi(Y)\right\rangle_l + \left\langle \overline{\nabla}_{T(e_i)} d\psi(X), \overline{\nabla}_{e_i} d\psi(Y)\right\rangle_l \\ & + \left\langle d\psi(X), \overset{T}{\Box}\, d\psi(Y)\right\rangle_l . \end{aligned} \tag{2.8}$$

and similarly

$$\begin{aligned} divZ_2 =& \left\langle d\psi(Y), \overline{\nabla}_{\mathrm{div}T^t} d\psi(X)\right\rangle_l + \left\langle \overline{\nabla}_{T(e_i)} d\psi(Y), \overline{\nabla}_{e_i} d\psi(X)\right\rangle_l \\ & + \left\langle d\psi(Y), \overset{T}{\Box}\, d\psi(X)\right\rangle_l . \end{aligned} \tag{2.9}$$

Therefore by Theorem 2.7 and equations (**??**) and (2.9), and Divergence Theorem we get the result. □

3. T-harmonic Maps

Definition 3.1. Let T be a tensor on Riemannian manifold (M^n, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map where h, l are Riemannian metrics on smooth manifolds M and $\overline{M}$, respectively and ∇, $\overline{\nabla}$ are Levi-Civita connections on M, $\overline{M}$, respectively. We denote the induced connection on the pullback bundle $\psi^* T\overline{M}$ by $\overline{\nabla}$ as well. We define a differential operator as follow:

$$\overset{\mathbf{T}}{\square}\psi = \sum_{i,j} T_{ij}\left(\overline{\nabla}_{e_i} d\psi(e_j) - d\psi(\nabla_{e_i} e_j)\right), \tag{3.1}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M and $T_{ij} = \langle Te_j, e_i\rangle_h$.

It is easily seen that the equation (3.1) is independent of choice of frames and so it is well defined.

In local coordinates $\{x^i\}$ for M and $\{y^\alpha\}$ for $\overline{M}$, $h = [h_{ij}]$ and $\psi = (\psi^\alpha)$, $\overset{\mathbf{T}}{\square}\psi$ has the following expression:

$$\begin{aligned}\overset{\mathbf{T}}{\square}\psi &= \left(\overset{T}{\square}\psi^\gamma + h^{ii'}h^{jj'}\left\langle T(\frac{\partial}{\partial x^j}), \frac{\partial}{\partial x^i}\right\rangle_h \frac{\partial\psi^\alpha}{\partial x^{i'}}\frac{\partial\psi^\beta}{\partial x^{j'}}\overline{\Gamma}^\gamma_{\alpha\beta}\circ\psi\right)\frac{\partial}{\partial y^\gamma}\circ\psi \\ &= \left(\overset{T}{\square}\psi^\gamma + \left\langle T\nabla\psi^\beta, \nabla\psi^\alpha\right\rangle_h \overline{\Gamma}^\gamma_{\alpha\beta}\circ\psi\right)\frac{\partial}{\partial y^\gamma}\circ\psi,\end{aligned} \tag{3.2}$$

where $\overline{\Gamma}^\gamma_{\alpha\beta}$'s are Christoffel symbols of the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$.

Remark 3.2. As the equation (3.2) shows, $\overset{\mathbf{T}}{\square}\psi = \overset{\mathbf{T^t}}{\square}\psi$. When ψ is a smooth function on M, $\overset{\mathbf{T}}{\square}\psi = \overset{T}{\square}\psi$.

Definition 3.3. Let T be a tensor on Riemannian manifold (M,h), $\psi : (M,h) \to (\overline{M}, l)$ be a smooth map from (M,h) to a Riemannian manifold $(\overline{M}, l)$. We define a T-energy functional for ψ on a compact domain $\Omega \subset M$ as follows:

$$E_T(\psi) = \frac{1}{2}\sum_{i,j}\int_\Omega T_{ij}\,\langle d\psi(e_i), d\psi(e_j)\rangle_l\, \mathrm{d}\Omega\,,$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on Ω. We say that ψ is a T-harmonic map if it is a critical point of the T-energy functional. That is for every variation $\{\psi_t\}_{t\in I}$ of ψ supported in a compact domain Ω the following equation should be satisfied:

$$\frac{d}{dt}\Big|_{t=0} E_T(\psi_t) = 0\,.$$

Theorem 3.4 (First variation formula of the T-energy functional)**.** *Let T be a tensor on Riemannian manifold (M,h), $\psi : (M,h) \to (\overline{M}, l)$ be a smooth map from (M,h) to a Riemannian manifold $(\overline{M}, l)$. Then*

$$\frac{d}{dt}\Big|_{t=0} E_T(\psi_t) = -\int_\Omega \left\langle V, \overset{\mathbf{T}}{\square}\psi + \frac{1}{2}d\psi\left(\mathrm{div}(T+T^t)\right)\right\rangle_l \mathrm{d}\Omega\,. \tag{3.3}$$

where V is the variation vector field of a smooth variation $\{\psi_t\}_{t\in I}$ supported in a compact domain Ω.

