J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.2.155 ISSN(Online) 2287-6081
Volume 30, Number 2 (May 2023), Pages 155–167

CONDITIONAL FOURIER-FEYNMAN TRANSFORM AND CONDITIONAL CONVOLUTION PRODUCT ASSOCIATED WITH VECTOR-VALUED CONDITIONING FUNCTION

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Abstract. In this paper, we use a vector-valued conditioning function to define a conditional Fourier-Feynman transform (CFFT) and a conditional convolution product (CCP) on the Wiener space. We establish the existences of the CFFT and the CCP for bounded functionals which form a Banach algebra. We then provide fundamental relationships between the CFFTs and the CCPs.

1. Introduction

Let $(C_0[0,T], m_w)$ denote the Wiener space, where $C_0[0,T]$ is the space of real valued continuous functions x on $[0,T]$ such that $x(0)=0$, and m_w is the Wiener measure. In [4, 5, 9, 13], the study of the conditional Wiener and the conditional Feynman integrals given finite dimensional conditioning functions depending on time parameters were performed. The concepts of the CFFT and the CCP were introduced by Park and Skoug in [11]. The structure of the CFFT and the CCP are based on the Feynman integral. In [11], Park and Skoug studied certain relationships between CFFT, $T_q(F|X)$, and the CCP, $(F*G|X)_q$ for functionals F and G on $C_0[0,T]$ with the one-dimensional conditioning function $X : C_0[0,T] \to \mathbb{R}$ defined by $X(x) = \int_0^T h(s)dx(s)$ with a nonzeo function in $L_2[0,T]$, where the integral $\int_0^T h(s)dx(s)$ means a stochastic integral.

In this paper, we study fundamental relationships which exist between the CFFT and the CCP for functionals on the Wiener space $C_0[0,T]$. But we use a vector-valued conditioning function $X_n : C_0[0,T] \to \mathbb{R}^n$ defined by $X_n(x) = (\int_0^T e_j(s)x(s), \dots, \int_0^T e_n(s)x(s))$ where $\{e_1, \dots, e_n\}$ is an orthogonal set of functions in $L_2[0,T]$.

Received by the editors January 1, 2023. Accepted February 23, 2023.
2020 *Mathematics Subject Classification.* 28C20, 46G12, 60J65.
Key words and phrases. Wiener space, conditional Fourier-Feynman transform, conditional convolution product, Paley-Wiener-Zygmund stochastic integral.
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2. Preliminaries

In this section, we introduce the concepts of the CFFT and the CCP for functionals on the complete Wiener measure space $(C_0[0,T], \mathcal{W}(C_0[0,T]), m_w)$, where $\mathcal{W}(C_0[0,T])$ denotes the σ-field of all Wiener measurable subsets. The definitions are based on the concept of the conditional Wiener integral associated with a vector-valued conditioning function.

We denote the Wiener integral of a Wiener integrable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_0[0,T]} F(x) dm_w(x),$$

and for $u \in L_2[0,T]$ and $x \in C_0[0,T]$, we let $\langle u, x\rangle = \int_0^T u(t)dx(t)$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral [6, 7, 8]. It is well-known that for each $v \in L_2[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for m_w-a.e. $x \in C_0[0,T]$ and is a Gaussian random variable with mean 0 and variance $\|v\|_2^2$ as a functional of $x \in C_0[0,T]$. If $\{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal set of functions in $L_2[0,T]$, then the random variables, $\{\langle \alpha_j, x\rangle\}_{j=1}^n$, are independent.

Let X be an $\mathbb{R}^n$-valued measurable function and let Y be a $\mathbb{C}$-valued integrable function on $(C_0[0,T], \mathcal{W}(C_0[0,T]), m_w)$. Let $\mathcal{F}(X)$ denote the σ-field generated by X. Then by the definition, the conditional expectation of Y given $\mathcal{F}(X)$, written $E(Y|X)$, is any real valued $\mathcal{F}(X)-$measurable function on $C_0[0,T]$ such that

$$\int_A Y(x) dm_w(x) = \int_A E(Y|X)(x) dm_w(x) \quad \text{for} \quad A \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_X)$ such that $E(Y|X) = \psi \circ X$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-field of Borel subsets in $\mathbb{R}^n$ and P_X is the probability distribution of X defined by $P_X(U) = m_w(X^{-1}(U))$ for $U \in \mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi}), \;\; \vec{\xi} \in \mathbb{R}^n$ is unique up to Borel null sets in $\mathbb{R}^n$. Following Tucker [12] and Yeh [13], the function $\psi(\vec{\xi})$, written $E(Y|X = \vec{\xi})$, is called the *conditional Wiener integral* of Y given X.

