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ON THE ROBUSTNESS OF CONTINUOUS TRAJECTORIES OF THE NONLINEAR CONTROL SYSTEM DESCRIBED BY AN INTEGRAL EQUATION

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Abstract. In this paper the control system described by Urysohn type integral equation is studied. It is assumed that control functions are integrally constrained. The trajectory of the system is defined as multivariable continuous function which satisfies the system's equation everywhere. It is shown that the set of trajectories is Lipschitz continuous with respect to the parameter which characterizes the bound of the control resource. An upper estimation for the diameter of the set of trajectories is obtained. The robustness of the trajectories with respect to the fast consumption of the remaining control resource is discussed. It is proved that every trajectory can be approximated by the trajectory obtained by full consumption of the control resource.

1. Introduction

The control system described by Urysohn type integral equation

$$x(\omega) = f(\omega, x(\omega)) + \int_E F(\omega, s, x(s), u(s))ds \tag{1.1}$$

is considered, where $\omega \in \Omega$, $x(\omega) \in \mathbb{R}^n$ is the state vector, $u(s) \in \mathbb{R}^m$ is the control vector, $\Omega \subset \mathbb{R}^k$, $E \subset \mathbb{R}^k$ are compact sets, $E \subseteq \Omega$.

Note that integral equations are adequate tool for description of the behavior of different processes arising in theory and applications. One of the outstanding scientist of the XX century W. Heisenberg in his well known book "Physics and Philosophy" underlines the importance of the integral equations by the following words: "The final equation of motion for matter will probably be some quantized nonlinear wave equation... This wave equation will probably be equivalent to rather

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complicated sets of integral equations..." (see, [4], p.68). The integral equations are often used for solution's concept extension of initial and boundary value problems for ordinary and partial differential equations. Separately, let us express that the theory of the integral equations is considered one of the origin of the contemporary functional analysis (see, [5], chapter 1, p.2).

For given $p > 1$ and $r \geq 0$ we denote

$$U_{p,r} = \Big\{ u(\cdot) \in L_p\big(E; \mathbb{R}^m\big) : \|u(\cdot)\|_p \leq r \Big\} \tag{1.2}$$

which is called the set of admissible control functions and every $u(\cdot) \in U_{p,r}$ is said to be an admissible control function, where $L_p\,(E; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) : E \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < \infty$, $\|u(\cdot)\|_p = \left(\int_E \|u(s)\|^p ds \right)^{1/p}$, $\|\cdot\|$ denotes the Euclidean norm.

It is obvious that the set of admissible control functions $U_{p,r}$ is the closed ball with radius r and centered at the origin in the space $L_p\,(E; \mathbb{R}^m)$.

In general, integral constraint on the control functions is inevitable, if the control resource of the system is exhausted by consumption, such as energy, fuel, finance, etc. (see, e.g., [1, 3, 8, 10])). For example, the motion of the flying object with rapidly changing mass is described by a control system with integral constraint on the control functions (see, e.g., [1, 10]). Different topological properties and approximate construction methods of the set of trajectories of the control systems with integral constraints on the control functions are discussed in [6, 7, 8] (see the references also therein).

It is assumed that the functions given in system (1.1) satisfy the following conditions:

1.A. The functions $f(\cdot) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $F(\cdot) : \Omega \times E \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous;

1.B. There exist $\gamma_0 \in [0, 1)$, $\gamma_1 \geq 0$, $\kappa_1 \geq 0$, $\gamma_2 \geq 0$, $\kappa_2 \geq 0$, $\gamma_3 \geq 0$ and $\kappa_3 \geq 0$ such that

$$\|f(\omega, x_1) - f(\omega, x_2)\| \leq \gamma_0 \|x_1 - x_2\|$$

is satisfied for every $(\omega, x_1) \in \Omega \times \mathbb{R}^n$ and $(\omega, x_2) \in \Omega \times \mathbb{R}^n$ and