Proof. Let $\Psi : I \times M \to \overline{M}$ be the variation $\{\psi_t\}_{t\in I}$ of ψ and $\overline{\nabla}$ denotes the induced connection on the pullback bundle $\Psi^* T\overline{M}$ as well. Let $e_t = \frac{\partial}{\partial t}$ be the standard coordinate vector field on I and $\{e_i\}_{i=1}^n$ be an orthonormal frame field on M. Since $[e_t, X] = 0$ for every $X \in \mathcal{X}(M)$,

$$\overline{\nabla}_{e_t} d\Psi(e_i)\Big|_{t=0} = \left(\overline{\nabla}_{e_i} d\Psi(e_t) + d\Psi[e_t, e_i]\right)\Big|_{t=0} = \overline{\nabla}_{e_i} V \ .$$

So we get that

(3.4)
$$\frac{d}{dt}\Big|_{t=0} E_T(\psi_t) = \frac{1}{2}\sum_{i,j}\int_\Omega T_{ij}\left\langle \overline{\nabla}_{e_i} V, d\psi(e_j)\right\rangle_l \mathrm{d}\Omega + \frac{1}{2}\sum_{i,j}\int_\Omega T_{ij}\left\langle \overline{\nabla}_{e_j} V, d\psi(e_i)\right\rangle_l \mathrm{d}\Omega \ .$$

Let X and Y be the following well defined smooth vector fields on Ω

$$X = \sum_{i,j} T_{ij}\left\langle V, d\psi(e_j)\right\rangle_l e_i \ , Y = \sum_{i,j} T_{ij}\left\langle V, d\psi(e_i)\right\rangle_l e_j \ .$$

We need to compute $divX$ and $divY$. Since $\mathrm{div}(.) = \sum_i \langle \nabla_{e_i}(.), e_i\rangle$ is independent of the choice of the orthonormal frame field, we can choose the frame $\{e_i\}_{i=1}^n$, such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. So at p,

$$\begin{aligned} divX &= \sum_i \langle \nabla_{e_i} X, e_i\rangle \\ &= \sum_{i,j}\left((\nabla_{e_i} T_{ij})\langle V, d\psi(e_j)\rangle_l + T_{ij}\left\langle \overline{\nabla}_{e_i} V, d\psi(e_j)\right\rangle + T_{ij}\left\langle V, \overline{\nabla}_{e_i} d\psi(e_j)\right\rangle\right) \quad (3.5) \\ &= \left\langle V, d\psi\left(\mathrm{div}T^t\right)\right\rangle + \sum_{i,j} T_{ij}\left\langle \overline{\nabla}_{e_i} V, d\psi(e_j)\right\rangle + \left\langle V, \overset{\mathbf{T}}{\Box}\psi\right\rangle \ , \end{aligned}$$

$$\begin{aligned} divY &= \sum_i \langle \nabla_{e_i} Y, e_i\rangle \\ &= \sum_{i,j}\left((\nabla_{e_j} T_{ij})\langle V, d\psi(e_i)\rangle_l + T_{ij}\left\langle \overline{\nabla}_{e_j} V, d\psi(e_i)\right\rangle + T_{ij}\left\langle V, \overline{\nabla}_{e_j} d\psi(e_i)\right\rangle\right) \\ &= \langle V, d\psi(\mathrm{div}T)\rangle + \sum_{i,j} T_{ij}\left\langle \overline{\nabla}_{e_j} V, d\psi(e_i)\right\rangle + \left\langle V, \sum_{i,j} T_{ij}\overline{\nabla}_{e_i} d\psi(e_j)\right\rangle \quad (3.6) \\ &= \langle V, d\psi(\mathrm{div}T)\rangle_l + \sum_{i,j} T_{ij}\left\langle \overline{\nabla}_{e_j} V, d\psi(e_i)\right\rangle + \left\langle V, \overset{\mathbf{T}}{\Box}\psi\right\rangle \ . \end{aligned}$$

Thus Divergence Theorem 1.1, and equations (3.4), (3.5) and (3.6) yield equation (3.3). □

Consequently, from Theorem 3.4, we get the following result.

Corollary 3.5. *Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$. Then ψ is T-harmonic map if and only if*

$$\overset{\mathbf{T}}{\square}\psi + \frac{1}{2} d\psi \left(\operatorname{div}(T + T^t)\right) = 0 \,. \tag{3.7}$$

We call L.H.S of equation (3.7), Amin-tension field $A_T(\psi) = \overset{\mathbf{T}}{\square}\psi + \frac{1}{2} d\psi \left(\operatorname{div}(T + T^t)\right)$ which is a generalization of the notion introduced in [3].

Remark 3.6. As we see when $T = I$, $A_I(\psi) = \tau(\psi)$ where $\tau(\psi)$ is the tension field and so I-harmonic condition is equivalent to being harmonic.

Example 3.7. Let $\psi : \mathbb{R}^n - \{0\} \to \mathbb{R}^n$ be defined as $\psi(x) = \frac{x}{|x|^2}$, T be a constant symmetric matrix, $\{\partial_i\}_{i=1}^n$ be the canonical orthonormal frame, and ∇ is the canonical Levi-Civita connection on $\mathbb{R}^n$. By straightforward computations we have

$$\overset{\mathbf{T}}{\square}\psi = \frac{1}{|x|^6} \left(|x|^2 \left(-4Tx - 2tr(T)x\right) + 8 \langle Tx, x\rangle \, x\right) \,. \tag{3.8}$$

Since T is a constant matrix, by Corollary 3.5 we have ψ is T-harmonic if and only if $\overset{\mathbf{T}}{\square}\psi = 0$. Now suppose that λ_1 and λ_2 be two distinct eigenvalues with eigenvectors V_1 and V_2 respectively. Substituting these eigenvectors in the equation $\overset{\mathbf{T}}{\square}\psi = 0$, we get $tr(T) = 2\lambda_1 = 2\lambda_2$ which is a contradiction. Therefore T just has one eigenvalue and so T is scalar matrix. So by equation (3.8), we get ψ is T-harmonic map if and only if T is scalar matrix and $n = 2$, and hence ψ is an harmonic map.

