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GROWTH ANALYSIS OF COMPOSITION OF INTEGER TRANSLATED ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we study the effect of integer translation on Nevanlinna's characteristic function of a meromorphic function. Also, we investigate comparative growth of integer translated entire and meromorphic functions under different conditions.

1. Introduction and Preliminary

Let f be a meromorphic function (i.e., regular except for poles) defined on the complex plane and let a be a complex number finite or infinite. To study how the values of f are distributed it is essential to explore the distribution of the solutions of the equation $f(z) = a$. This includes an estimate of the number of roots $n_f(r, a)$, counted with or without multiplicity in a disc $|z| \leq r$ for any non-negative real number r, estimates on the growth and the asymptotics of the number of such solutions in terms of r, the comparison of the various estimates when the constant a varies, etc.

An oldest such result is the Fundamental Theorem of Algebra stating that a polynomial of degree n has n complex roots (counting with proper multiplicity). This theorem allows us to write any polynomial $f(z)$ in the form

$$f(z) = cz^p \prod_{r=p+1}^{n} \left(1 - \frac{z}{z_r}\right),$$

where z_r are the zeros of $f(z)$ other than those at the origin. Precisely, when $z \to \infty$ the growth of a polynomial $f(z)$ of degree $n \geq 1$ is equal to n. Then

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = d,$$

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where d is the coefficient of the monomial of highest degree of f. Thus, growth of the polymonial f can be read on its coefficients. Borel [1] first gave the concept order of growth for an entire function f by the quantity

$$\rho_f = \{\inf \mu : \log M_f(r) = O(r^{\mu}), 0 \leq \mu \leq \infty\},$$

where the maximum modulus $M_f(r) = \max_{|z|=r} |f(z)|$. With such a definition, the order of a polynomial is zero, the exponential function e^z has order one and the function e^{e^z} has infinite order. Later he reformulated the definition as : An entire function f of order ρ_f $(0 < \rho_f < \infty)$ satisfies

$$\limsup_{r\to\infty} \frac{\log n_f(r,a)}{\log r} \leq \rho_f$$

for every finite complex value 'a', with equality holding except possibly for one value a. Here, $n_f(r,a)$ is the number of roots with multiplicity of the equation $f(z) = a$ in the disc $|z| \leq r$. Borel's definition of order was used later by Nevanlinna [9] who generalized it to the setting of meromorphic functions that are not necessarily entire.

For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n_f(t,a)$ $(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of a non-constant meromorphic function f in $|z| \leq t$, where an ∞-point is a pole of f. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

and

$$\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r.$$

The function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ are called the counting function for a-points (distinct a-points) of f . In many occassions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively . We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta,$$

where

$$\begin{aligned} \log^+ x &= \log x \qquad \text{if } x \geq 1 \\ &= 0 \qquad \text{if } 0 \leq x \leq 1. \end{aligned}$$

For $a \in \mathbb{C}$, we denote by $m_f(r,a)$ the function $m_{\frac{1}{f-a}}(r)$ and we mean by $m_f(r,\infty)$ the function $m_f(r)$ which is called the proximity function of f. The function $T_f(r) = m_f(r) + N_f(r)$ is called the Nevanlinna's Characteristic function of f. If $f(z)$ is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of $f(z)$ is defined as

$$T_f(r) = m_f(r).$$

Further for the entire $f(z)$ we have

$$T_f(r) \leq \log^+ M_f(r) \leq \frac{R+r}{R-r} T_f(R) \quad \text{cf. [6]}, \tag{1}$$

where $0 < r \leq R$.

We shall assume that the reader is familiar with the basic results and the standard notations of the Nevanlinna value distribution theory, see [6, 12] for more details. For all $r \in \mathbb{R}$, we define $\exp^{[1]} r = e^r$ and $\exp^{[k]} r = \exp\left(\exp^{[k-1]} r\right)$, $k \in \mathbb{N}$. We also define for all r sufficiently large $\log^{[1]} r = \log r$, $\log^{[k]} r = \log\left(\log^{[k-1]} r\right)$, $k \in \mathbb{N}$. Moreover, we denote by $\exp^{[0]} r = r$, $\log^{[0]} r = r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$.

Let us recall some well-known definitions.

Definition 1. The Nevanlinna's deficiency of a finite or infinite complex number 'a' with respect to a meromorphic function g are defined as

$$\delta_g(a) = 1 - \limsup_{r\to\infty} \frac{N_g(r,a)}{T_g(r)} = \liminf_{r\to\infty} \frac{m_g(r,a)}{T_g(r)}.$$

Definition 2. The order ρ_f and lower order λ_f of a meromorphic function f are given by

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r}, \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}$$

respectively. Also for an integer $l \geq 2$, the generalised order $\rho_f^{[l]}$ and the generalised lower order $\lambda_f^{[l]}$ of a meromorphic function are respectively defined by

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r}, \quad \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r} \quad \text{cf. [10]}.$$

and if f is entire, then $T_f(r)$ can be replaced with $\log M_f(r)$.

Juneja, Kapoor and Bajpai [7] defined (p,q)-order and (p,q)-lower order of an entire function respecitvely as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$. When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$. Clearly, $\rho_f(2,1) = \rho_f^{[2]} = \rho_f$ and $\lambda_f(2,1) = \lambda_f^{[2]} = \lambda_f$.