$$\begin{aligned} \|F(\omega_1, s, x_1, u_1) - F(\omega_2, s, x_2, u_2)\| &\leq [\gamma_1 + \kappa_1(\|u_1\| + \|u_2\|)] \|\omega_1 - \omega_2\| \\ &+ [\gamma_2 + \kappa_2(\|u_1\| + \|u_2\|)] \|x_1 - x_2\| + [\gamma_3 + \kappa_3(\|x_1\| + \|x_2\|)] \|u_1 - u_2\| \end{aligned}$$

for every $(\omega_1, s, x_1, u_1) \in \Omega \times E \times \mathbb{R}^n \times \mathbb{R}^m$ and $(\omega_2, s, x_2, u_2) \in \Omega \times E \times \mathbb{R}^n \times \mathbb{R}^m$;

1.C. There exist $p > 1$ and $r_* > 0$ such that the inequality

$$\gamma_0 + \kappa_2 \mu(E) + 2\gamma_* r_* [\mu(E)]^{(p-1)/p} < 1$$

is satisfied where $\gamma_* = \max\{\gamma_1, \gamma_2, \gamma_3\}$, $\mu(E)$ stands for the Lebesgue measure of the set E.

If the function $(\omega, s, x, u) \to F(\omega, s, x, u)$, $(\omega, s, x, u) \in \Omega \times E \times \mathbb{R}^n \times \mathbb{R}^m$ is Lipschitz continuous with respect to (ω, x, u), then it satisfies the condition 1.B.

Let $u(\cdot) \in U_{p,r}$. A continuous function $x(\cdot) : \Omega \to \mathbb{R}^n$ satisfying the integral equation (1.1) for every $\omega \in \Omega$, is said to be a trajectory of the system (1.1) generated by the admissible control function $u(\cdot) \in U_{p,r}$. The set of trajectories of the system (1.1) generated by all admissible control functions $u(\cdot) \in U_{p,r}$ is denoted by symbol $\mathbb{Z}_{p,r}$ and is called the set of trajectories of the system (1.1).

The paper is organized as follows. In Section 2 the basic properties of the system's trajectory are presented which are used in following arguments. In Section 3 it is proved that the set of trajectories $\mathbb{Z}_{p,r}$ is Lipschitz continuous with respect to r (Theorem 3.1). In Section 4 an upper evaluation for the diameter of the set of trajectories is given (Theorem 4.1). In section 5 the robustness of the system's trajectory with respect to the fast consumption of the remaining control resource is established (Theorem 5.1). It is proved that every trajectory can be approximated by the trajectory obtained by full consumption of the control resource (Theorem 5.2).

2. Basic Properties of the Trajectories

We set

$$L(p, r) = \gamma_0 + \kappa_2 \mu(E) + 2\gamma_* r [\mu(E)]^{(p-1)/p} \tag{2.1}$$

From condition 1.C it follows that

$$L(p, r_*) < 1. \tag{2.2}$$

Then there exist $\alpha_* > 0$ such that $L(p, r) < 1$ for every $r \in [0, r_* + \alpha_*]$. Denote

$$L_*(p) = \gamma_0 + \kappa_2 \mu(E) + 2\gamma_* (r_* + \alpha_*) [\mu(E)]^{(p-1)/p} \tag{2.3}$$

From (2.1), (2.2) and (2.3) it follows that

$$0 \le L_*(p) < 1\,, \quad L_*(p) - \gamma_0 > 0\,. \tag{2.4}$$

From now on, it will be assumed that $r \in [0, r_* + \alpha_*]$.

For given $\omega \in \Omega$ we set

$$\mathbb{Z}_{p,r}(\omega) = \{x(\omega) \in \mathbb{R}^n : x(\cdot) \in \mathbb{Z}_{p,r}\}\,. \tag{2.5}$$

The set $\mathbb{Z}_{p,r}(\omega)$ is close to the attainable set notion used in control and dynamical systems theory and consists of points to which arrive the trajectories of the system at ω (see, e.g., [2, 11]).