By Corollary 3.5, we can get the following result.

Corollary 3.8. *Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$, and f be a smooth function on M. Then ψ is fT-harmonic map if and only if*

$$f\overset{\mathbf{T}}{\square}\psi + \frac{1}{2} d\psi \left((T + T^t)(\nabla f) + f div(T + T^t)\right) = 0 \,.$$

Example 3.9. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be defined as $\psi(x_1, \ldots, x_n) = x_1$, T be a constant matrix, and f be a smooth function. By Example 2.4 and equation (3.2), $\overset{\mathbf{T}}{\square}\psi = 0$. By Corollary 3.8, ψ is fT-harmonic function if and only if

$$d\psi \left((T + T^t)(\nabla f)\right) = 0 \,. \tag{3.9}$$

Let $T\partial_i = \sum_j T_{ji}\partial_j$ and $\nabla f = \sum_i (\nabla_{\partial_i} f)\partial_i$. Since $d\psi(\partial_i) = \delta_{i1}$, by equation (3.9), ψ is fT-harmonic function if and only if $\sum_i (\nabla_{\partial_i} f)(T_{i1} + T_{1i}) = 0$, which is a first order homogeneous linear PDE with constant coefficients. If for every i, $T_{i1} + T_{1i} = 0$, then f is an arbitrary function. If for some i_0, $T_{i_0 1} + T_{1 i_0} \neq 0$, then by analytical solution of this PDE, we have $f = F(c_1, \ldots, \hat{c}_{i_0}, \ldots, c_n)$ where $c_i = x_i - \frac{T_{i1}+T_{1i}}{T_{i_0 1}+T_{1 i_0}} x_{i_0}$ and F is a smooth function.

Proposition 3.10. *Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$, and f_i, $i = 1, 2$, be smooth positive functions on M. Then*

- *i) if ψ is f_1T-harmonic map and f_2T-harmonic map, then $(T + T^t)(\nabla \ln \frac{f_1}{f_2}) \in \ker \mathrm{d}\psi$.*
- *ii) if $(T + T^t)(\nabla \ln \frac{f_1}{f_2}) \in \ker \mathrm{d}\psi$, then ψ is f_1T-harmonic map if and only if it is f_2T-harmonic map.*

Proof. At first we prove (i). By Corollary 3.8, we have ψ is f_1T-harmonic map if and only if

$$f_1 \overset{\mathbf{T}}{\Box}\psi + \frac{1}{2} d\psi\left((T + T^t)\nabla f_1 + f_1 div(T + T^t)\right) = 0\,, \tag{3.10}$$

and ψ is f_2T-harmonic map if and only if

$$f_2 \overset{\mathbf{T}}{\Box}\psi + \frac{1}{2} d\psi\left((T + T^t)\nabla f_2 + f_2 div(T + T^t)\right) = 0\,. \tag{3.11}$$

So by equations (3.10) and (3.11), we get

$$\frac{1}{f_1} d\psi\left((T + T^t)\nabla f_1 + f_1 div(T + T^t)\right) = \frac{1}{f_2} d\psi\left((T + T^t)\nabla f_2 + f_2 div(T + T^t)\right)\,.$$

Therefore $(T + T^t)(\nabla \ln \frac{f_1}{f_2}) \in \ker \mathrm{d}\psi$. In a similar way, we get (ii). □

Lemma 3.11. *Let T be a tensor on Riemannian manifold (M, h), $\psi : (M, h) \to (\overline{M}, l)$ be a smooth map from (M, h) to a Riemannian manifold $(\overline{M}, l)$, and f be a smooth positive function on M. Assume that $\varphi : (M, fh) \to (\overline{M}, l)$ where $\varphi(p) = \psi(p)$ for every $p \in M$. Then*

- *i)*

$$\overset{\mathbf{T}}{\Box}\varphi = \frac{1}{f}\left(\overset{\mathbf{T}}{\Box}\psi - \frac{1}{2} d\psi\left((T + T^t)\overset{h}{\nabla}\ln f - (\overset{h}{tr}T)\overset{h}{\nabla}\ln f\right)\right)\,,$$

- *ii) φ is a T-harmonic map if and only if*

$$\overset{\mathbf{T}}{\Box}\psi + \frac{1}{2} d\psi\left(\left(\frac{n}{2} - 1\right)\left((T + T^t)\overset{h}{\nabla}\ln f\right) + \overset{h}{\mathrm{div}}\,(T + T^t)\right) = 0\,.$$

