J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.2.139 ISSN(Online) 2287-6081
Volume 30, Number 2 (May 2023), Pages 139–154

GENERALIZED RELATIVE ORDER (α, β) ORIENTED SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we wish to prove some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of their generalized relative order (α, β) and generalized relative lower order (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1. Introduction

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions which are available in [7, 8, 13]. We also use the standard notations and definitions of the theory of entire functions which are available in [12] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. In this connection the following definition is relevant:

Definition 1.1 ([2]). A non-constant entire function f is said to have the *Property* (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [2].

When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [7, p.4]) plays the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

Received by the editors October 29, 2022. Revised March 31, 2023. Accepted April 04, 2023.
2020 *Mathematics Subject Classification.* 30D35, 30D30, 30D20.
Key words and phrases. entire function, meromorphic function, growth, composition, generalized relative order (α, β).
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wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r,a)$.

For an entire function f, the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.Further we assume that throughout the present paper α_1, α_2, α_3, α_4, β, β_1, β_2, β_3 and β_4 denote the functions belonging to L_1 unless otherwise specifically stated.

The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

introduced by Sheremeta [10], is called a generalized order (α, β) of an entire function f. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction. For the purpose of further applications, Biswas et al. [4] rewrite the definition of the generalized order (α, β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [10]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [5]:

Definition 1.2 ([4])**.** The generalized order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\begin{aligned} \rho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \\ \text{and } \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ where } \alpha \in L_1. \end{aligned}$$

If f is a meromorphic function, then

$$\begin{aligned} \rho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \\ \text{and } \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha \in L_2. \end{aligned}$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [7]$\}$, for an entire function f, one may easily verify that

$$\begin{aligned} \rho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \\ \text{and } \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \end{aligned}$$

when $\alpha \in L_2$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 9]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [4] introduce the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function with respect to another entire function in the following way:

Definition 1.3 ([4])**.** Let $\alpha,\ \beta\in L_1$. The generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function f with respect to an entire function g denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\begin{aligned}\rho_{(\alpha,\beta)}[f]_g &= \limsup_{r\to+\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}\\ \text{and } \lambda_{(\alpha,\beta)}[f]_g &= \liminf_{r\to+\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.\end{aligned}$$

In this paper, we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized relative order (α,β) and generalized relative lower order (α,β).

2. Main Results

In this section first we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([3])**.** *If f is a meromorphic function and g is an entire function then for all sufficiently large values of r,*

$$T_{f\circ g}(r)\leqslant\{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.2 ([6])**.** *Let f be an entire function which satisfies the Property (A), $\beta>0$, $\delta>1$ and $\alpha>2$. Then*

$$\beta T_f(r)<T_f\left(\alpha r^{\delta}\right).$$

Now we present the main results of the paper.

Theorem 2.3. *Let f be a meromorphic function and g, h be any two entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\rho_{(\alpha_1,\beta_1)}[f]_h<+\infty$, $\rho_{(\alpha_2,\beta_2)}[g]<+\infty$ and h satisfies the Property (A). Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$ and $A\geq0$ be any number, then*
(i) If $\beta_1(\alpha_2^{-1}(\log r))\leq r$ and $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\left\{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))}=0 \text{ and}$$

(ii) If either $\beta_1(r) = B\alpha_2(r)$ *where* B *is any positive constant and* $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty$ *or* $\beta_1(\alpha_2^{-1}(r)) \in L^0$ *and* $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$*, then*

$$\lim_{r\to+\infty} \frac{\left\{\exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))} = 0.$$

Proof. Let us suppose that $\Delta > 2$ and $\delta \to 1+$ in Lemma 2.2. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.2 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [9] } that for all sufficiently large values of r,

$$\begin{aligned} T_h^{-1}(T_{f\circ g}(\beta_2^{-1}(\log r))) &\leqslant T_h^{-1}(\{1+o(1)\}\,T_f(M_g(\beta_2^{-1}(\log r)))) \\ \text{i.e., } \alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) &\leqslant \alpha_1(\Delta(T_h^{-1}(T_f(M_g(\beta_2^{-1}(\log r)))))^{\delta}) \\ \text{i.e., } \alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) &\leqslant (1+o(1))\alpha_1(T_h^{-1}(T_f(M_g(\beta_2^{-1}(\log r))))) \end{aligned}$$

$$\text{i.e., } \alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))$$

$$\leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \tag{2.1}$$

Further from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large values of r that

$$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r))))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\gamma(r). \tag{2.2}$$