Let $\mathcal{G} = \{e_1, \ldots, e_n\}$ be an orthonormal set of functions in $L_2[0,T]$. For each $j \in \{1, \ldots, n\}$, let $\gamma_j(x) = \langle e_j, x\rangle$, and let $\beta_j(t) = \int_0^t e_j(s)ds$ for $t \in [0,T]$. Then the stochastic PWZ integrals $\{\gamma_1(x), \ldots, \gamma_n(x)\}$ form a set of independent standard Gaussian random variables on $C_0[0,T]$ with $E_x[x(t)\gamma_j(x)] = \beta_j(t)$ for all $j \in \{1, \ldots, n\}$.

Given an orthonormal set $\mathcal{G} = \{e_1, \ldots, e_n\}$ of functions in $L_2[0,T]$, let $X_{\mathcal{G}} : C_0[0,T] \longrightarrow \mathbb{R}^n$ be defined by

$$X_{\mathcal{G}}(x) = (\langle e_j, x\rangle, \ldots, \langle e_n, x\rangle) = (\gamma_1(x), \ldots, \gamma_n(x)). \tag{2.1}$$

Define a projection map $\mathcal{P}_{\mathcal{G}}$ from $L_2[0,T]$ into Span$\mathcal{G}$ by

$$\mathcal{P}_{\mathcal{G}} v = \sum_{j=1}^{n} (v, e_j)_2 e_j \in \mathrm{Span}\mathcal{G}$$

where $(\cdot,\cdot)_2$ denotes the inner product on the Hilbert space $L_2[0,T]$.

For each $x \in C_0[0,T]$ and $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, let

$$x_{\mathcal{G}} = \langle \mathcal{P}_{\mathcal{G}} I_{[0,t]}, x\rangle = \sum_{j=1}^{n} \gamma_j(x)\beta_j \quad \text{and} \quad \vec{\xi}_{\mathcal{G}} = \sum_{j=1}^{n} \xi_j (e_j, I_{[0,t]})_2 = \sum_{j=1}^{n} \xi_j \beta_j,$$

where $I_{[0,t]}$ denotes the indicator function of the interval $[0,t]$.

In [10], Park and Skoug proved the facts that the process $\{x(t) - x_{\mathcal{G}}(t), 0 \le t \le T\}$ and the Gaussian random variable $\gamma_j(x)$ are stochastically independent for each $j \in \{1, \ldots, n\}$, and that the processes $\{x(t) - x_{\mathcal{G}}(t), 0 \le t \le T\}$ and $\{x_{\mathcal{G}}(t), 0 \le t \le T\}$ are also stochastically independent. Using these basic results, Park and Skoug established the following evaluation formula to express conditional Wiener integrals in terms of ordinary Wiener integrals.

Theorem 2.1 ([10])**.** *Let $F \in L_1(C_0[0,T])$. Then*

$$E(F|X_{\mathcal{G}} = \vec{\xi}) = E_x\bigg[F\bigg(x - \sum_{j=1}^{n} \gamma_j(x)\beta_j + \sum_{j=1}^{n} \xi_j\beta_j\bigg)\bigg] \tag{2.2}$$

for a.e. $\vec{\xi} \in \mathbb{R}^n$.

3. Conditional Fourier-Feynman Transform and Conditional Convolution Product given $\mathbb{R}^n$-valued Conditioning Function

In order to define the CFFT and the CCP, we need the concept of the scale-invariant measurability on the Wiener space $C_0[0,T]$. A subset B of $C_0[0,T]$ is called a *scale-invariant measurable* (SIM) *set* if $\rho B \in \mathcal{W}(C_0[0,T])$ for all $\rho > 0$, and an SIM set N is called a *scale-invariant null set* if $m_w(\rho N) = 0$ for all $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold *scale-invariant almost everywhere* (SI-a.e.). A functional F is said to be SIM *provided F*

is defined on an SIM set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_0[0,T])$-measurable for every $\rho>0$. For more detailed studies of the scale-invariant measurability, see [2].