The translation of a meromorphic function $f(z)$ be denoted by $f(z+n)$, where $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us denote this family by $f_n(z)$, i.e.,

$$f_n(z) = \{f(z+n) : n \in \mathbb{N}\}.$$

It is clear that the number of poles of f may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, i.e.,

$$N_{f(z+n)}(r) = N_f(r) + e_n, \tag{2}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$. Also,

$$\begin{aligned} m_{f(z+n)}(r) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta} + n\right)\right| d\theta \\ &= m_f(r) + e'_n \ , \end{aligned} \tag{3}$$

where e'_n (may be distinct from e_n) be such that $e'_n \to 0$ as $r \to \infty$. Therefore, from (2) and (3) we get

$$N_{f(z+n)}(r) + m_{f(z+n)}(r) = N_f(r) + e_n + m_f(r) + e'_n$$
$$\text{i.e., } T_{f(z+n)}(r) = T_f(r) + e_n + e'_n.$$

Now if n varies, then the Nevanlinna's Characteristic function for the family $f_n(z)$ is

$$T_{f_n}(r) = nT_f(r) + \sum_n \left(e_n + e'_n\right). \tag{4}$$

Again it is obvious that the maximum modulus $M_f(r)$ of an entire function $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, and for the family $f_n(z)$ we obtain

$$M_{f_n}(r) = nM_f(r) + \sum_n e_n^{''} ,$$

where $e_n^{''}$ (may be distinct from e_n, e_n') be such that $e_n^{''} \to 0$ as $r \to \infty$. This implies,

$$\log M_{f_n}(r) = \log M_f(r) + O(1) . \tag{5}$$

Now from (4) we obtain

$$\log^{[p-1]} T_{f_n}(r) = \log^{[p-1]} n + \log^{[p-1]} T_f(r) + \log^{[p-1]} \left(1 + \frac{\sum_n (e_n + e_n')}{nT_f(r)} \right) ,$$

where $e_n \to 0$, $e_n' \to 0$ as $r \to \infty$. Since $T_f(r)$ is an increasing function of r, therefore

$$\log^{[p-1]} T_{f_n}(r) = \log^{[p-1]} T_f(r) + O(1) . \tag{6}$$

This implies,

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_n}(r)}{\log^{[q]} r} = \limsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

$$\text{or, } \rho_{f_n}(p,q) = \rho_f(p,q) .$$

Similary, it can be proved that

$$\lambda_{f_n}(p,q) = \lambda_f(p,q) .$$

Datta and Tamang [4]; Biswas and Datta [3]; Tamang and Biswas [11] etc. investigated the changes to Nevanlinna's Characteristic function of the integer translated meromorphic functions. They developed a new technique to describe the comparative growth of composition of entire and meromorphic functions. Their work motivated us to study further on this topic. But our investgations focus on finding the comparative growth related to (p,q)-order. Using (p,q)-order we prove some new result

2. Main Results

In this section, we start with some existing results on comparative growth.

Lemma 1 ([8]). *Let g be an entire function with $\lambda_g < \infty$ and assume that $a_i(i = 1, 2, ...n; n \leq \infty)$ are entire functions satisfying $T_{a_i}(r) = o\{T_g(r)\}$.*

If $\sum_{i=1}^{n}\delta_g(a_i) = 1$, *then*

$$\lim_{r\to\infty}\frac{T_g(r)}{\log M_g(r)} = \frac{1}{\pi}.$$

Lemma 1 ([2])**.** *If f be a meromorphic function and g be an entire fucntion, then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \le \{1 + o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 3 ([7, 5])**.** *Let f be an entire function with non zero finite generalised order $\rho_f^{[l]}$ (non zero finite generalised lower order $\lambda_f^{[l]}$). If $p-q = l-1$, then the (p,q)-order $\rho_f(p,q)$ (lower (p,q)-order $\lambda_f(p,q)$) of f will be equal to 1. If $p-q \ne l-1$, then $\rho_f(p,q)$ ($\lambda_f(p,q)$ is either zero or infinity).*

Now, we will prove the main results of our paper. Their proofs mainly depend on the results stated above.

Theorem 1. *Let f be meromorphic and g be entire such that $\rho_f(p,q)$ and λ_g are both finite, where p, q are any two positive integers with $p > q$. Also, suppose that there exist entire functions a_i $(i = 1, 2, ..., k; k \le \infty)$ satisfying $T_{a_i}(r) = o\{T_g(r)\}$ as $r \to \infty$ and $\sum_{i=1}^{k}\delta_g(a_i) = 1$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for any $R > 2r$*

$$\limsup_{r\to\infty}\frac{\log^{[p-1]} T_{f_m\circ g_n}(r)}{\log M_{g_n}(R)} \le \frac{3\rho_f(p,q)}{\pi}.$$

Proof. Let $f_m \circ g_n = h_t$, where h is a meromorphic function and $t \in \mathbb{N}$. So, h_t can be expressed as $h_t(z) = \{h(z+t) : t \in \mathbb{N}\}$. Then,

$$T_{h_t}(r) = tT_h(r) + \sum_t (e_t + e'_t),$$

where $e_t \to 0$, $e'_t \to 0$ as $r \to \infty$. That is,

$$T_{f_m\circ g_n}(r) = tT_{f\circ g}(r) + \sum_t (e_t + e'_t). \tag{7}$$

Now by Lemma 2 and the inequality $T_g(r) \le \log^+ M_g(r)$, we get from (7) for any positive ϵ and for all sufficiently large values of r that

$$\begin{aligned} \log^{[p-1]} T_{f_m \circ g_n}(r) &\leq \log^{[p-1]} T_f(M_g(r)) + O(1) \\ or, \log^{[p-1]} T_{f_m \circ g_n}(r) &\leq (\rho_f(p,q)+\epsilon) \log^{[q]}(M_g(r)) + O(1) \\ or, \log^{[p-1]} T_{f_m \circ g_n}(r) &\leq (\rho_f(p,q)+\epsilon) \log M_g(r) + O(1). \end{aligned} \tag{8}$$