Now the following cases may arise:

Case I. Let $\beta_1(\alpha_2^{-1}(\log r)) \leq r$. Now we get from (2.1) for all sufficiently large values of r that

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))$$

$$\leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\alpha_2(M_g(\beta_2^{-1}(\log r))))) \tag{2.3}$$

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))$$

$$\leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)})). \tag{2.4}$$

Case II. Let $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant. Then we have from (2.1) that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) \leqslant (1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\alpha_2(M_g(\beta_2^{-1}(\log r)))$$

$$\begin{aligned} &\text{i.e., } \alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) \\ &\leqslant (1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\log r \end{aligned}$$

$$\text{i.e., } \exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))) \leqslant r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{2.5}$$

Case III. Let $\beta_1(\alpha_2^{-1}(r)) \in L_1$ and $\lim_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$. Then we have from (2.3) that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))$$

$$i.e.,\ \exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))$$

$$\leqslant \exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))). \tag{2.6}$$

Now when $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r} = +\infty$, we obtain from (2.2) and (2.4) of Case I that for all sufficiently large values of r,

$$\frac{\left\{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))}$$

$$\leqslant \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)^{1+A}[\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}))]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\left\{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))} = 0,$$

This proves the first part of the theorem.

Again combining (2.2) and (2.5) of Case II, we get for all sufficiently large values of r that

$$\frac{\left\{\exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)}$$

Since $\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r} = +\infty$,

$$\frac{r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{\gamma(r)} \to 0$$

as $r \to +\infty$. Thus it follows from above that

$$\lim_{r\to+\infty}\frac{\left\{\exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))} = 0. \tag{2.7}$$

Further combining (2.2) and (2.6) of Case III it follows that for all sufficiently large values of r,

$$\frac{\left\{\exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))}$$

$$\leqslant \frac{\left[\exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))\right]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)\gamma(r)}.$$

Since $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$,

$$\frac{\left[\exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}\left[f\right]_h+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))\right]^{1+A}}{\gamma(r)}\to 0$$

as $r\to+\infty$. Thus from above we obtain that

$$\lim_{r\to+\infty}\frac{\left\{\exp(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))\right\}^{1+A}}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\gamma(r)))))}=0. \tag{2.8}$$

Hence the second part of the theorem follows from (2.7) and (2.8).

Thus the theorem follows. □

Remark 2.4. Theorem 2.3 improves and extends Theorem 3 of [11].

Remark 2.5. In Theorem 2.3 if we take the condition $\rho_{(\alpha_1,\beta_1)}\left[f\right]_h>0$ instead of $0<\lambda_{(\alpha_1,\beta_1)}\left[f\right]_h\leq\rho_{(\alpha_1,\beta_1)}\left[f\right]_h<+\infty$, the theorem remains true with " limit inferior" in place of "limit".

Theorem 2.6. *Let f be a meromorphic function and g, h, k be any three entire functions such that $\rho_{(\alpha_1,\beta_1)}\left[f\right]_h<+\infty$, $\lambda_{(\alpha_2,\beta_2)}\left[g\right]_k>0$, $\rho_{(\alpha_2,\beta_2)}\left[g\right]<+\infty$ and h satisfies the Property (A). Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$ and $A\geq 0$ be any number.*
(i) If $\beta_1(\alpha_2^{-1}(\log r))\leq r$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^{1+A}}{\alpha_2(\exp(T_g(\beta_2^{-1}(\gamma(r)))))}=0.$$

(ii) If either $\beta_1(r)=B\alpha_2(r)$ where B is any positive constant and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$ or $\beta_1(\alpha_2^{-1}(r))\in L^0$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_2(\exp(T_g(\beta_2^{-1}(\gamma(r)))))}=0.$$

The proof of Theorem 2.6 would run parallel to that of Theorem 2.3. We omit the details.

Remark 2.7. In Theorem 2.6, if we take the condition $\rho_{(\alpha_2,\beta_2)}\left[g\right]_k>0$ instead of $\lambda_{(\alpha_2,\beta_2)}\left[g\right]_k>0$, the theorem remains true with "limit inferior" instead of "limit".