Let $\mathbb{C}_+=\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>0\}$ and let $\widetilde{\mathbb{C}}_+=\{\lambda\in\mathbb{C}\setminus\{0\}:\mathrm{Re}(\lambda)\geq 0\}$. Let $X_{\mathcal{G}}:C_0[0,T]\to\mathbb{R}^n$ be given by (2.1) and let F be a $\mathbb{C}$-valued SIM functional such that the Wiener integral $E_x[F(\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda>0$. For $\lambda>0$ and $\vec{\xi}$ in $\mathbb{R}^n$, let $J_F(\lambda;\vec{\xi})=E(F(\lambda^{-1/2}\,\cdot\,)|X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot\,)=\vec{\xi})$ denote the conditional Wiener integral of $F(\lambda^{-1/2}\,\cdot\,)$ given $X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot\,)$. If for a.e. $\vec{\xi}\in\mathbb{R}^n$, there exists a function $J_F^*(\lambda;\vec{\xi})$, analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda;\vec{\xi})=J_F(\lambda;\vec{\xi})$ for all $\lambda>0$, then $J_F^*(\lambda;\,\cdot\,)$ is defined to be the conditional analytic Wiener integral of F over $C_0[0,T]$ given $X_{\mathcal{G}}$ with parameter λ. For $\lambda\in\mathbb{C}_+$, we write $E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}}=\vec{\xi})=J_F^*(\lambda;\vec{\xi})$. If for fixed real $q\in\mathbb{R}\setminus\{0\}$, the limit

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}}=\vec{\xi})$$

exists for a.e. $\vec{\xi}\in\mathbb{R}^n$, then we will denote the value of this limit by $E^{\mathrm{anf}_q}(F|X_{\mathcal{G}}=\vec{\xi})$, and we call it the conditional analytic Feynman integral of F over $C_0[0,T]$ given $X_{\mathcal{G}}$ with parameter q.

Let F be a $\mathbb{C}$-valued SIM functional on $C_0[0,T]$ such that the Wiener integral $E[F(y+\lambda^{-1/2}\,\cdot\,)]\equiv E_x[F(y+\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda>0$. Then one can easily see from (2.2) that for all $\lambda>0$,

(3.1)
$$E(F(\lambda^{-1/2}\,\cdot\,)|X_{\mathcal{G}}(\lambda^{-1/2}\,\cdot\,)=\vec{\xi})=E_x\bigg[F\bigg(\lambda^{-1/2}x-\lambda^{-1/2}\sum_{j=1}^n\gamma_j(x)\beta_j+\sum_{j=1}^n\xi_j\beta_j\bigg)\bigg].$$

Thus we have that

$$E^{\mathrm{anw}_\lambda}(F|X_{\mathcal{G}}=\vec{\xi})=E_x^{\mathrm{anw}_\lambda}\bigg[F\bigg(x-\sum_{j=1}^n\gamma_j(x)\beta_j+\sum_{j=1}^n\xi_j\beta_j\bigg)\bigg]$$

and

$$E^{\mathrm{anf}_q}(F|X_{\mathcal{G}}=\vec{\xi})=E_x^{\mathrm{anf}_q}\bigg[F\bigg(x-\sum_{j=1}^n\gamma_j(x)\beta_j+\sum_{j=1}^n\xi_j\beta_j\bigg)\bigg] \tag{3.2}$$

where $E_x^{\mathrm{anw}_\lambda}[F(x)]$ and $E_x^{\mathrm{anf}_q}[F(x)]$ denote the analytic Wiener and the analytic Feynman integrals of functionals F on $C_0[0,T]$, respectively, see [1, 5].

We are now ready to state the definitions of the CFFT and the CCP of functionals on $C_0[0,T]$.

Definition 3.1. Let $F : C_0[0,T] \to \mathbb{C}$ be an SIM functional on $C_0[0,T]$ such that the Wiener integral $E[F(y+\lambda^{-1/2}\,\cdot\,)]$ exists as a finite number for all $\lambda > 0$. Let $X_{\mathcal{G}} : C_0[0,T] \to \mathbb{R}^n$ be given by (2.1). For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let $T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi})$ denote the conditional analytic Wiener integral of $F(y+\cdot)$ given $X_{\mathcal{G}}$, that is to say,

$$\begin{aligned} T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi}) &= E^{\mathrm{anw}_\lambda}(F(y+\,\cdot\,)|X_{\mathcal{G}} = \vec{\xi}) \\ &= E_x^{\mathrm{anw}_\lambda}\bigg[F\bigg(y + x - \sum_{j=1}^n \gamma_j(x)\beta_j + \sum_{j=1}^n \xi_j\beta_j\bigg)\bigg]. \end{aligned}$$

We define the L_1 analytic CFFT $T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi})$ of F given $X_{\mathcal{G}}$ by the formula

$$T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi}) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi}).$$

We also define the CCP of SIM functionals F and G given $X_{\mathcal{G}}$ by the formula

$$\begin{aligned} &[(F * G)_\lambda|X_{\mathcal{G}}](y,\vec{\xi}) \\ &= \begin{cases} E^{\mathrm{anw}_\lambda}\bigg(F\bigg(\dfrac{y+\cdot}{\sqrt{2}}\bigg)G\bigg(\dfrac{y-\cdot}{\sqrt{2}}\bigg)\bigg|X_{\mathcal{G}} = \vec{\xi}\bigg), & \lambda \in \mathbb{C}_+ \\ E^{\mathrm{anf}_q}\bigg(F\bigg(\dfrac{y+\cdot}{\sqrt{2}}\bigg)G\bigg(\dfrac{y-\cdot}{\sqrt{2}}\bigg)\bigg|X_{\mathcal{G}} = \vec{\xi}\bigg), & \lambda = -iq,\ q \in \mathbb{R}\setminus\{0\}. \end{cases} \end{aligned}$$

4. CFFT and CCP for Functionals in a Banach Algebra

In this section, we will establish the existences of the CFFT and the CCP for bounded functionals in the Cameron and Storvick's Banach algebra $\mathcal{S}(L_2[0,T])$.