Since by (1), we have $T_g(r) \leq \log^+ M_g(r) \leq 3T_g(2r)$, then (8) gives

$$\log^{[p-1]} T_{f_m \circ g_n}(r) \leq 3(\rho_f(p,q)+\epsilon) T_g(2r).$$

In view of (5), for any $R > 2r$ the above inequality implies

$$\frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log M_{g_n}(R)} \leq \frac{3(\rho_f(p,q)+\epsilon) T_g(2r)}{\log M_g(2r) + O(1)}. \tag{9}$$

Since $\epsilon > 0$ is arbitrary, therfore by Lemma 1 and (9) we obtain

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log M_{g_n}(R)} \leq \frac{3\rho_f(p,q)}{\pi}.$$

This completes the proof. □

Similarly, the following theorem also can be proved.

Theorem 2. *Let f be meromorphic and g be entire such that $\lambda_f(p,q)$ and λ_g are both finite, where p, q are any two positive integers with $p > q$. Also, suppose that there exist entire functions a_i $(i = 1, 2, ..., k; k \leq \infty)$ satisfying $T_{a_i}(r) = o\{T_g(r)\}$ as $r \to \infty$ and $\sum_{i=1}^{k} \delta_g(a_i) = 1$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for any $R > 2r$*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log M_{g_n}(R)} \leq \frac{3\lambda_f(p,q)}{\pi}.$$

Theorem 3. *Let f and g be any two meromorphic functions such that $\rho_f(p,q) < \infty$ and $\lambda_{f\circ g}(p,q) = \infty$, where p, q are positive integers with $p > q > 1$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for every positive number β,*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[p-1]} T_{f_m}(r^\beta)} = \infty.$$

Proof. If possible let the value of the limit is not infinite. Then, we can find a constant $\alpha > 0$ such that for a sequence of values of r tending to infinity

$$\log^{[p-1]} T_{f_m \circ g_n}(r) \leq \alpha \log^{[p-1]} T_{f_m}\left(r^\beta\right). \tag{10}$$

Again for $q > 1$, from the definition of $\rho_{f_m}(p,q)$ it follows for any positive ϵ and for all sufficiently large values of r that

$$\log^{[p-1]} T_{f_m}\left(r^{\beta}\right) \leq \left(\rho_{f_m}(p,q)+\epsilon\right)\log^{[q]} r^{\beta}$$

$$(11) \qquad or,\ \log^{[p-1]} T_{f_m}\left(r^{\beta}\right) \leq \left(\rho_f(p,q)+\epsilon\right)\log^{[q]} r + O(1),\ \text{as } \rho_{f_m}=\rho_f.$$

Thus, from (10) and (11) we have for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} T_{f_m\circ g_n}(r)}{\log^{[q]} r} \leq \frac{\alpha\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} r + O(1)}{\log^{[q]} r}.$$

This implies on using (7) that

$$\frac{\log^{[p-1]} t + \log^{[p-1]} T_{f\circ g}(r) + \log^{[p-1]}\left(1+\frac{\sum_t (e_t+e_t')}{tT_{f\circ g}(r)}\right)}{\log^{[q]} r}$$

$$\leq \frac{\alpha\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} r + O(1)}{\log^{[q]} r}$$

$$or,\ \frac{\log^{[p-1]} T_{f\circ g}(r) + O(1)}{\log^{[q]} r} \leq \frac{\alpha\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} r + O(1)}{\log^{[q]} r}$$

$$or,\ \frac{\log^{[p-1]} T_{f\circ g}(r)}{\log^{[q]} r} \leq \frac{\alpha\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} r + O(1)}{\log^{[q]} r}.$$

Since $\epsilon > 0$ is arbitrary, therefore,

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} T_{f\circ g}(r)}{\log^{[q]} r} \leq \alpha\rho_f(p,q) < \infty$$

$$i.e,\ \lambda_{f\circ g}(p,q) < \infty,$$

which is a contradiction. Hence, the theorem. □

Remark 1. Theorem 3 is also valid with "limit superior" instead of "limit" if $\lambda_{f\circ g}(p,q)=\infty$ is replaced by $\rho_{f\circ g}(p,q)=\infty$ and the remaining other conditions are same.

Corollary 1. *Under the assumptions of Remark* 1

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} T_{f_m\circ g_n}(r)}{\log^{[p-2]} T_{f_m}(r^{\beta})} = \infty.$$

Proof. From Remark 1, we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log^{[p-1]} T_{f_m \circ g_n}(r) &> K \log^{[p-1]} T_{f_m}\left(r^{\beta}\right) \\ i.e., \log^{[p-2]} T_{f_m \circ g_n}(r) &> \left\{\log^{[p-2]} T_{f_m}\left(r^{\beta}\right)\right\}^{K}, \end{aligned}$$

from which the Corollary follows. □

Corollary 2. *Under the same conditions of Theorem* 3 *if* $q = 1$,

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[p-1]} T_{f_m}(r^{\beta})} = \infty.$$

Corollary 3. *Under the same conditions of Remark* 1 *if* $q = 1$,

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[p-1]} T_{f_m}(r^{\beta})} = \infty.$$

Remark 2. The condition $\lambda_{f\circ g}(p,1) = \infty$ in Corollary 2 is necessary as we see in the following example.