Proof of case (i).
Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on (M, fh), that is

$$f\,\langle e_i, e_j\rangle_h = \delta_{ij}\,. \tag{3.12}$$

Then we have

$$T_{ij} = f\,\langle Te_j, e_i\rangle_h\ . \tag{3.13}$$

We put $g_i = f^{\frac{1}{2}} e_i$. Then by equations (3.12) and (3.13), we get

$$\langle g_i, g_j\rangle_h = \delta_{ij} \quad , T_{ij} = \langle T(g_j), g_i\rangle_h\ . \tag{3.14}$$

Therefore by Definition 3.1 and equation (3.14), we have

$$\begin{aligned}
\overset{\mathbf{T}}{\Box}\varphi &= \sum_{i,j} T_{ij}\left(\overset{l}{\overline{\nabla}}_{e_i}\, d\varphi(e_j) - d\varphi(\overset{fh}{\nabla}_{e_i}\, e_j)\right)\\
&= \sum_{i,j} T_{ij}\left(\overset{l}{\overline{\nabla}}_{\frac{g_i}{f^{\frac{1}{2}}}}\, d\varphi(\frac{g_j}{f^{\frac{1}{2}}}) - d\varphi(\overset{fh}{\nabla}_{\frac{g_i}{f^{\frac{1}{2}}}}\, \frac{g_j}{f^{\frac{1}{2}}})\right)\\
&= \sum_{i,j} \frac{1}{f} T_{ij}\left(\overset{l}{\overline{\nabla}}_{g_i}\, d\varphi(g_j) - d\varphi(\overset{fh}{\nabla}_{g_i}\, g_j)\right)\\
&= \sum_{i,j} \frac{1}{f} T_{ij}\Bigg(\overset{l}{\overline{\nabla}}_{g_i}\, d\varphi(g_j) - d\varphi\Big(\overset{h}{\nabla}_{g_i}\, g_j + \frac{1}{2f} df(g_j) g_i + \frac{1}{2f} df(g_i) g_j\\
&\qquad - \frac{1}{2f}\langle g_i, g_j\rangle_h \overset{h}{\nabla} f\Big)\Bigg)\\
&= \frac{1}{f}\left(\overset{\mathbf{T}}{\Box}\psi - \frac{1}{2f} T_{ij} d\psi\left(df(g_j) g_i + df(g_i) g_j - \langle g_i, g_j\rangle_h \overset{h}{\nabla} f\right)\right)\\
&= \frac{1}{f}\left(\overset{\mathbf{T}}{\Box}\psi - \frac{1}{2f} d\psi\left(\sum_i (df(T(g_i))g_i + df(g_i)T(g_i) - \langle g_i, T(g_i)\rangle_h \overset{h}{\nabla} f)\right)\right)\\
&= \frac{1}{f}\left(\overset{\mathbf{T}}{\Box}\psi - \frac{1}{2f} d\psi\left((T + T^t)\overset{h}{\nabla} f - (\overset{h}{trT})\overset{h}{\nabla} f\right)\right)\ .
\end{aligned}$$

Proof of case (ii).
By Corollary 3.5 we get

$$\overset{\mathbf{T}}{\Box}\varphi + \frac{1}{2} d\varphi\left(\overset{fh}{\operatorname{div}}\,(T + T^t)\right) = 0\,, \tag{3.15}$$

At first we compute $\overset{fh}{\operatorname{div}}\,T$. As before, let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on (M, fh) and $g_i = f^{\frac{1}{2}} e_i$. So

$$\begin{aligned}
\overset{fh}{\operatorname{div}}\, T &= \sum_i (\overset{fh}{\nabla}_{e_i} T)e_i = \frac{1}{f}\sum_i (\overset{fh}{\nabla}_{g_i} T)g_i = \frac{1}{f}\sum_i \left(\overset{fh}{\nabla}_{g_i} Tg_i - T\overset{fh}{\nabla}_{g_i} g_i\right) \\
&= \frac{1}{f}\sum_i \left(\overset{h}{\nabla}_{g_i} Tg_i + \frac{1}{2f}\left(df(T(g_i))g_i + df(g_i)T(g_i) - \langle g_i, T(g_i)\rangle_h \overset{h}{\nabla} f\right)\right. \\
&\qquad \left. -T\left(\overset{h}{\nabla}_{g_i} g_i + \frac{1}{2f}\left(df(g_i)g_i + df(g_i)g_i - \langle g_i, g_i\rangle_h \overset{h}{\nabla} f\right)\right)\right) \\
&= \frac{1}{f}\left(\overset{h}{\operatorname{div}}\, T + \frac{1}{2f}\left(\sum_i (df(T(g_i))g_i) - (\overset{h}{tr}\, T)\overset{h}{\nabla} f + (n-1)T\overset{h}{\nabla} f\right)\right),
\end{aligned}$$

and similarly, we compute $\overset{fh}{\operatorname{div}}\, T^t$, and then by substituting in equation (3.15), and by use of case (i) we get the result. □

Proposition 3.12. *Let T be a tensor on Riemannian manifold (M,h), $\psi:(M,h)\to(\overline{M},l)$ be a smooth map from (M,h) to a Riemannian manifold $(\overline{M},l)$, and f_1 and f_2 be smooth positive functions on M. Assume that $\varphi:(M,f_2h)\to(\overline{M},l)$ be $\varphi(p)=\psi(p)$ for every $p\in M$. Then*

i) if ψ is f_1T-harmonic map and φ is T-harmonic map, then f_1 and f_2 satisfy the equation $(T+T^t)\overset{h}{\nabla}\ln\frac{f_2^{(\frac{n}{2}-1)}}{f_1}\in\ker d\psi$.

ii) if f_1 and f_2 satisfy the equation $(T+T^t)\overset{h}{\nabla}\ln\frac{f_2^{(\frac{n}{2}-1)}}{f_1}\in\ker d\psi$, then, ψ is f_1T-harmonic map if and only if φ is T-harmonic map.