From the conditions 1.A-1.C it follows the validity of the following propositions.

Proposition 2.1. *Every admissible control function $u(\cdot) \in U_{p,r}$ generates a unique trajectory $x(\cdot) \in C(\Omega;\mathbb{R}^n)$ of the system (1.1) where $C(\Omega;\mathbb{R}^n)$ is the space of continuous functions $x(\cdot) : \Omega \to \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(\omega)\| : \omega \in \Omega\}$.*

Proposition 2.2. *There exists $g_* > 0$ such that the inequality*

$$\|x(\cdot)\|_C \le g_*$$

is satisfied for every $x(\cdot) \in \mathbb{Z}_{p,r}$ and $r \in [0, r_* + \alpha_*]$.

Proposition 2.3. *The set of trajectories $\mathbb{Z}_{p,r}$ is a precompact subset of the space $C(\Omega;\mathbb{R}^n)$.*

The proofs of the Proposition 2.1, Proposition 2.2 and Proposition 2.3 are similar to the proofs of the Theorem 3.1, Theorem 4.1 and Theorem 5.1 of [6] respectively.

Proposition 2.4. *Let $u_1(\cdot) \in U_{p,r_1}$, $u_2(\cdot) \in U_{p,r_2}$ where $r_1 \in [0, r_* + \alpha_*]$ and $r_2 \in [0, r_* + \alpha_*]$. Then*

$$\int_E [\gamma_2 + \kappa_2(\|u_1\| + \|u_2\|)]\, ds \le L_*(p) - \gamma_0$$

where $L_*(p)$ is defined by (2.3).

The proof of the proposition follows from Hölder's inequality.

For given metric space $(Y, d_Y(\cdot,\cdot))$ the Hausdorff distance between the sets $S \subset Y$ and $W \subset Y$ is denoted by $h_Y(S,W)$ and defined as

$$h_Y(S,W) = \max\{\sup_{x\in S} d_Y(x,W),\ \sup_{y\in W} d_Y(y,S)\}$$

where $d_Y(x,W) = \inf\{d_Y(x,y) : y \in W\}$.

Let $b(Y)$ be a family of all nonempty bounded subsets of given metric space $(Y, d_Y(\cdot,\cdot))$. By virtue of [2, 9] we have that $(b(Y), h_Y(\cdot,\cdot))$ is a pseudometric space where $h_Y(\cdot,\cdot)$ stands for Hausdorff distance between the subsets of the space $(Y, d_Y(\cdot,\cdot))$.

Now, let $(Y, d_Y(\cdot,\cdot))$ and $(T, d_T(\cdot,\cdot))$ be metric spaces, $\Phi(\cdot) : Y \to T$ be a given set valued map and $y_* \in Y$. If $h_T(\Phi(y), \Phi(y_*)) \to 0$ as $y \to y_*$, then the map $\Phi(\cdot)$ is called continuous at y_*.

If there exists $R > 0$ such that

$$h_T(\Phi(y_1), \Phi(y_2)) \le R \cdot d_Y(y_1, y_2)$$

for every $y_1 \in Y$ and $y_2 \in Y$, then the map $\Phi(\cdot)$ is called Lipschitz continuous with Lipschitz constant R.

By symbol $h_n(A, U)$ we denote the Hausdorff distance between the sets $A \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, and by symbol $h_C(V, Q)$ we denote the Hausdorff distance between the sets $V \subset C(\Omega; \mathbb{R}^n)$ and $Q \subset C(\Omega; \mathbb{R}^n)$.

Proposition 2.5. *For each fixed $r \in [0, r_* + \alpha_*]$ the set valued $\omega \to \mathbb{Z}_{p,r}(\omega)$, $\omega \in \Omega$, is continuous, i.e. $h_n(\mathbb{Z}_{p,r}(\omega), \mathbb{Z}_{p,r}(\omega_*)) \to 0$ as $\omega \to \omega_*$ for every fixed $\omega_* \in \Omega$ where the set $\mathbb{Z}_{p,r}(\omega)$ is defined by (2.5).*

The validity of the Proposition 2.5 can be specified analogously to the proof of the Proposition 5.2 from [6].