The Banach algebra $\mathcal{S}(L_2[0,T])$ consists of functionals on $C_0[0,T]$ having the form

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x\rangle\}df(u) \tag{4.1}$$

for SI-a.e. $x \in C_0[0,T]$, where the associated measure f is an element of the Banach algebra $\mathcal{M}(L_2[0,T])$, the space of $\mathbb{C}$-valued countably additive (and hence finite) Borel measures on $L_2[0,T]$. More precisely, since we shall identify functionals which coincide SI-a.e. on $C_0[0,T]$, the space $\mathcal{S}(L_2[0,T])$ can be regarded as the space of all s-equivalence classes of functionals of the form (4.1). It was also shown in [1] that the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L_2[0,T])$ is a Banach algebra with the norm

$$\|F\| \equiv \|f\| = \int_{L_2[0,T]} d|f|(u).$$

In particular, it was shown in [3] that the Banach algebra $\mathcal{S}(L_2[0,T])$ contains many functionals of interest in Feynman integration theory. For a more detailed study of the Banach algebra $\mathcal{S}(L_2[0,T])$, see [1, 3].

Using the fact that the PWZ stochastic integral $\langle w, x\rangle$ of a function w in $L_2[0,T]$ is a Gaussian random variable, as a functional of x, with mean zero and variance $\|w\|_2^2$, and the change of variable theorem, we have the following lemma.

Lemma 4.1. *For each $w \in L_2[0,T]$ and any $\rho > 0$,*

$$E_x[\exp\{i\rho\langle w, x\rangle\}] = \exp\big\{-\rho^2\|w\|_2^2\big\}. \tag{4.2}$$

From the bilinearity of the PWZ stochastic integral $\langle\cdot,\cdot\rangle$ and equation (4.2) with w replaced with $w - \sum_{j=1}^n (w, e_j)_2 e_j$, we have the following lemma.

Lemma 4.2. *Let $\{e_1, \dots, e_n\}$ be an orthonormal set of functions in $L_2[0,T]$. Then for each $w \in L_2[0,T]$ and any $\rho > 0$,*

$$E_x\bigg[\exp\bigg\{i\rho\bigg\langle w, x - \sum_{j=1}^n \gamma_j(x)\beta_j\bigg\rangle\bigg\}\bigg] = \exp\bigg\{-\frac{\rho^2}{2}\bigg[\|w\|^2 - \sum_{j=1}^n (w, e_j)_2^2\bigg]\bigg\}. \tag{4.3}$$

In particular, for any $q \in \mathbb{R}\setminus\{0\}$ and any $\rho > 0$,

$$E_x^{\mathrm{an}f_q}\bigg[\exp\bigg\{i\rho\bigg\langle w, x - \sum_{j=1}^n \gamma_j(x)\beta_j\bigg\rangle\bigg\}\bigg] = \exp\bigg\{-\frac{i\rho^2}{2q}\bigg[\|w\|^2 - \sum_{j=1}^n (w, e_j)_2^2\bigg]\bigg\}. \tag{4.4}$$

In our first theorem of this section, we establish the existences of the CFFT $T_q^{(1)}(F|X_{\mathcal{G}})$ of functionals F in the Banach algebra $\mathcal{S}(L_2[0,T])$.

Theorem 4.3. *Let $F \in \mathcal{S}(L_2[0,T])$ be given by equation (4.1), and let $X_{\mathcal{G}}$ be given by equation (2.1). Then for a.e. $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned} &T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi}) \\ &= \int_{L_2[0,T]} \exp\bigg\{i\langle u, y\rangle - \frac{i}{2q}\bigg[\|u\|_2^2 - \sum_{j=1}^n (u, e_j)_2^2\bigg] + i\sum_{j=1}^n \xi_j (u, e_j)_2\bigg\} df(u) \end{aligned} \tag{4.5}$$

for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$.