Proof. By Corollary 3.8 we have ψ is f_1T-harmonic map if and only if

$$\overset{\mathbf{T}}{\Box}\psi + \frac{1}{2}d\psi\left((T+T^t)\overset{h}{\nabla}\ln f_1 + \overset{h}{\operatorname{div}}\,(T+T^t)\right) = 0\,, \tag{3.16}$$

and by Lemma 3.11, φ is T-harmonic map if and only if

$$\overset{\mathbf{T}}{\Box}\psi + \frac{1}{2}d\psi\left(\left(\frac{n}{2}-1\right)(T+T^t)\overset{h}{\nabla}\ln f_2 + \overset{h}{\operatorname{div}}\,(T+T^t)\right) = 0\,. \tag{3.17}$$

Therefore by equalizing equations (3.16) and (3.17), we prove case (i). In a similar way, we get (ii). □

Theorem 3.13. *Let T be a tensor on Riemannian manifold (M,h), $\psi:(M,h)\to(\overline{M},l)$ be a smooth map from (M,h) to a Riemannian manifold $(\overline{M},l)$, f_1 and f_2 be smooth positive functions on M, and ψ be a conformal immersion $\psi^*l=f_2h$. Then ψ is f_1T-harmonic map if and only if*

$$\begin{cases} \sum_i B(Te_i,e_i)=0, \\ (T+T^t)\overset{h}{\nabla}\ln(f_1f_2) + \overset{h}{\operatorname{div}}\,(T+T^t) - (trT)\overset{h}{\nabla}\ln f_2 = 0\,, \end{cases} \tag{3.18}$$

where B is the second fundamental form of conformal immersion ψ and $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on (M, h).

Proof. Since ψ is a conformal immersion, so the second fundamental form

$$B(e_i, e_j) = \overline{\nabla}_{e_i} d\psi(e_j) - d\psi(\overset{f_2 h}{\nabla}_{e_i} e_j), \tag{3.19}$$

is normal to tangent space of submanifold $\psi(M) \subset \overline{M}$. Thus by equation (3.19), we have

(3.20)

$$\begin{aligned}
\overset{\mathbf{T}}{\Box}\psi &= \sum_{i,j} T_{ij} \left(\overline{\nabla}_{e_i} d\psi(e_j) - d\psi(\overset{h}{\nabla}_{e_i} e_j) \right) = \sum_{i,j} T_{ij} \left(B(e_i, e_j) + d\psi(\overset{f_2 h}{\nabla}_{e_i} e_j - \overset{h}{\nabla}_{e_i} e_j) \right) \\
&= \sum_{i,j} T_{ij} \left(B(e_i, e_j) + \frac{1}{2f_2} d\psi \left(df_2(e_j) e_i + df_2(e_i) e_j - \delta_{ij} \overset{h}{\nabla} f_2 \right) \right) \\
&= \sum_{i} \left(B(e_i, Te_i) + \frac{1}{2f_2} d\psi \left(df_2(Te_i) e_i + df_2(e_i) Te_i - T_{ii} \overset{h}{\nabla} f_2 \right) \right) \\
&= \sum_i B(Te_i, e_i) + \frac{1}{2f_2} d\psi \left(\sum_i \left\langle \overset{h}{\nabla} f_2, T(e_i) \right\rangle_h e_i + T \overset{h}{\nabla} f_2 - (trT) \overset{h}{\nabla} f_2 \right) \\
&= \sum_i B(Te_i, e_i) + \frac{1}{2} d\psi \left((T + T^t) \overset{h}{\nabla} \ln f_2 - (trT) \overset{h}{\nabla} \ln f_2 \right) .
\end{aligned}$$

By Corollary 3.8 we have ψ is $f_1 T$-harmonic map if and only if

$$\overset{\mathbf{T}}{\Box}\psi + \frac{1}{2} d\psi \left((T + T^t) \overset{h}{\nabla} \ln f_1 + \overset{h}{\mathrm{div}} (T + T^t) \right) = 0 . \tag{3.21}$$

Now substituting equation (3.20) in equation (3.21), we get

(3.22)

$$\sum_i B(Te_i, e_i) + \frac{1}{2} d\psi \left((T + T^t) \overset{h}{\nabla} \ln(f_1 f_2) - (trT) \overset{h}{\nabla} \ln f_2 + \overset{h}{\mathrm{div}} (T + T^t) \right) = 0 .$$

By noting normal and tangential part of equation (3.22), we get system of equations (3.18). □

Remark 3.14 (Proposition 1.1 of [15])**.** By Theorem 3.13, a conformal immersion $\psi^* l = f_2 h$, is f_1-harmonic map if and only if the mean curvature vector field $H = 0$, that is ψ is minimal and $f_1 = C f_2^{\frac{n}{2}-1}$ for some constant C. In particular, an isometric immersion is f-harmonic if and only if f is constant and hence ψ is harmonic.