Theorem 2.8. *Let f be a meromorphic function and g, h, k,l, m be five entire functions such that $\lambda_{(\alpha_3,\beta_3)}\left[l\right]_m>0$, $\lambda_{(\alpha_4,\beta_4)}\left[k\right]>0$, $\rho_{(\alpha_1,\beta_1)}\left[f\right]_h<\infty$, $\rho_{(\alpha_2,\beta_2)}\left[g\right]<$*

$\lambda_{(\alpha_4,\beta_4)}[k]$ *and* h, m *both satisfy the Property (A). Also let* C *and* D *be any two positive constants.*

(i) Any one of the following four conditions are assumed to be satisfied:

(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*

(b) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*;*

(c) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$*;*

(d) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) > \exp(\alpha_4(r))$*, then*

$$\lim_{r\to+\infty} \frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))} = \infty.$$

(ii) Any one of the following two conditions are assumed to be satisfied:

(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$*;*

(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))} = \infty.$$

(iii) Any one of the following two conditions are assumed to be satisfied:

(a) $\beta_3(r) = D\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L_1$*;*

(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L_1$*, then*

$$\lim_{r\to+\infty} \frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))} = \infty.$$

(iv) If $\alpha_2(\beta_1^{-1}(r)) \in L_1$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$*, then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))} = \infty.$$

Proof. CASE I. Let $\beta_1(r) = C(\exp(\alpha_2(r)))$. Then we have from (2.1) that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) \leqslant C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{2.9}$$

CASE II. Let $\exp(\alpha_2(r)) > \beta_1(r)$. Then we have from (2.1) that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)) < (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{2.10}$$

CASE III. Let $\alpha_2(\beta_1^{-1}(r)) \in L_1$. Then we get from(2.1) that for all sufficiently large values of r,

$$\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))))) \leqslant r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{2.11}$$

Now suppose that $\Lambda > 2$ and $\delta \to 1+$ in Lemma 2.2. Since $T_m^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.2 and the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ {cf. [7]} for an entire function f that for all sufficiently large values of r,

$$\alpha_3(T_m^{-1}(3T_{l\circ k}(\beta_4^{-1}(\log r)))) \geq \alpha_3\Big(T_m^{-1}\Big(T_l\Big(\frac{1}{8}M_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\Big)$$

$$i.e., \ \alpha_3(\Lambda(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))^{\delta})) \geq \alpha_3\Big(T_m^{-1}\Big(T_l\Big(\frac{1}{8}M_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\Big)$$

$$i.e., \ \alpha_3(\Delta(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))^{\delta})) \geq \alpha_3\Big(T_m^{-1}\Big(T_l\Big(\frac{1}{8}M_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\Big)$$

$$i.e., \ \alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))) \geq (1+o(1))\Big(\alpha_3\Big(T_m^{-1}\Big(T_l\Big(\frac{1}{8}M_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big)\Big)\Big)\Big)$$

$$i.e., \ \alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))) \geq (1+o(1))(\lambda_{(\alpha_3,\beta_3)}[l]_m - \varepsilon)\beta_3\Big(M_k\Big(\frac{\beta_4^{-1}(\log r)}{4}\Big)\Big). \tag{2.12}$$

Case IV. Let $\beta_3(r) = D\exp(\alpha_4(r))$ Then from (2.12) it follows that for all sufficiently large values of r,

$$\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))) \geq D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[l]_m - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{2.13}$$

Case V. Let $\beta_3(r) > \exp(\alpha_4(r))$. Now from (2.12) it follows that for all sufficiently large values of r,

$$\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))) > (1+o(1))(\lambda_{(\alpha_3,\beta_3)}[l]_m - \varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{2.14}$$

Case VI. Let $\alpha_4(\beta_3^{-1}(r)) \in L_1$. Then from (2.12) we obtain that for all sufficiently large values of r,

$$\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))))) \geqslant r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}. \tag{2.15}$$

Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$ we can choose $\varepsilon(>0)$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_4,\beta_4)}[k] - \varepsilon. \tag{2.16}$$

Now combining (2.9) of Case I and (2.13) of Case IV it follows that for all sufficiently large values of r,

$$\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))} \geq \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}\left[l\right]_m-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}\left[f\right]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (2.16) and above we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}=\infty. \tag{2.17}$$