Example 1. Let $f = \exp z$, $g = z$ and $p = 2$, $q = 1$, $\beta = 1$. Then $\rho_f(p,1) = \lambda_{f\circ g}(p,1) = 1$. Now, $f_m(z) = \exp(z+m)$, $g_n(z) = z+n$ and $(f_m \circ g_n)(z) = f_m(g_n(z)) = f_m(z+n) = \exp(z+m+n)$, where $m, n \in \mathbb{N}$. So, $T_{f_m\circ g_n}(r) = \frac{r+m+n}{\pi} + O(1)$, $T_{f_m}(r) = \frac{r+m}{\pi} + O(1)$. Therefore,

$$\begin{aligned} \lim_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[p-1]} T_{f_m}(r^{\beta})} &= \lim_{r\to\infty} \frac{\log T_{f_m \circ g_n}(r)}{\log T_{f_m}(r)} \\ &= \lim_{r\to\infty} \frac{\frac{r+m+n}{\pi} + O(1)}{\frac{r+m}{\pi} + O(1)} \\ &= 1 \neq \infty, \end{aligned}$$

which is contrary to Corollary 2.

Remark 3. Considering $f = \exp z$, $g = z$ and $p = 2$, $q = 1$, $\beta = 1$ one can easily verify that $\rho_{f\circ g}(p,1) = \infty$ in Corollary 3 is essential.

Theorem 4. *Let f be meromorphic and g be entire such that $\rho_g(s,t) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$, where p, q, s, t are positive integer with $p > q$ and $s > t$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for any $R > r$*

$$(i) \limsup_{r\to\infty} \frac{\left\{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)\right\}^2}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right) \; \log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)} = 0, \;\; \text{if } q \geq s$$

and

$$(ii)\limsup_{r\to\infty}\frac{\log^{[p+s-q-2]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)\ \log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[p-2]}T_{f_m}\left(\exp^{[q-1]}r\right)\ \log^{[q]}M_{g_n}\left(\exp^{[t-1]}R\right)}=0,\ \textit{if } q<s.$$

Proof. From the definition of (p,q)-lower order of f_m we have for any positive ϵ and for all sufficiently large values of r,

$$\log^{[p-1]}T_{f_m}\left(\exp^{[q-1]}r\right)\geq\left(\lambda_{f_m}(p,q)-\epsilon\right)\log^{[q]}\left(\exp^{[q-1]}r\right)$$

$$\begin{aligned}
or,\log^{[p-1]}T_{f_m}\left(\exp^{[q-1]}r\right) &\geq \left(\lambda_{f_m}(p,q)-\epsilon\right)\log r\\
or,\log^{[p-2]}T_{f_m}\left(\exp^{[q-1]}r\right) &\geq r^{\left(\lambda_{f_m}(p,q)-\epsilon\right)}
\end{aligned}$$

$$(12)\qquad or,\log^{[p-2]}T_{f_m}\left(\exp^{[q-1]}r\right)\geq r^{\left(\lambda_f(p,q)-\epsilon\right)},\ \text{as } \lambda_{f_m}=\lambda_f.$$

Again by Lemma 2 and the inequality $T_g(r)\leq\log^+M_g(r)$, we get from (7) for all sufficiently large values of r that

$$\begin{aligned}
\log^{[p-1]}T_{f_m\circ g_n}(r) &\leq \log^{[p-1]}T_f\left(M_g(r)\right)+O(1)\\
or,\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right) &\leq \log^{[p-1]}T_f\left(M_g\left(\exp^{[t-1]}r\right)\right)+O(1)
\end{aligned}$$

$$or,\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)$$

$$(13)\qquad \leq\left(\rho_f(p,q)+\epsilon\right)\log^{[q]}M_g\left(\exp^{[t-1]}r\right)+O(1).$$

This implies on using (5) that for any $R>r$,

$$\begin{aligned}
\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[t-1]}R\right)} &\leq \frac{\left(\rho_f(p,q)+\epsilon\right)\log^{[q]}M_g\left(\exp^{[t-1]}r\right)+O(1)}{\log^{[q]}M_g\left(\exp^{[t-1]}R\right)+O(1)}\\
or,\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[t-1]}R\right)} &\leq \frac{\left(\rho_f(p,q)+\epsilon\right)\log^{[q]}M_g\left(\exp^{[t-1]}r\right)+O(1)}{\log^{[q]}M_g\left(\exp^{[t-1]}r\right)+O(1)}\\
or,\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[t-1]}R\right)} &\leq \left(\rho_f(p,q)+\epsilon\right)+\frac{O(1)}{\log^{[q]}M_g\left(\exp^{[t-1]}r\right)+O(1)}.
\end{aligned}$$

Since $\epsilon\,(>0)$ is arbitrary, therefore,

$$(14)\qquad \limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[t-1]}R\right)}\leq\rho_f(p,q).$$

We may consider the following two cases:

Case-I: Let $q \geq s$. Then from (13),

$$\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq (\rho_f(p,q)+\epsilon)\log^{[s-1]} M_g\left(\exp^{[t-1]} r\right) + O(1). \tag{15}$$

Now for all sufficiently large values of r,

$$\log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq (\rho_g(s,t)+\epsilon)\log^{[t]}\left(\exp^{[t-1]} r\right)$$
$$or, \log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq (\rho_g(s,t)+\epsilon)\log r$$
$$or, \log^{[s-1]} M_g\left(\exp^{[t-1]} r\right) \leq r^{(\rho_g(s,t)+\epsilon)}. \tag{16}$$

Therefore, from (15) and (16) we obtain

$$\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq (\rho_f(p,q)+\epsilon)\, r^{(\rho_g(s,t)+\epsilon)} + O(1). \tag{17}$$

Again by (12) and (17),

$$\frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)} \leq \frac{(\rho_f(p,q)+\epsilon)\, r^{(\rho_g(s,t)+\epsilon)} + O(1)}{r^{\lambda_f(p,q)-\epsilon}}$$
$$\leq (\rho_f(p,q)+\epsilon)\,.\frac{1}{r^{2\left(\frac{\lambda_f(p,q)\,-\,\rho_g(s,t)}{2}-\epsilon\right)}} + \frac{O(1)}{r^{\lambda_f(p,q)-\epsilon}}. \tag{18}$$

Since $\rho_g(s,t) < \lambda_f(p,q)$, we can choose $\epsilon\,(>0)$ in such a way that

$$0 < \epsilon < \frac{\lambda_f(p,q)-\rho_g(s,t)}{2}$$
$$i.e, 0 < \rho_g(s,t)+\epsilon < \lambda_f(p,q)-\epsilon. \tag{19}$$

Applying (19) we get from (18) that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)} = 0. \tag{20}$$

Finally, from (14) and (20) we have

$$\limsup_{r\to\infty} \frac{\left\{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)\right\}^2}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)\ \log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)}$$
$$\leq \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)}.\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)}$$
$$= \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)}.\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)}$$
$$\leq 0.\rho_f(p,q) = 0.$$

This proves the first part.

Case-II: Let $q < s$. Then from (13) we have for all sufficiently large values of r that

$$\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq \left(\rho_f(p,q)+\epsilon\right)\exp^{[s-q]}\log^{[s]} M_g\left(\exp^{[t-1]} r\right)+O(1). \tag{21}$$

Now for all sufficiently large values of r

$$\log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq \left(\rho_g(s,t)+\epsilon\right)\log^{[t]}\exp^{[t-1]} r$$

$$or, \log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq \left(\rho_g(s,t)+\epsilon\right)\log r$$

$$or, \exp^{[s-q]}\log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq \exp^{[s-q]}\log r^{(\rho_g(s,t)+\epsilon)}$$

$$or, \exp^{[s-q]}\log^{[s]} M_g\left(\exp^{[t-1]} r\right) \leq \exp^{[s-q-1]} r^{(\rho_g(s,t)+\epsilon)}. \tag{22}$$

Therefore from (21) and (22),

$$\log^{[p-1]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq \left(\rho_f(p,q)+\epsilon\right)\exp^{[s-q-1]} r^{(\rho_g(s,t)+\epsilon)}+O(1)$$

$$or, \log^{[p]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq \exp^{[s-q-2]} r^{(\rho_g(s,t)+\epsilon)}+O(1)$$

$$or, \log^{[p+s-q-2]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right) \leq r^{(\rho_g(s,t)+\epsilon)}+O(1). \tag{23}$$

Again, combining (12) and (23) we obtain

$$\begin{aligned}\frac{\log^{[p+s-q-2]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)} &\leq \frac{r^{(\rho_g(s,t)+\epsilon)}+O(1)}{r^{\lambda_f(p,q)-\epsilon}}\\ &\leq \frac{1}{r^{2\left(\frac{\lambda_f(p,q)\,-\,\rho_g(s,t)}{2}-\epsilon\right)}}+\frac{O(1)}{r^{\lambda_f(p,q)-\epsilon}},\end{aligned}$$

which further implies on using (19) that

$$\limsup_{r\to\infty}\frac{\log^{[p+s-q-2]} T_{f_m \circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)}=0. \tag{24}$$

Finally, from (14) and (24) we get

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p+s-q-2]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)\ \log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)\ \log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)} \\
\leq\ & \limsup_{r\to\infty} \frac{\log^{[p+s-q-2]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)} . \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)} \\
=\ & \limsup_{r\to\infty} \frac{\log^{[p+s-q-2]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)} . \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)} \\
\leq\ & 0.\rho_f(p,q) = 0.
\end{aligned}$$

This completes the proof. □

Remark 4. The condition $\rho_g(s,t) < \lambda_f(p,q)$ in Theorem 4 is necessary which is evident from the following example.