Proposition 3.15. *Let $\psi_1 : (M, h) \to (\overline{M}, l)$ and $\psi_2 : (\overline{M}, l) \to (\overline{\overline{M}}, k)$ be smooth maps between Riemannian manifolds M, $\overline{M}$ and $\overline{M}$, $\overline{\overline{M}}$, and ∇, $\overline{\nabla}$, $\overline{\overline{\nabla}}$ be Levi-Civita* *connections on M, $\overline{M}$, $\overline{\overline{M}}$, respectively, f a smooth function , and T a tensor on Riemannian manifold M. Then $\psi_2 \circ \psi_1$ is a fT-harmonic map if and only if*

$$f \sum_i \left(\overline{\overline{\nabla}}_{d\psi_1(e_i)} d\psi_2 \right) d\psi_1(T(e_i)) + d\psi_2 \left(f \overset{\mathbf{T}}{\Box} \psi_1 + \frac{1}{2} d\psi_1 \left((T+T^t)\nabla f + f div(T+T^t) \right) \right) = 0 \,,$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M. Especially, If ψ_1 is a fT-harmonic map, then $\psi_2 \circ \psi_1$ is a fT-harmonic map if and only if

$$f \sum_i \left(\overline{\overline{\nabla}}_{d\psi_1(e_i)} d\psi_2 \right) d\psi_1(T(e_i)) = 0 \,.$$

Proof. Assume an local orthonormal frame $\{e_i\}_{i=1}^n$ on M such that $(\nabla_{e_i} e_j)(p) = 0$ at a fix point $p \in M$ for every i, j. Then by Definition 3.1, we have

$$\begin{aligned} \overset{\mathbf{T}}{\Box}(\psi_2 \circ \psi_1) &= \sum_{i,j} T_{ij} \overline{\overline{\nabla}}_{e_i} d(\psi_2 \circ \psi_1)(e_j) \\ &= \sum_{i,j} T_{ij} \left(\left(\overline{\overline{\nabla}}_{d\psi_1(e_i)} d\psi_2 \right) d\psi_1(e_j) + d\psi_2 \left(\overline{\nabla}_{e_i} d\psi_1(e_j) \right) \right) \\ &= \sum_i \left(\overline{\overline{\nabla}}_{d\psi_1(e_i)} d\psi_2 \right) d\psi_1(T(e_i)) + d\psi_2 \left(\overset{\mathbf{T}}{\Box} \psi_1 \right) . \end{aligned} \tag{3.23}$$

By Corollary 3.8 we have $\psi_2 \circ \psi_1$ is a fT-harmonic map if and only if

$$f \overset{\mathbf{T}}{\Box}(\psi_2 \circ \psi_1) + \frac{1}{2} d(\psi_2 \circ \psi_1) \left((T+T^t)\nabla f + f div(T+T^t) \right) = 0 \,. \tag{3.24}$$

Therefore substituting equation (3.23) in equation (3.24) we get the result. □

Remark 3.16. Note that by Proposition 3.15, we can not get that ψ_2 is f-harmonic map if ψ_1 and $\psi_2 \circ \psi_1$ are fT-harmonic maps, even if ψ_1 is an identity map and T is definite symmetric tensor. We show it as the following. Let $\psi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be identity map, and $\psi_2 : \mathbb{R}^2 \to \mathbb{R}$, $\psi_2(x_1, x_2) = x_1$. By Example 3.9, if $f = f(x_1)$ is non constant, then ψ_2 is not a f-harmonic map. By Corollary 3.8, ψ_1 and ψ_2 are fT-harmonic maps if $T\nabla f + f divT = 0$. So for every j, $f'(x_1)T_{j1} + f(x_1)\sum_i \dfrac{\partial T_{ji}}{\partial x_i} = 0$. If $T_{21} = 0$, we have $T_{11} = \frac{k}{|f(x_1)|} e^{g(x_2)}$ where g is a smooth function and k is some constant, and $T_{22} = T_{22}(x_1)$. Therefore we can choose a definite diagonal tensor T to prove the claim.

4. fL_k-HARMONIC HYPERSURFACES

In this section, as application of fT-harmonic maps for conformal immersions, we consider fL_k-harmonic hypersurfaces in space forms, [4], which is fT-harmonic hypersurfaces when T is P_k transformation.

Theorem 4.1. *Let $\psi : M^n \to R^{n+1}(c)$ be an isometric immersion from a connected oriented Riemannian manifold M into a simply connected space form $R^{n+1}(c)$, and f be a smooth positive function on M. Then ψ is an fL_k-harmonic hypersurface if and only if $H_{k+1} = 0$ and $P_k \nabla f = 0$.*

Proof. As we know the second fundamental form of ψ is $B(X, Y) = \langle S(X), Y\rangle N$ where S is the shape operator and X, Y are vector fields on M, and N as the unit normal direction. The P_k's transformation are symmetric and free-divergence in space forms. Now by putting $T = P_k$ in Theorem 3.13, we get $\sum_i \langle S \circ P_k(e_i), (e_i)\rangle N = 0$ where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M, and $P_k\nabla f = 0$. Since $tr(S \circ P_k) = (k + 1)s_{k+1}$, $s_{k+1} = 0$. □