Further combining (2.9) of Case I and (2.14) of Case V it follows that for all sufficiently large values of r,

$$\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))} > \frac{(1+o(1))(\lambda_{(\alpha_3,\beta_3)}\left[l\right]_m-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}\left[f\right]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Hence from (2.16) and above we get that

$$\liminf_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}=\infty. \tag{2.18}$$

Similarly combining (2.10) of Case II and (2.13) of Case IV, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)}=\infty. \tag{2.19}$$

Likewise combining (2.10) of Case II and (2.14) of Case V it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}=\infty. \tag{2.20}$$

Hence the first part of the theorem follows from (2.17), (2.18), (2.19) and (2.20).

Again combining (2.9) of Case I and (2.15) of Case VI we obtain that for all sufficiently large values of r,

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}$$
$$\geq \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}\left[f\right]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (2.16) and above we obtain that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}=\infty. \tag{2.21}$$

Now in view of (2.10) of Case II and (2.15) of Case VI we get that for all sufficiently large values of r,

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}$$
$$> \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{(1+o(1))(\rho_{(\alpha_1,\beta_1)}\,[f]_h+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (2.16) and above we obtain that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right))}=\infty, \tag{2.22}$$

Therefore the second part of the theorem follows from (2.21) and (2.22).

Further combining (2.11) of Case III and (2.13) of Case IV it follows that for all sufficiently large values of r,

$$\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))}\geq$$

$$\frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}\,[l]_m-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}. \tag{2.23}$$

Now in view of (2.16) we obtain from (2.23) that

$$\lim_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))}=\infty. \tag{2.24}$$

Similarly combining (2.11) of Case III and (2.14) of Case V we get that

$$\lim_{r\to+\infty}\frac{\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))}=\infty. \tag{2.25}$$

Hence the third part of the theorem follows from (2.24) and (2.25).

Again combining (2.11) of Case III and (2.15) of Case VI we obtain that for all sufficiently large values of r,

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))}\geq\frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Now in view of (2.16) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(T_m^{-1}(T_{l\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}\left(T_{f\circ g}\left(\beta_2^{-1}(\log r)\right)\right)))))}=\infty.$$

This proves the fourth part of the theorem.

Thus the theorem follows. $\square$

Theorem 2.9. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < +\infty$, $\rho_{(\alpha_2,\beta_2)}[g] < +\infty$ and h satisfies the Property (A). If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then*

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}(T_{f\circ g}(r)))))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. In view of (2.2) it follows that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r). \tag{2.26}$$

Again in view of (2.1), we get that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}(T_{f\circ g}(r))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(M_g(r)).$$

Since $\alpha_2(\beta_1^{-1}(r)) \in L_1$, we obtain from above that for all sufficiently large values of r,

$$\begin{aligned} \alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))) &\leq (1+o(1))\alpha_2(M_g(r)) \\ \text{i.e., } \alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}(T_{f\circ g}(r))))) &\leq (1+o(1))(\rho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \end{aligned}$$

Now combining (2.26) and above we get that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}(T_{f\circ g}(r)))))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Hence the theorem follows. □

Theorem 2.10. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < +\infty$, $\lambda_{(\alpha_2,\beta_2)}[g] < +\infty$ and h satisfies the Property (A). If $\alpha_2(\beta_1^{-1}(r)) \in L_1$, then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(T_h^{-1}(T_{f\circ g}(r)))))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

The proof of Theorem 2.10 would run parallel to that of Theorem 2.9. We omit the details.

Theorem 2.11. *Let f be meromorphic function and g, h, k be any three entire functions such that $\rho_{(\alpha_1,\beta_1)}[f\circ g]_h < \infty$ and $\lambda_{(\alpha_3,\beta_3)}[g]_k > 0$. Then*

$$\lim_{r\to\infty}\frac{\left\{\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r))))\right\}^2}{\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(\log r))))\cdot\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(r))))} = 0.$$

Proof. For arbitrary positive ε we have that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r)))) \leq (\rho_{(\alpha_1,\beta_1)}\,[f\circ g]_h + \varepsilon)\log r. \tag{2.27}$$