Proof. Using (4.1), (3.1) with F replaced with $F(y+\cdot)$, the Fubini theorem, (4.3) with w and ρ replaced with u and $\lambda^{-1/2}$, it follows that for $(\lambda,\vec{\xi})\in(0,+\infty)\times\mathbb{R}^n$,

$$\begin{aligned}
J_{F(y+\cdot)}(\lambda;\vec{\xi}) &\equiv E\big(F(y+\lambda^{-1/2}\cdot)\big|X_{\mathcal{G}}(\lambda^{-1/2}\cdot)=\vec{\xi}\big)\\
&= \int_{L_2[0,T]} \exp\bigg\{i\langle u,y\rangle + i\bigg\langle u,\sum_{j=1}^n \xi_j\beta_j\bigg\rangle\bigg\}\\
&\qquad\times E_x\bigg[\exp\bigg\{i\lambda^{-1/2}\bigg\langle u, x-\sum_{j=1}^n\gamma_j(x)\beta_j\bigg\rangle\bigg\}\bigg]df(u)\\
&= \int_{L_2[0,T]} \exp\bigg\{i\langle u,y\rangle - \frac{1}{2\lambda}\bigg[\|u\|_2^2-\sum_{j=1}^n(u,e_j)_2^2\bigg] + i\sum_{j=1}^n\xi_j(u,e_j)_2\bigg\}df(u).
\end{aligned}$$

Let

$$\begin{aligned}
&J^*_{F(y+\cdot)}(\lambda;\vec{\xi})\\
&= \int_{L_2[0,T]} \exp\bigg\{i\langle u,y\rangle - \frac{1}{2\lambda}\bigg[\|u\|_2^2-\sum_{j=1}^n(u,e_j)_2^2\bigg] + i\sum_{j=1}^n\xi_j(u,e_j)_2\bigg\}df(u)
\end{aligned}\tag{4.6}$$

for $\lambda\in\mathbb{C}_+$. Since $\mathrm{Re}(\lambda)>0$ for all $\lambda\in\mathbb{C}_+$, it follows that

$$\big|J^*_{F(y+\cdot)}(\lambda;\vec{\xi})\big| \le \int_{L_2[0,T]} d|f|(u) = \|f\| < +\infty. \tag{4.7}$$

Hence, applying the dominated convergence theorem, we see that $J^*_F(\lambda;\vec{\xi})$ is a continuous function of $\lambda\in\widetilde{\mathbb{C}}_+$. Also, applying the Morera theorem, one can see that $J^*_{F(y+\cdot)}(\lambda;\vec{\xi})$ is analytic on $\mathbb{C}_+$. Therefore, the conditional analytic Wiener integral $T_\lambda(F|X_{\mathcal{G}})(y,\vec{\xi}) = E^{\mathrm{anw}_\lambda}(F(y+\cdot)|X_{\mathcal{G}}=\vec{\xi}) = J^*_{F(y+\cdot)}(\lambda;\vec{\xi})$ exists and is given by the right hand side of (4.6). Finally, by the dominated convergence theorem (the use of which is justified by (4.7)), the L_1 analytic CFFT $T_q^{(1)}(F|X_{\mathcal{G}}=\vec{\xi})$ of F exists and is given by the formula (4.5). □

From the definition of the conditional Feynman integral and the L_1 analytic CFFT, it follows that $T_q^{(1)}(F|X_{\mathcal{G}})(0,\vec{\xi}) = E^{\mathrm{anf}_q}(F|X_{\mathcal{G}}=\vec{\xi})$. We thus have the following corollary.

Corollary 4.4. *Let F and $X_{\mathcal{G}}$ be as in Theorem 4.3. Then the conditional Feynman integral $E^{\mathrm{anf}_q}(F|X_{\mathcal{G}}=\vec{\xi})$ of F exists for all $q\in\mathbb{R}\setminus\{0\}$ and a.e. $\vec{\xi}\in\mathbb{R}^n$, and is* *given by the formula*

$$E^{\mathrm{an}f_q}(F|X_{\mathcal{G}}=\vec{\xi})$$
$$=\int_{L_2[0,T]}\exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{j=1}^{n}(u,e_j)_2^2\bigg]+i\sum_{j=1}^{n}\xi_j(u,e_j)_2\bigg\}df(u).$$