3. Lipschitz Continuity of the set of Trajectories with respect to the Control Resource Bound

In this section it will be shown that the set valued map $r \to \mathbb{Z}_{p,r}$, $r \in [0, r_* + \alpha_*]$, is Lipschitz continuous. Denote

$$B_C(1) = \{x(\cdot) \in C([t_0, \theta]; \mathbb{R}^n) : \|x(\cdot)\|_C \le 1\}, \tag{3.1}$$

$$R_0 = \frac{(\kappa_3 + 2g_*\gamma_3)[\mu(E)]^{(p-1)/p}}{1 - L_*(p)} \tag{3.2}$$

where $L_*(p)$ is defined by (2.3), g_* is given in Proposition 2.2.

Theorem 3.1. *The set valued map $r \to \mathbb{Z}_{p,r}$, $r \in [0, r_* + \alpha_*]$ is Lipschits continuous with Lipschitz constant R_0, i.e.*

$$h_C(\mathbb{Z}_{p,r_1}, \mathbb{Z}_{p,r_2}) \le R_0 \cdot |r_1 - r_2|$$

for every $r_1 \in [0, r_* + \alpha_*]$ and $r_2 \in [0, r_* + \alpha_*]$ where R_0 is defined by (3.2).

Proof. Without loss of generality we assume that $r_1 < r_2$ which implies

$$\mathbb{Z}_{p,r_1} \subset \mathbb{Z}_{p,r_2} \ . \tag{3.3}$$

Choose an arbitrary $x_*(\cdot) \in \mathbb{Z}_{p,r_2}$ generated by the control function $u_*(\cdot) \in U_{p,r_2}$. Define a new control function $v_*(\cdot) : E \to \mathbb{R}^m$, setting

$$v_*(s) = \frac{r_1}{r_2} u_*(s)\,, \quad s \in E\ . \tag{3.4}$$

Since $u_*(\cdot) \in U_{p,r_2}$, then from (3.4) it follows that $v_*(\cdot) \in U_{p,r_1}$. Let $y_*(\cdot) : \Omega \to \mathbb{R}^n$ be the trajectory of the system (1.1) generated by the control function $v_*(\cdot) \in U_{p,r_1}$. It is obvious that $y_*(\cdot) \in \mathbb{Z}_{p,r_1}$. From (1.1), (3.4), condition 1.B, Proposition 2.2, Proposition 2.4, inclusion $u_*(\cdot) \in U_{p,r_2}$ and Hölder's inequality we have

$$\begin{aligned}
&\|x_*(\omega) - y_*(\omega)\| \le \gamma_0 \|x_*(\omega) - y_*(\omega)\| + \int_E [\kappa_2 + \gamma_2(\|u_*(s)\| + \|v_*(s)\|)] \\
&\qquad \cdot \|x_*(s) - y_*(s)\| ds + \int_E [\kappa_3 + \gamma_3(\|x_*(s)\| + \|y_*(s)\|)] \cdot \|u_*(s) - v_*(s)\| ds \\
&\le \gamma_0 \|x_*(\cdot) - y_*(\cdot)\|_C + \int_E [\kappa_2 + \gamma_2(\|u_*(s)\| + \|v_*(s)\|)]\, ds \\
&\qquad \cdot \max\{\|x_*(s) - y_*(s)\| : s \in E\} + (\kappa_3 + 2\gamma_3 g_*) \cdot \int_E \|u_*(s) - \frac{r_1}{r_2} u_*(s)\| ds \\
&\le \gamma_0 \|x_*(\cdot) - y_*(\cdot)\|_C + (L_*(p) - \gamma_0) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C \\
&\qquad + (\kappa_3 + 2\gamma_3 g_*)\, \frac{|r_1 - r_2|}{r_2} \int_E \|u_*(s)\| ds \\
&\le L_*(p) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C + (\kappa_3 + 2\gamma_3 g_*) \cdot [\mu(E)]^{(p-1)/p} r_2 \cdot \frac{|r_1 - r_2|}{r_2} \\
&= L_*(p) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C + (\kappa_3 + 2\gamma_3 g_*) \cdot [\mu(E)]^{(p-1)/p} \cdot |r_1 - r_2|
\end{aligned}$$