Example 2. Let $f = g = \exp z$ and $p = s = 2$, $q = t = 1$, $R = 2r$. Then, $\rho_g(s,t) = \lambda_f(p,q) = \rho_f(p,q) = 1$. Now $f_m(z) = \exp(z+m)$, $g_n(z) = \exp(z+n)$ and $(f_m o g_n)(z) = \exp(\exp(z+n)+m)$. Then, $T_{f_m\circ g_n}(r) \geq T_{f_m\circ g_n}\left(\frac{r}{2}\right) \sim \frac{\exp\left(\frac{r+n}{2}\right)}{\left(2\pi^3 \frac{r+n}{2}\right)^{\frac{1}{2}}} + O(1)\ (r\to\infty)$, $T_{f_m}(r) \leq \log M_{f_m}(r) = r + m$. Therefore,

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p+s-q-2]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)\ \log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[p-2]} T_{f_m}\left(\exp^{[q-1]} r\right)\ \log^{[q]} M_{g_n}\left(\exp^{[t-1]} R\right)} \\
=\ & \limsup_{r\to\infty} \frac{\{\log T(r, f_m\circ g_n)\}^2}{T(r,f_m)\ \log M_{g_n}(2r)} \\
\geq\ & \limsup_{r\to\infty} \frac{\left\{\frac{r+n}{2} - \frac{1}{2}\log(r+n) + O(1)\right\}^2}{(r+m)(2r+n)} \\
=\ & \frac{1}{8} \neq 0,
\end{aligned}$$

which is contrary to Theorem 4.

Theorem 5. *Let f be meromorphic and g be entire such that $\rho_f(p,q) < \infty$ and $\rho_{f\circ g}(s,t) < \infty$, where p,q,s,t are all positive integers with $p > q$ and $s > t$. Also, let $0 < \lambda_g < \infty$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for any two positive integers a, b with $a - b = 1, a > 2$ and any $R > r$,*

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right)\ \log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[b-1]} R\right) \log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right)} \\
\leq\ & \rho_f(p,q) . \rho_{f\circ g}(s,t) .
\end{aligned}$$

Proof. From (7) for any positive ϵ and for all sufficiently large values of r, we have

$$\begin{aligned}\log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right) &\leq \log^{[s-1]} T_{f\circ g}\left(\exp^{[t-1]} r\right) + O(1)\\ or,\quad \log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right) &\leq \left(\rho_{f\circ g}(s,t)+\epsilon\right)\log^{[t]}\exp^{[t-1]} r + O(1)\end{aligned}$$

$$or,\quad \log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right) \leq \left(\rho_{f\circ g}(s,t)+\epsilon\right)\log r + O(1). \tag{25}$$

Again for all sufficiently large values of r,

$$\begin{aligned}\log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right) &\geq \left(\lambda_{g_n}(a,b)-\epsilon\right)\log^{[b]}\exp^{[b-1]} r\\ or,\quad \log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right) &\geq \left(\lambda_{g}(a,b)-\epsilon\right)\log^{[b]}\exp^{[b-1]} r \text{ , as } \lambda_{g_n}=\lambda_g\end{aligned}$$

$$or,\quad \log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right) \geq \left(\lambda_{g}(a,b)-\epsilon\right)\log r. \tag{26}$$

Combining (25) and (26),

$$\frac{\log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right)} \leq \frac{\left(\rho_{f\circ g}(s,t)+\epsilon\right)\log r + O(1)}{\left(\lambda_{g}(a,b)-\epsilon\right)\log r}.$$

Since $\epsilon\,(>0)$ is arbitrary, therefore,

$$\limsup_{r\to\infty}\frac{\log^{[s-1]} T_{f_m\circ g_n}\left(\exp^{[t-1]} r\right)}{\log^{[a-1]} T_{g_n}\left(\exp^{[b-1]} r\right)} \leq \frac{\rho_{f\circ g}(s,t)}{\lambda_g(a,b)}. \tag{27}$$

Again by Lemma 2 and the inequality $T_g(r) \leq \log^+ M_g(r)$, we get from (7) for all sufficiently large values of r that

$$\begin{aligned}\log^{[p-1]} T_{f_m\circ g_n}(r) &\leq \log^{[p-1]} T_f\left(M_g(r)\right) + O(1)\\ or,\quad \log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right) &\leq \log^{[p-1]} T_f\left(M_g\left(\exp^{[b-1]} r\right)\right) + O(1)\end{aligned}$$

$$or,\quad \log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right) \leq \left(\rho_f(p,q)+\epsilon\right)\log^{[q]} M_g\left(\exp^{[b-1]} r\right) + O(1).$$

In view of (5), for any $R>r$ the above inequality implies that

$$\begin{aligned}\frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[b-1]} R\right)} &\leq \frac{\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} M_g\left(\exp^{[b-1]} r\right)+O(1)}{\log^{[q]} M_g\left(\exp^{[b-1]} R\right)+O(1)}\\ or,\quad \frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[b-1]} R\right)} &\leq \frac{\left(\rho_f(p,q)+\epsilon\right)\log^{[q]} M_g\left(\exp^{[b-1]} r\right)+O(1)}{\log^{[q]} M_g\left(\exp^{[b-1]} r\right)+O(1)}\\ or,\quad \frac{\log^{[p-1]} T_{f_m\circ g_n}\left(\exp^{[b-1]} r\right)}{\log^{[q]} M_{g_n}\left(\exp^{[b-1]} R\right)} &\leq \left(\rho_f(p,q)+\epsilon\right) + \frac{O(1)}{\log^{[q]} M_g\left(\exp^{[b-1]} r\right)+O(1)}.\end{aligned}$$

Since $\epsilon\,(>0)$ is arbitrary, therefore,

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[b-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[b-1]}R\right)}\le\rho_f\left(p,q\right). \tag{28}$$