Remark 4.5. Given a functional F in $\mathcal{S}(L_2[0,T])$ with the corresponding measure $f\in\mathcal{M}(L_2[0,T])$, and given a nonzero real number q and a vector $\vec{\xi}\in\mathbb{R}^n$, define a set function $f_{q,\vec{\xi}}:\mathcal{B}(L_2[0,T])\to\mathbb{C}$ by the formula

$$f_{q,\vec{\xi}}(U)=\int_{U}\exp\bigg\{-\frac{i}{2q}\bigg[\|u\|_2^2-\sum_{j=1}^{n}(u,e_j)_2^2\bigg]+i\sum_{j=1}^{n}\xi_j(u,e_j)_2\bigg\}df(u) \tag{4.8}$$

for each U in $\mathcal{B}(L_2[0,T])$, the Borel σ-field on $L_2[0,T]$. Then $f_{q,\vec{\xi}}$ is clearly a member of $\mathcal{M}(L_2[0,T])$ and $\|f_{q,\vec{\xi}}\|=\|f\|$ for any $q\in\mathbb{R}\setminus\{0\}$ and $\vec{\xi}\in\mathbb{R}^n$. Then equation (4.5) can be written by

$$T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi})=\int_{L_2[0,T]}\exp\{i\langle u,y\rangle\}df_{q,\vec{\xi}}(u) \tag{4.9}$$

for SI-a.e. $y\in C_0[0,T]$, and so the L_1 analytic CFFT $T_q^{(1)}(F|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi})$ of F is an element of $\mathcal{S}(L_2[0,T])$ for each $\vec{\xi}\in\mathbb{R}^n$.

In our next theorem, we also establish the existence the CCP of functionals F and G in $\mathcal{S}(L_2[0,T])$.

Theorem 4.6. *Let F and G be the functionals in $\mathcal{S}(L_2[0,T])$ with corresponding Borel measures f and g, respectively, in $\mathcal{M}(L_2[0,T])$, and let $X_{\mathcal{G}}$ be given by equation (2.1). Then for a.e. $\vec{\xi}\in\mathbb{R}^n$,*

$$\begin{aligned}[(F*G)_q|X_{\mathcal{G}}](y,\vec{\xi})=\int_{L_2[0,T]}\int_{L_2[0,T]}\exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle\\ -\frac{i}{4q}\bigg[\|u-v\|_2^2-\sum_{j=1}^{n}(u-v,e_j)_2^2\bigg]+\frac{i}{\sqrt{2}}\sum_{j=1}^{n}\xi_j(u-v,e_j)_2\bigg\}df(u)dg(v)\end{aligned} \tag{4.10}$$

for all $q\in\mathbb{R}\setminus\{0\}$ and SI-a.e. $y\in C_0[0,T]$.

Proof. By using similar methods as those in the proof of Theorem 4.3, it follows equation (4.10) immediately by the definition of the CCP. □

Remark 4.7. Given two functionals F and G in $\mathcal{S}(L_2[0,T])$ with the corresponding measures f and g in $\mathcal{M}(L_2[0,T])$, and given a nonzero real q and a vector $\vec{\xi}\in\mathbb{R}^n$, define a set function $\varphi_{q,\vec{\xi}} : \mathcal{B}(L_2[0,T] \times L_2[0,T]) \to \mathbb{C}$ by the formula

$$\begin{aligned}\varphi_{q,\vec{\xi}}(V) = \iint_V \exp\bigg\{ -\frac{i}{4q}\bigg[\|u-v\|_2^2 - \sum_{j=1}^n (u-v,e_j)_2^2\bigg] \\ + \frac{i}{\sqrt{2}}\sum_{j=1}^n \xi_j (u-v,e_j)_2\bigg\} df(u)dg(v)\end{aligned} \tag{4.11}$$

for each V in $\mathcal{B}(L_2[0,T] \times L_2[0,T])$, the Borel σ-field on $L_2[0,T] \times L_2[0,T]$. Then $\varphi_{q,\vec{\xi}}$ is a complex measure on $\mathcal{B}(L_2[0,T] \times L_2[0,T])$. Define a function $\phi : L_2[0,T] \times L_2[0,T] \to L_2[0,T]$ by $\phi(u,v) = (u+v)/\sqrt{2}$. Then ϕ is a continuous function, and so it is $\mathcal{B}(L_2[0,T] \times L_2[0,T])$-measurable. Thus the set function $\varphi_{q,\vec{\xi}} \circ \phi^{-1} : L_2[0,T] \to \mathbb{C}$ is in $\mathcal{M}(L_2[0,T])$ obviously. Under these setting, equation (4.10) can be rewritten by

$$[(F * G)_q | X_{\mathcal{G}}](y, \vec{\xi}) = \int_{L_2[0,T]} \exp\{i\langle w, y\rangle\} d\varphi_{q,\vec{\xi}} \circ \phi^{-1}(w)$$

for SI-a.e. $y \in C_0[0,T]$. Thus the CCP $[(F * G)_q | X_{\mathcal{G}}](\,\cdot\,, \vec{\xi})$ of F and G is an element of $\mathcal{S}(L_2[0,T])$ for each $\vec{\xi} \in \mathbb{R}^n$.

5. Relationships between the CFFT and the CCP

In this section, we establish basic relationships between the CFFTs and the CCPs. The following theorem is one of our main assertions; namely that the CFFT of the CCP is the product of the CFFTs.