for every $\omega \in \Omega$ and hence

$$\|x_*(\cdot) - y_*(\cdot)\|_C \le L_*(p) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C + (\kappa_3 + 2\gamma_3 g_*) \cdot [\mu(E)]^{(p-1)/p} \cdot |r_1 - r_2|\,.$$

From (2.4), (3.2) and the last inequality we obtain

$$\|x_*(\cdot) - y_*(\cdot)\|_C \le \frac{(\kappa_3 + 2\gamma_3 g_*) \cdot [\mu(E)]^{(p-1)/p}}{1 - L_*(p)} \cdot |r_1 - r_2| = R_0 \cdot |r_1 - r_2|\,. \tag{3.5}$$

So, we have that for arbitrarily chosen $x_*(\cdot) \in \mathbb{Z}_{p,r_2}$ there exists $y_*(\cdot) \in \mathbb{Z}_{p,r_1}$ such that the inequality (3.5) holds which implies that

$$\mathbb{Z}_{p,r_2} \subset \mathbb{Z}_{p,r_2} + R_0 |r_1 - r_2| \cdot B_C(1) \tag{3.6}$$

where $B_C(1)$ is defined by (3.1).

The inclusions (3.3) and (3.6) complete the proof. $\square$

4. Diameter of the Set of Trajectories

Let $(Y, d_Y(\cdot,\cdot))$ be a metric space and $G \subset Y$. The diameter of the set G is denoted by $diam(G)$ and is defined as

$$diam(G) = \sup\{d_Y(x,g) : x \in G, g \in G\}.$$

Theorem 4.1. *The inequality*

$$diam(\mathbb{Z}_{p,r}) \leq \frac{2(\kappa_3 + 2\gamma_3 g_*) r [\mu(E)]^{(p-1)/p}}{1 - L_*(p)}$$

is verified where g_* is defined in Proposition 2.2, $L_*(p)$ is defined by (2.3).

Proof. Let $x(\cdot) \in \mathbb{Z}_{p,r}$ and $z(\cdot) \in \mathbb{Z}_{p,r}$ be arbitrarily chosen trajectories generated by the control functions $u(\cdot) \in U_{p,r}$ and $v(\cdot) \in U_{p,r}$ respectively. Then from (1.1), condition 1.B, Proposition 2.2, Proposition 2.4 and Hölder's inequality we have

$$\begin{aligned}
\|x(\omega) - z(\omega)\| &\leq \gamma_0 \|x(\omega) - z(\omega)\| + \int_E [\kappa_2 + \gamma_2(\|u(s)\| + \|v(s)\|)] \\
&\quad \cdot \max\{\|x(s) - y(s)\| : s \in E\} + (\kappa_3 + 2\gamma_3 g_*) \cdot \int_E \|u(s) - v(s)\| ds \\
&\leq \gamma_0 \|x(\cdot) - z(\cdot)\|_C + (L_*(p) - \gamma_0) \cdot \|x(\cdot) - y(\cdot)\|_C \\
&\quad + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E)]^{(p-1)/p} \\
&= L_*(p) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E)]^{(p-1)/p}
\end{aligned}$$

for every $\omega \in \Omega$ and hence

$$\|x(\cdot) - z(\cdot)\|_C \leq L_*(p) \cdot \|x_*(\cdot) - y_*(\cdot)\|_C + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E)]^{(p-1)/p}.$$