Then, from (27) and (28) we obtian

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[b-1]}r\right).\log^{[s-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[b-1]}R\right).\log^{[a-1]}T_{g_n}\left(\exp^{[b-1]}r\right)}\\ \le\;&\limsup_{r\to\infty}\frac{\log^{[s-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[a-1]}T_{g_n}\left(\exp^{[b-1]}r\right)}.\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(\exp^{[b-1]}r\right)}{\log^{[q]}M_{g_n}\left(\exp^{[b-1]}R\right)}\\ \le\;&\frac{\rho_{f\circ g}\left(s,t\right)}{\lambda_g\left(a,b\right)}.\rho_f\left(p,q\right).\end{aligned} \tag{29}$$

Since $\lambda_g^{[2]}=\lambda_g$ is non-zero finite and $a-b=1$, therefore by Lemma 3 we get $\lambda_g\left(a,b\right)=1$. Hence, the proof. □

Corollary 4. *Under the same conditions of Theorem* 5 *when* $a=2$,

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(r\right).\log^{[s-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(R\right).\log T_{g_n}\left(r\right)}\le\frac{\rho_{f\circ g}\left(s,t\right).\rho_f\left(p,q\right)}{\lambda_g}.$$

Remark 5. The condition $\lambda_g>0$ in Corollary 4 is essential as we see in the following example:

Example 3. Let $f=\exp z$ and $g=z$. Also, let $a=2$ and $p=q=s=t=1, R=2r$. Then, $\lambda_g=0$ and $\rho_f=\rho_{f\circ g}=1$. Now $f_m\left(z\right)=\exp\left(z+m\right),\ g_n\left(z\right)=z+n$ and $\left(f_m o g_n\right)\left(z\right)=\exp\left(z+n+m\right)$.So, $T_{f_m\circ g_n}\left(r\right)=\frac{r+n+m}{\pi}+O\left(1\right), M_{g_n}\left(r\right)\sim r+n$ and $T_{g_n}\left(r\right)=\log\left(r+n\right)+O\left(1\right)$. Hence,

$$\begin{aligned}&\limsup_{r\to\infty}\frac{\log^{[p-1]}T_{f_m\circ g_n}\left(r\right).\log^{[s-1]}T_{f_m\circ g_n}\left(\exp^{[t-1]}r\right)}{\log^{[q]}M_{g_n}\left(R\right).\log T_{g_n}\left(r\right)}\\ =\;&\limsup_{r\to\infty}\frac{\left(T_{f_m\circ g_n}\left(r\right)\right)^2}{\log M_{g_n}\left(2r\right).\log T_{g_n}\left(r\right)}\\ =\;&\limsup_{r\to\infty}\frac{\left(\frac{r+n+m}{\pi}+O\left(1\right)\right)^2}{\log\left(2r+n\right)\ \left(\log\left(r+n\right)+O\left(1\right)\right)}\\ =\;&\infty,\end{aligned}$$

which is contrary to Corollary 4.

Theorem 6. *Let f be meromorphic and g be entire such that $\rho_f\left(p,q\right)<\infty$, where p,q are any two positive integers with $p>q$. If $f_m\left(z\right)=\left\{f\left(z+m\right):m\in\mathbb{N}\right\}$ and*

$g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, *then for any* $R > r$,

$$(i)\ \ \limsup_{r\to\infty} \frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} \leq 1$$

and

$$(ii)\ \ \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} \leq \rho_f(p,q).$$

Proof. Using Lemma 2 and the inequality $T_g(r) \leq \log^+ M_g(r)$, we get from (7) for any positive ϵ and for all sufficiently large values of r that

$$\log^{[p-1]} T_{f_m \circ g_n}(r) \leq \log^{[p-1]} T_f(M_g(r)) + O(1)$$

$$or,\ \ \log^{[p-1]} T_{f_m \circ g_n}(r) \leq (\rho_f(p,q) + \epsilon) \log^{[q]} M_g(r) + O(1) \tag{30}$$

$$or, \log^{[p]} T_{f_m \circ g_n}(r) \leq \log^{[q+1]} M_g(r) + O(1).$$

In view of (5), this implies that

$$\frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} \leq \frac{\log^{[q+1]} M_g(r) + O(1)}{\log^{[q+1]} M_g(R) + O(1)}$$

and for any $R > r$,

$$\limsup_{r\to\infty} \frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} \leq 1.$$

Again, from (5) and (30) for any $R > r$ we obtain

$$\frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} \leq \frac{(\rho_f(p,q) + \epsilon) \log^{[q]} M_g(r) + O(1)}{\log^{[q]} M_g(r) + O(1)}.$$

Since $\epsilon\,(> 0)$ is arbitrary, therefore,

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} \leq \rho_f(p,q).$$

Hence, the results. □

Remark 6. The condition $\rho_f(p,q) < \infty$ in Theorem 6 is necessary which is evident from the following example:

Example 4. Let $f = \exp^{[2]} z$, $g = \exp z$ and $p = 2$, $q = 1, R = 2r$. Then $\rho_f(p,q) = \infty$. Now $f_m(z) = \exp^{[2]}(z+m)$, $g_n(z) = \exp(z+n)$ and $(f_m \circ g_n)(z) = \exp^{[2]}(\exp(z+n) + m)$ .Therefore,

$$
\begin{aligned}
\log T_{f_m \circ g_n}(r) &\geq \log^{[2]} M_{f_m \circ g_n}(\frac{r}{2}) + O(1) \\
or, \quad \log T_{f_m \circ g_n}(r) &\geq \log^{[2]} \left(\exp^{[2]} \left(\exp\left(\frac{r}{2}+n\right)+m\right)\right) + O(1) \\
or, \quad \log T_{f_m \circ g_n}(r) &\geq \exp\left(\frac{r}{2}+n\right) + O(1) \\
or, \quad \log^{[2]} T_{f_m \circ g_n}(r) &\geq \left(\frac{r}{2}+n\right) + O(1)
\end{aligned}
$$

and

$$
\begin{aligned}
\log M_{g_n}(R) &= \log M_{g_n}(2r+n) = (2r+n) + O(1) \\
or, \log^{[2]} M_{g_n}(R) &= \log^{[2]} M_{g_n}(2r) = \log(2r+n) + O(1).
\end{aligned}
$$

So,

$$
\begin{aligned}
\limsup_{r\to\infty} \frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} &= \limsup_{r\to\infty} \frac{\log^{[2]} T_{f_m \circ g_n}(r)}{\log^{[2]} M_{g_n}(R)} \\
&\geq \limsup_{r\to\infty} \frac{\left(\frac{r}{2}+n\right)+O(1)}{\log(2r+n)+O(1)} \\
&= \infty
\end{aligned}
$$

$$
\Rightarrow \limsup_{r\to\infty} \frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} = \infty
$$

and

$$
\begin{aligned}
\limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} &= \limsup_{r\to\infty} \frac{\log T_{f_m \circ g_n}(r)}{\log M_{g_n}(R)} \\
&\geq \limsup_{r\to\infty} \frac{\exp\left(\frac{r}{2}+n\right)+O(1)}{(2r+n)+O(1)} \\
&= \infty
\end{aligned}
$$

$$
\Rightarrow \limsup_{r\to\infty} \frac{\log^{[p-1]} T_{f_m \circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} = \infty,
$$

are contrary to Theorem 6.

In the line of Theorem 6 the following theorem can be deduced.

Theorem 7. *Let f be meromorphic and g be entire such that $\lambda_f(p,q) < \infty$, where p, q are any two positive integers with $p > q$. If $f_m(z) = \{f(z+m) : m \in \mathbb{N}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{N}\}$, then for any $R > r$,*

$$
(i) \quad \liminf_{r\to\infty} \frac{\log^{[p]} T_{f_m \circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} \leq 1
$$

and

$$(ii)\quad \liminf_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} \leq \lambda_f(p,q).$$

Remark 7. The condition $\lambda_f(p,q) < \infty$ in Theorem 7 is necessary which is evident from the following example:

Example 5. Let $f = \exp^{[2]} z$, $g = \exp z$ and $p = 2$, $q = 1, R = 2r$. Then $\lambda_f(p,q) = \infty$. Now $f_m(z) = \exp^{[2]}(z+m)$, $g_n(z) = \exp(z+n)$ and $(f_m \circ g_n)(z) = \exp^{[2]}(\exp(z+n)+m)$. Therefore,

$$\begin{aligned}
\log T_{f_m\circ g_n}(r) &\geq \log^{[2]} M_{f_m\circ g_n}(\frac{r}{2}) + O(1)\\
or,\ \log T_{f_m\circ g_n}(r) &\geq \log^{[2]}\left(\exp^{[2]}\left(\exp\left(\frac{r}{2}+n\right)+m\right)\right) + O(1)\\
or,\ \log T_{f_m\circ g_n}(r) &\geq \exp\left(\frac{r}{2}+n\right) + O(1)\\
or,\ \log^{[2]} T_{f_m\circ g_n}(r) &\geq \left(\frac{r}{2}+n\right) + O(1)
\end{aligned}$$

and

$$\begin{aligned}
\log M_{g_n}(R) &= \log M_{g_n}(2r+n) = (2r+n) + O(1)\\
or, \log^{[2]} M_{g_n}(R) &= \log^{[2]} M_{g_n}(2r+n) = \log(2r+n) + O(1).
\end{aligned}$$

So,

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[p]} T_{f_m\circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} &= \liminf_{r\to\infty} \frac{\log^{[2]} T_{f_m\circ g_n}(r)}{\log^{[2]} M_{g_n}(R)}\\
&\geq \liminf_{r\to\infty} \frac{\left(\frac{r}{2}+n\right)+O(1)}{\log(2r+n)+O(1)}\\
&= \infty \Rightarrow \liminf_{r\to\infty} \frac{\log^{[p]} T_{f_m\circ g_n}(r)}{\log^{[q+1]} M_{g_n}(R)} = \infty
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} &= \liminf_{r\to\infty} \frac{\log T_{f_m\circ g_n}(r)}{\log M_{g_n}(R)}\\
&\geq \liminf_{r\to\infty} \frac{\left(\frac{r}{2}+n\right)+O(1)}{(2r+n)+O(1)}\\
&= \infty \Rightarrow \liminf_{r\to\infty} \frac{\log^{[p-1]} T_{f_m\circ g_n}(r)}{\log^{[q]} M_{g_n}(R)} = \infty,
\end{aligned}$$

are contrary to Theorem 7.

3. Concluding Remarks

We have used here a different approach to measure the comparative growth of composite entire and meromorphic functions. The same technique may be applied to investigate by using the notion of different extended and modified growth indicators. Very few research has been done in this topic. The topic deserves further study.

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5. Conflicts of Interest

The authors declare no conflict of interest.

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