Remark 4.2. As it is well known totally umbilic hypersurfaces of dimension equal or greater than two in the Euclidean space are hyperplanes and hyperspheres, and in the Hyperbolic space are obtained by intersecting with affine hyperplanes, especially are hyperspheres and hyperbolic spaces of codimension one, and in the Euclidean sphere are hyperspheres, hence are of constant principal curvatures. So Theorem 4.1 implies that a totally umbilic hypersurface M of $R^{n+1}(c)$ is fL_k-harmonic if and only if, f is arbitrary smooth positive function on M if $k \neq 0$, and f is constant positive function if $k = 0$, and in both cases M is an open piece of $\mathbb{R}^n$ when $c = 0$ or an open piece of $\mathbb{H}^n(-1)$ when $c = -1$ or an open piece of $\mathbb{S}^n(1)$ when $c = 1$.

Remark 4.3. Consider the cylinder $\mathbb{S}^1(r) \times \mathbb{R} \subset \mathbb{R}^3$. Since $H_1 \neq 0$ and $H_2 = 0$, by Theorem 4.1, it is an L_1-harmonic hypersurface but not harmonic.

Theorem 4.4. *Let $\psi : M \to R^3(c)$ be an isometric immersion from a connected oriented Riemannian surface M into a simply connected space form $R^3(c)$ and f be a smooth positive function on M. Then ψ is an fL_1-harmonic surface if and only if the principal curvatures are zero and $2H_1$, and $H_1\nabla_v f = 0$ for every vector v in the distribution of space of principal vectors of zero's principal curvature.*

Proof. By Theorem 4.1, M is an fL_1-harmonic hypersurface if and only if $s_2 = 0$ and $S\nabla f = s_1\nabla f$. Since $s_2 = 0$, principal curvatures are zero and s_1. Let $\{e_1, e_2\}$

be a local orthonormal principal vector fields, corresponding to principal curvatures zero and s_1, respectively. So by $S\nabla f = s_1\nabla f$, we get $s_1\nabla_{e_1} f = 0$. The proof of converse is straightforward. □

Theorem 4.5. *Let $\psi : M \to R^3(c)$,$(c = 0, 1)$, be an isometric immersion from a complete connected oriented Riemannian surface M into a simply connected space form $R^3(c)$. If $c = 0$, then ψ is fL_1-harmonic surface if and only if $\psi(M)$ is a cylinder over planar curve and $H_1\nabla_v f = 0$ for every vector v in the distribution of space of principal vectors of zero's principal curvature. If $c = 1$, then ψ is fL_1-harmonic surface if and only if $\psi(M)$ is $\mathbb{S}^2(1)$ and f is arbitrary smooth positive function on M.*

Proof. If $H_2 = 0$, we have constant sectional curvature $K = c$, and so ψ is a space form. By Hartman-Nirenberg theorem and Liebmann theorem, the only complete oriented two dimensional space form with constant sectional curvature $K = c$ in $R^3(c)$ is: a cylinder over planar curve if $c = 0$; $\mathbb{S}^2(1)$ if $c = 1$ (cf. [10, 11, 14]). Now by Remark 4.2, Theorem 4.1 and Theorem 4.4, we get the result. □

As a result of Theorem 4.1, we can get the following corollary for isoparametric hypersurfaces in space forms (see proof of Theorem 4.5 of [3]).

Corollary 4.6. *Let $\psi : M^n \to R^{n+1}(c)$, be an isoparametric hypersurface immersed into simply connected space form $R^{n+1}(c)$. If $c = 0$, then ψ is an fL_k-harmonic hypersurface if and only if $\psi(M)$ is an open piece of $\mathbb{R}^n$, and f is arbitrary smooth positive function on M if $k \neq 0$, and f is constant positive function if $k = 0$, or $\psi(M)$ is an open piece of generalized right cylinder $\mathbb{S}^m(r) \times \mathbb{R}^{n-m}$ with $r > 0$ and $m \leq k$, and f is an arbitrary smooth positive function on M if $m < k$, and f is positive constant on each integral submanifold of distribution of space of principal vectors of zero's principal curvature if $m = k$. If $c = -1$, then ψ is an fL_k-harmonic hypersurface if and only if $\psi(M)$ is an open piece of $\mathbb{H}^n(-1)$, and f is arbitrary smooth positive function on M if $k \neq 0$, and f is constant positive function if $k = 0$. If $c = 1$ and M has at most two principal curvatures, then ψ is an fL_k-harmonic hypersurface if and only if $\psi(M)$ is an open piece of $\mathbb{S}^n(1)$, and f is arbitrary smooth positive function on M if $k \neq 0$, and f is constant positive function if $k = 0$, or $\psi(M)$ is an open piece of $\mathbb{S}^m(\frac{1}{\sqrt{\alpha^2+1}}) \times \mathbb{S}^{n-m}(\frac{\alpha}{\sqrt{\alpha^2+1}})$ with $\alpha > 0$, and α and f satisfy the following equations:*

$$\sum_i \binom{m}{i}\binom{n-m}{k+1-i}(-\alpha^2)^i = 0\,,$$
$$\sum_i \binom{m-1}{i}\binom{n-m}{k-i}(-\alpha^2)^i\nabla_v f = \sum_i \binom{m}{i}\binom{n-m-1}{k-i}(-\alpha^2)^i\nabla_w f = 0\,,$$

for every vector v and w in the distribution of space of principal vectors corresponding to α and $-\frac{1}{\alpha}$ principal curvatures, respectively.