Theorem 5.1. *Let F, G, and $X_{\mathcal{G}}$ be as in Theorem 4.6. Then for all $q \in \mathbb{R} \setminus \{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}&T_q^{(1)}\Big([(F * G)_q | X_{\mathcal{G}}](\,\cdot\,, \vec{\xi}^{(1)})\Big| X_{\mathcal{G}}\Big)(y, \vec{\xi}^{(2)}) \\ &= T_q^{(1)}(F|X_{\mathcal{G}})\bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)} + \vec{\xi}^{(1)}}{\sqrt{2}}\bigg) T_q^{(1)}(G|X_{\mathcal{G}})\bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)} - \vec{\xi}^{(1)}}{\sqrt{2}}\bigg).\end{aligned}$$

Proof. Using (4.9) with F and f replaced with $[(F * G)_q | X_{\mathcal{G}}]$ and $\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1}$ respectively, (4.8) with f replaced with $\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1}$, (4.11), the Fubini theorem, and (4.5) together with simple calculations, it follows that

$$\begin{aligned}&T_q^{(1)}\Big(\big[(F * G)_q \big| X_{\mathcal{G}}\big](\,\cdot\,, \vec{\xi}^{(1)})\Big| X_{\mathcal{G}}\Big)(y, \vec{\xi}^{(2)}) \\ &= \int_{L_2[0,T]} \exp\{i\langle w, y\rangle\} d(\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1})_{q,\vec{\xi}^{(2)}}(w)\end{aligned}$$

$$
\begin{aligned}
&= \int_{L_2[0,T]} \exp\bigg\{ i\Big\langle u, \frac{y}{\sqrt{2}}\Big\rangle \\
&\quad - \frac{i}{2q}\bigg[\|u\|_2^2 - \sum_{j=1}^{n}(u,e_j)_2^2\bigg] + i\sum_{j=1}^{n}\frac{\xi_j^{(2)}+\xi_j^{(1)}}{\sqrt{2}}(u,e_j)_2\bigg\} df(u) \\
&\quad \times \int_{L_2[0,T]} \exp\bigg\{ i\Big\langle v, \frac{y}{\sqrt{2}}\Big\rangle \\
&\quad - \frac{i}{2q}\bigg[\|v\|_2^2 - \sum_{j=1}^{n}(v,e_j)_2^2\bigg] + i\sum_{j=1}^{n}\frac{\xi_j^{(2)}-\xi_j^{(1)}}{\sqrt{2}}(v,e_j)_2\bigg\} dg(v) \\
&= T_q^{(1)}(F|X_{\mathcal{G}})\bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}+\vec{\xi}^{(1)}}{\sqrt{2}}\bigg) T_q^{(1)}(G|X_{\mathcal{G}})\bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}-\vec{\xi}^{(1)}}{\sqrt{2}}\bigg)
\end{aligned}
$$

as desired. □

In order to provide our second main assertion of this paper, we need the following lemma.

Lemma 5.2. *Let F, G, and $X_{\mathcal{G}}$ be as in Theorem 4.6. Then for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$
\begin{aligned}
&\bigg[\Big(T_q^{(1)}(F|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi}^{(1)}) * T_q^{(1)}(G|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi}^{(2)})\Big)_{-q}\bigg|X_{\mathcal{G}}\bigg](y,\vec{\xi}^{(3)}) \\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}\bigg[\|u+v\|_2^2 - \sum_{j=1}^{n}(u+v)_2^2\bigg] \\
&\quad + i\sum_{j=1}^{n}\bigg(\xi_j^{(1)} + \frac{\xi_j^{(3)}}{\sqrt{2}}\bigg)(u,e_j)_2 + i\sum_{j=1}^{n}\bigg(\xi_j^{(2)} - \frac{\xi_j^{(3)}}{\sqrt{2}}\bigg)(v,e_j)_2\bigg\} df(u)dg(v)
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_{\mathcal{G}}\bigg)(y,\vec{\xi}) \\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}\bigg[\|u+v\|_2^2 - \sum_{j=1}^{n}(u+v)_2^2\bigg] \\
&\qquad + i\sum_{j=1}^{n}\xi_j(u+v,e_j)_2\bigg\} df(u)dg(v).
\end{aligned}
\tag{5.2}
$$