From the last inequality and (2.4) we conclude that

$$\|x(\cdot) - z(\cdot)\|_C \leq \frac{2(\kappa_3 + 2\gamma_3 g_*) r [\mu(E)]^{(p-1)/p}}{1 - L_*(p)}. \tag{4.1}$$

Since $x(\cdot) \in \mathbb{Z}_{p,r}$ and $z(\cdot) \in \mathbb{Z}_{p,r}$ are arbitrarily chosen trajectories, then from (4.1) we have the proof of the theorem. □

5. Robustness of the Trajectories

In this section the robustness of a trajectory of the system (1.1) with respect to the fast consumption of the remaining control resource will be discussed.

Theorem 5.1. *Let $\varepsilon > 0$ be a given number, $x(\cdot) \in \mathbb{Z}_{p,r}$ be a trajectory of the system (1.1) generated by the control function $u(\cdot) \in U_{p,r}$ and let $\|u(\cdot)\|_p = r_0 < r$. Assume that $E_* \subset E$ is the Lebesgue measurable set, the control function $w(\cdot) : E \to \mathbb{R}^m$ is defined*

$$w(s) = \begin{cases} u(s) & \text{if} \quad s \in E \setminus E_* \ , \\ u_*(s) & \text{if} \quad s \in E_* \end{cases}$$

such that $\|w(\cdot)\|_p = r$ and let $y(\cdot) : \Omega \to \mathbb{R}^m$ be the trajectory of the system (1.1) generated by the control function $w(\cdot) \in U_{p,r}$. If

$$\mu(E_*) \leq \left[\frac{1 - L_*(p)}{2\,(\kappa_3 + 2\gamma_* g_*)\, r} \cdot \varepsilon \right]^{p/(p-1)} \tag{5.1}$$

then

$$\|x(\cdot) - y(\cdot)\|_C \leq \varepsilon$$

where g_* is defined in Proposition 2.2, $L_*(p)$ is defined by (2.3).

Proof. According to (1.1), condition 1.B, inclusions $u(\cdot) \in U_{p,r}$, $w(\cdot) \in U_{p,r}$, Proposition 2.2, Proposition 2.4 and Hölder's inequality, we have

$$\begin{aligned} \|x(\omega) - y(\omega)\| &\leq \gamma_0 \|x(\cdot) - y(\cdot)\|_C + \int_E [\kappa_2 + \gamma_2(\|u(s)\| + \|w(s)\|)]\, ds \\ &\quad \cdot \max\{\|x(s) - y(s)\| : s \in E\} + (\kappa_3 + 2\gamma_3 g_*) \cdot \int_{E_*} \|u(s) - w(s)\| ds \\ &\leq \gamma_0 \|x(\cdot) - y(\cdot)\|_C + (L_*(p) - \gamma_0) \cdot \|x(\cdot) - y(\cdot)\|_C \\ &\quad + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E_*)]^{(p-1)/p} \\ &= L_*(p) \cdot \|x(\cdot) - y(\cdot)\|_C + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E_*)]^{(p-1)/p} \end{aligned}$$

for every $\omega \in \Omega$ and consequently

$$\|x(\cdot) - y(\cdot)\|_C \leq L_*(p) \cdot \|x(\cdot) - y(\cdot)\|_C + 2\,(\kappa_3 + 2\gamma_3 g_*)\, r \cdot [\mu(E_*)]^{(p-1)/p} .$$

From the last inequality and (2.4) we conclude that

$$\|x(\cdot) - z(\cdot)\|_C \leq \frac{2(\kappa_3 + 2\gamma_3 g_*) r [\mu(E_*)]^{(p-1)/p}}{1 - L_*(p)} .$$

The last inequality and (5.1) imply that

$$\|x(\cdot) - z(\cdot)\|_C \leq \frac{2(\kappa_3 + 2\gamma_3 g_*) r}{1 - L_*(p)} [\mu(E_*)]^{(p-1)/p} \leq \varepsilon .$$

The proof is completed. $\square$

Denote

$$V_{p,r} = \Big\{ u(\cdot) \in L_p\big(E;\mathbb{R}^m\big) : \|u(\cdot)\|_p = r \Big\}$$

and let $\mathbb{Z}^*_{p,r}$ be the set of trajectories of the system (1.1) generated by all admissible control functions $u(\cdot) \in V_{p,r}$.