Again for all sufficiently large values of r we get

$$\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(\log r)))) \geq (\lambda_{(\alpha_3,\beta_3)}\,[g]_k - \varepsilon)\log r. \tag{2.28}$$

Similarly for all sufficiently large values of r we have

$$\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(r)))) \geq (\lambda_{(\alpha_3,\beta_3)}\,[g]_k - \varepsilon) r. \tag{2.29}$$

From (2.27) and (2.28) we have that for all sufficiently large values of r,

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(\log r))))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}\,[f\circ g]_h + \varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}\,[g]_k - \varepsilon)\log r}.$$

As $\varepsilon(>0)$ is arbitrary we obtain from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(\log r))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}\,[f\circ g]_h}{\lambda_{(\alpha_3,\beta_3)}\,[g]_k}. \tag{2.30}$$

Again from (2.27) and (2.29) we get that for all sufficiently large values of r,

$$\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(r))))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}\,[f\circ g]_h + \varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}\,[g]_k - \varepsilon) r}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(T_k^{-1}(T_g(\beta_3^{-1}(r))))} = 0. \tag{2.31}$$

Thus the theorem follows from (2.30) and (2.31). $\square$

Theorem 2.12. *Let f be meromorphic function and g, h be any two entire functions such that $\rho_{(\alpha_1,\beta_1)}\,[f]_h < \infty$ and $\lambda_{(\alpha_3,\beta_3)}\,[f\circ g]_h = \infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha_3(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(T_h^{-1}(T_{f\circ g}(r))) \leq \Delta\cdot\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_3(r))))). \tag{2.32}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}\,[f]_h$, it follows that for all sufficiently large values of r,

$$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_3(r))))) \leq (\rho_{(\alpha_1,\beta_1)}\,[f]_h + \epsilon)\beta_3(r). \tag{2.33}$$

Thus from (2.32) and (2.33), we have that for a sequence of values of r tending to infinity,

$$\alpha_3(T_h^{-1}(T_{f\circ g}(r))) \leq \Delta(\rho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r)$$

$$i.e., \ \frac{\alpha_3(T_h^{-1}(T_{f\circ g}(r)))}{\beta_3(r)} \leq \frac{\Delta(\rho_{(\alpha_1,\beta_1)}[f]_h + \epsilon)\beta_3(r)}{\beta_3(r)}$$

$$i.e., \ \liminf_{r+\infty} \frac{\alpha_3(T_h^{-1}(T_{f\circ g}(r)))}{\beta_3(r)} = \lambda_{(\alpha_3,\beta_3)}[f\circ g]_h < \infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 2.13. Theorem 2.12 is also valid with "limit superior" instead of "limit" if $\lambda_{(\alpha_3,\beta_3)}[f\circ g]_h = \infty$ is replaced by $\rho_{(\alpha_3,\beta_3)}[f\circ g]_h = \infty$ and the other conditions remain the same.

Analogously one may also state the following theorem without its proof as it may be carried out in the line of Theorem 2.12.

Theorem 2.14. *Let f be meromorphic function and g, h be any two entire functions such that $\rho_{(\alpha_1,\beta_1)}[g]_h < \infty$ and $\rho_{(\alpha_3,\beta_3)}[f\circ g]_h = \infty$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha_3(T_h^{-1}(T_{f\circ g}(r)))}{\alpha_1(T_h^{-1}(T_g(\beta_1^{-1}(\beta_3(r)))))} = \infty.$$

Remark 2.15. Theorem 2.14 is also valid with "limit" instead of "limit superior" if $\rho_{(\alpha_3,\beta_3)}[f\circ g]_h = \infty$ is replaced by $\lambda_{(\alpha_3,\beta_3)}[f\circ g]_h = \infty$ and the other conditions remain the same.

3. Conclusion

Throughout this paper, we have generalized some results using the concept of generalized relative order (α,β) of entire and meromorphic functions.The technique used to define generalized relative order (α,β) is newly developed idea and this concept is very much significant. Defining new idea of relative order of growths in the complex plane, we have discussed some growth properties of entire and meromorphic functions.This technique may also be applied in the study of growth of solutions of complex differential equations with entire or meromorphic coefficients. These works will be very much helpful for the future researchers.

Acknowledgment

The authors are very much grateful to the reviewers for their valuable suggestions to bring the paper in its present form.

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