Proof. In view of Remark 4.5, we observe that

$$
T_q^{(1)}(F|X_{\mathcal{G}})(y,\vec{\xi}^{(1)}) = \int_{L_2[0,T]} \exp\{i\langle u,y\rangle\} df_{q,\vec{\xi}^{(1)}}(u)
$$

and

$$T_q^{(1)}(G|X_{\mathcal{G}})(y,\vec{\xi}^{(2)}) = \int_{L_2[0,T]} \exp\{i\langle v,y\rangle\}dg_{q,\vec{\xi}^{(2)}}(v)$$

where $f_{q,\vec{\xi}^{(1)}}$ is the complex measure in $\mathcal{M}(L_2[0,T])$ given by (4.8) with $\vec{\xi}$ replaced with $\vec{\xi}^{(1)}$, and $g_{q,\vec{\xi}^{(2)}}$ is the complex measure in $\mathcal{M}(L_2[0,T])$ given by the formula:

$$g_{q,\vec{\xi}^{(2)}}(U) = \int_U \exp\bigg\{-\frac{i}{2q}\bigg[\|v\|_2^2 - \sum_{j=1}^n (v,e_j)_2^2\bigg] + i\sum_{j=1}^n \xi_j(v,e_j)_2\bigg\}dg(v)$$

for each $U \in \mathcal{B}(L_2[0,T])$. Then using (4.10) with F, G, $\vec{\xi}$, f and g replaced with $T_q^{(1)}(F|X_{\mathcal{G}})(\cdot,\vec{\xi}^{(1)})$, $T_q^{(1)}(G|X_{\mathcal{G}})(\cdot,\vec{\xi}^{(2)})$, $\vec{\xi}^{(3)}$, $f_{q,\vec{\xi}^{(1)}}$ and $g_{q,\vec{\xi}^{(2)}}$ respectively, and (4.9) with $\vec{\xi}$ replaced with $\vec{\xi}^{(1)}$, it follows equation (5.1) immediately.

Next, using the definition of the L_1 analytic CFFT, (3.2) with F replaced with $F((y+\cdot)/\sqrt{2})G((y+\cdot)\sqrt{2})$, and the Fubini theorem, it follows that

$$\begin{aligned}
&T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_{\mathcal{G}}\bigg)(y,\vec{\xi})\\
&= E^{\mathrm{an}f_q}\bigg[F\bigg(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\bigg[x - \sum_{j=1}^n \gamma_j(x)\beta_j + \sum_{j=1}^n \xi_j\beta_j\bigg]\bigg)\\
&\qquad \times G\bigg(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\bigg[x - \sum_{j=1}^n \gamma_j(x)\beta_j + \sum_{j=1}^n \xi_j\beta_j\bigg]\bigg)\bigg]\\
&= \int_{L_2[0,T]}\int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle + \bigg\langle u+v, \sum_{j=1}^n \xi_j\beta_j\bigg\rangle\bigg\}\\
&\qquad \times E^{\mathrm{an}f_q}\bigg[\exp\bigg\{\frac{i}{\sqrt{2}}\bigg\langle u+v, x - \sum_{j=1}^n \gamma_j(x)\beta_j\bigg\rangle\bigg\}df(u)dg(v).
\end{aligned} \tag{5.3}$$

Applying (4.4) with w and ρ replaced with $u+v$ and $1/\sqrt{2}$ in the last expression of (5.3), it follows equation (5.2) as desired. □

Let $(\mathbb{R}^n)^4$ denote the product of four copies of $\mathbb{R}^n$. A close examination of the right-hand sides of (5.1) and (5.2) shows that they are equal if $(\vec{\xi},\vec{\xi}^{(1)},\vec{\xi}^{(2)},\vec{\xi}^{(3)}) \in (\mathbb{R}^n)^4$ is in the solution set of the system

$$\begin{cases} \vec{\xi} - \sqrt{2}\vec{\xi}^{(1)} - \vec{\xi}^{(3)} = \vec{0} \\ \vec{\xi} - \sqrt{2}\vec{\xi}^{(2)} + \vec{\xi}^{(3)} = \vec{0}. \end{cases} \tag{5.4}$$

Theorem 5.3. *Let F, G, and $X_{\mathcal{G}}$ be as in Theorem 4.6 and let $(\vec{\xi}, \vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \vec{\xi}^{(3)})$ satisfy the system* (5.4)*. Then for all $q \in \mathbb{R} \setminus \{0\}$ and SI-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}&\Big(\big[T_q(F|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi}^{(1)}) * T_q(G|X_{\mathcal{G}})(\,\cdot\,,\vec{\xi}^{(2)})\big]_{-q}\Big|X_{\mathcal{G}}\Big)(y,\vec{\xi}^{(3)})\\ &= T_q^{(1)}\bigg(F\Big(\frac{\cdot}{\sqrt{2}}\Big)G\Big(\frac{\cdot}{\sqrt{2}}\Big)\bigg|X_{\mathcal{G}}\bigg)(y,\vec{\xi}).\end{aligned}$$

Acknowledgement

The authors would like to express their gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper.

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