Theorem 4.7. *Let $\psi : M^n \to R^{n+1}(c)$ be an isometric immersion from a connected oriented Riemannian manifold M into a simply connected space form $R^{n+1}(c)$, and f be a smooth positive function on M. If all principal curvature are non negative (it is called weakly convex), then ψ is an fL_k-harmonic hypersurface if and only if $\psi(M)$ is an open piece of $\mathbb{R}^n$ when $c = 0$ or an open piece of $\mathbb{H}^n(-1)$ when $c = -1$ or an open piece of $\mathbb{S}^n(1)$ when $c = 1$, and in all cases, f is arbitrary smooth positive function on M if $k \neq 0$, and f is constant positive function if $k = 0$.*

Proof. Since all all principal curvature are non negative, so if $H_{k+1} = 0$, then all principal curvature are zero. That is the hypersurface M is totally geodesic. Now by Remark 4.2, we get the result. □

Let us recall that every compact hypersurface immersed into the Euclidean space or in the hyperbolic space or in the Euclidean hemisphere has an elliptic point (cf. [1, 2]), that is, a point where all the principal curvatures are positive (or negative). Therefore for every k, $k = 0, \ldots, n$, k-th mean curvature is not identically zero. So we have the following non-existence result as a consequence of Theorem 4.1.

Corollary 4.8. *There exists no compact orientable fL_k-harmonic hypersurface either in $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}_+$.*

References

1. L. J. Alías & N. Gürbüz: An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures. Geom. Ded. **121** (2006), 113-127. https://doi.org/10.1007/s10711-006-9093-9
2. L. J. Alías & S. M. B. Kashani: Hypersurfaces in space forms satisfying the condition $L_k x = Ax + b$. Taiwanese J. Math. **14** (2010), 1957-1977. https://doi.org/10.11650/twjm/1500406026
3. M. Aminian & S.M.B. Kashani: L_k-biharmonic hypersurfaces in space forms. Acta Math. Vietnam. **42** (2017), 471-490. https://doi.org/10.1007/s40306-016-0195-7

4. M. Aminian & M. Namjoo: fL_k-harmonic maps and fL_k-harmonic morphisms. Acta Math. Vietnam. (2020). https://doi.org/10.1007/s40306-020-00392-0
5. É. Cartan: Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques. Math. Z. **45** (1939), 335-367. https://doi.org/10.1007/BF01580289
6. ———: Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions. Rev. Univ. Tucuman, Ser. A **1** (1940), 177-191.
7. B.-Y. Chen: Total mean curvature and submanifold of finite type. 2nd ed., Series in Pure Math., World Scientific, 2014. https://doi.org/10.1142/9237
8. S. Y. Cheng & S.T. Yau: Hypersurfaces with constant scalar curvature. Math. Ann. **255** (1977), 195-204. https://doi.org/10.1007/BF01425237
9. J. Eells & J. H. Sampson: Harmonic mappings of Riemannian manifolds. Amer. J. Math. **86** (1964), 109-160. https://doi.org/10.2307/2373037
10. José A. Gálvez: Surfaces of constant curvature in 3-dimensional space forms. Mat. Contemp. **37** (2009), 1-42. https://www.semanticscholar.org/paper/Surfaces-of-constant-curvature-in-3-dimensional-.-G
11. P. Hartman & L. Nirenberg: On spherical image maps whose Jacobians do not change sign. Amer. J. Math. **81** (1959), 901-920. https://doi.org/10.2307/2372995
12. G. Y. Jiang: 2-harmonic maps and their first and second variational formulas. Chinese Ann. Math. **7A** (1986), 388-402, the English translation, Note di Mathematica **28** (2008) 209-232. http://siba-ese.unisalento.it/index.php/notemat/article/view/1770, DOI Code:10.1285/I15900932V28N1SUPPLP209
13. H. F. M̈unzner: Isoparametrische hyperflächen in sphären. I. Math. Ann. **251** (1980), II, **256** (1981).
14. B. O' Neill & E. Stiel: Isometric immersions of constant curvature manifolds. Michigan Math. J. **10** (1963), 335-339. https://projecteuclid.org/journals/michigan-mathematical-journal/volume-10/issue-4/Isometric-immersions-of-constant-curvature-manifolds/10.1307/mmj/1028998967.full DOI Code:10.1307/mmj/1028998967
15. Y.-L. Ou: f-harmonic morphisms between Riemannian manifolds. Chin. Ann. Math., Series B **35** (2014), 225-236. https://doi.org/10.1007/s11401-014-0825-0
16. R. C. Reilly: Variational properties of functions of the mean curvatures for hypersurfaces in space forms. J. Diff. Geom. **8** (1973), 465-477. https://projecteuclid.org/journals/journal-of-differential-geometry/volume-8/issue-3/Variational-properties-of-functions-of-the-mean-curvatures-for-hypersurfaces/10.4310/jdg/1214431802.full DOI Code:10.4310/jdg/1214431802

Professor: Department of Mathematics, Vali-e-Asr University, Rafsanjan, Iran
Email address: mehran.aminian@vru.ac.ir