Theorem 5.2. *The equality $cl(\mathbb{Z}_{p,r}) = cl(\mathbb{Z}^*_{p,r})$ is satisfied where cl denotes the closure of a set.*

Proof. Since $\mathbb{Z}^*_{p,r} \subset \mathbb{Z}_{p,r}$ then we have

$$cl(\mathbb{Z}^*_{p,r}) \subset cl(\mathbb{Z}_{p,r})\,. \tag{5.2}$$

Let ν be an arbitrarily chosen number, $E_* \subset E$ be the Lebesgue measurable set such that

$$\mu(E_*) \leq \left[\frac{1 - L_*(p)}{2\,(\kappa_3 + 2\gamma_* g_*)\, r} \cdot \nu \right]^{p/(p-1)} \tag{5.3}$$

where g_* is defined in Proposition 2.2, $L_*(p)$ is defined by (2.3).

Now, let us choose an arbitrary trajectory $x(\cdot) \in \mathbb{Z}_{p,r}$ generated by the control function $u(\cdot) \in U_{p,r}$ and let $\|u(\cdot)\|_p = r_0 < r$. Suppose that $\int_{E\setminus E_*} \|u(s)\|^p ds = r_1^p$. It is obvious that $r_1 \leq r_0$. Define new control function $v_*(\cdot) : E \to \mathbb{R}^m$, setting

$$v_*(s) = \begin{cases} u(s) & \text{if} \quad s \in E \setminus E_*\ , \\ \left[\dfrac{r^p - r_1^p}{\mu(E_*)} \right]^{1/p} \cdot e_* & \text{if} \quad s \in E_* \end{cases} \tag{5.4}$$

where $e_* \in \mathbb{R}^m$ is an arbitrary vector such that $\|e_*\| = 1$. It is not difficult to verify that $\|v_*(\cdot)\|_p = r$ and hence $v_*(\cdot) \in V_{p,r}$. Let $z_*(\cdot) : E \to \mathbb{R}^m$ be the trajectory of the system (1.1) generated by the control function $v_*(\cdot) \in V_{p,r}$. Then $z_*(\cdot) \in \mathbb{Z}^*_{p,r}$ and from (5.3), (5.4) and Theorem 5.1 it follows that $\|x(\cdot) - z_*(\cdot)\|_C \leq \nu$. Since $z_*(\cdot) \in \mathbb{Z}^*_{p,r}$ and $\nu > 0$ are arbitrarily chosen, then we obtain that $x(\cdot) \in cl(\mathbb{Z}^*_{p,r})$ which implies that $\mathbb{Z}_{p,r} \subset cl\,(\mathbb{Z}^*_{p,r})$. This inclusion yields

$$cl(\mathbb{Z}_{p,r}) \subset cl(\mathbb{Z}^*_{p,r})\,. \tag{5.5}$$

(5.2) and (5.5) complete the proof. □

Conclusion

The Lipschitz continuity of the set of trajectories allows to establish an error estimation for the set of trajectories if there is an inaccuracy in determining the upper bound of the total control resource. Evaluation of the diameter permits to estimate the set of trajectories in general. From the robustness of the trajectory with respect to the remaining control resource it follows that if you have a needless control resource and you want to get rid of it, then spending the remaining control resource on the domain with sufficiently small measure, you will obtain a small deviation from the initial trajectory. This also illustrates that the consuming the control resource with big quants on the domains with sufficiently small measures is not effective way to change the system's trajectory.

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