J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2023.30.3.309 ISSN(Online) 2287-6081
Volume 30, Number 3 (August 2023), Pages 309–336

UTILIZING WEAK $\psi - \varphi$ CONTRACTION ON FUZZY METRIC SPACES

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Abstract. We establish some common fixed point theorems satisfying weak $\psi - \varphi$ contraction on partially ordered non-Archimedean fuzzy metric spaces. By using this results we show the existence of fixed point on the domain of words and apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and divide and Conquer algorithms, respectively and also give an example to show the usefulness of our hypothesis. Our results generalize, extend and improve several well-known results of the existing literature in fixed point theory.

1. Introduction

George and Veeramani [13] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [19] with the help of continuous t-norm and defined the Hausdorff topology of fuzzy metric spaces. In [18], Istratescu introduced the concept of non-Archimedean fuzzy metric space.

In [15], Guo and Lakshmikantham introduced the notion of coupled fixed point for single-valued mappings. Using this notion, Gnana-Bhaskar and Lakshmikantham [2] established some coupled fixed point theorems by defining mixed monotone property. After that, Lakshmikantham and Ciric [20] extended the notion of mixed monotone property to mixed $g-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Gnana-Bhaskar and Lakshmikantham [2]. Later on, Choudhury and Kundu [3] improved the results of Lakshmikantham and Ciric [20], by defining the notion of compatibility in the context of coupled coincidence point. Subsequently

Received by the editors October 06, 2022. Revised June 05, 2023. Accepted July 08, 2023.
2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, coincidence point, coupled fixed point, coupled coincidence point, weak ψ-φ contraction, partially ordered non-Archimedean fuzzy metric spaces, domain of words.

Hussain et al. [17] introduced a new concept of generalized compatibility of a pair of mappings F, $G : X^2 \to X$ defined on a product space and proved some coupled coincidence point results. Hussain et al. [17] also deduce some coupled fixed point results without mixed monotone property. Erhan et al. [12], announced that the results were established in Hussain et al. [17] can be derived from the coincidence point results in the literature. In [7], Deshpande and Handa introduced the concepts of generalized compatibility and generalized weakly compatibility for the pair $\{F, G\}$ of mappings F, $G : X^2 \to X$ in the setting of fuzzy metric space and also introduced the concept of common fixed point of the mappings F, $G : X^2 \to X$. Deshpande and Handa [7], proved a common fixed point theorem for generalized weakly compatible pair F, $G : X^2 \to X$, without mixed monotone property of any of the mappings, on a non complete fuzzy metric space, which is not partially ordered. For more details one can consult [1, 5 − 12, 16, 17].

On the other hand Gordji et al. [14] proved some fixed point theorems for (ψ, φ)-weak contractive mappings in a complete metric space on a partially ordered metric space.

In this paper, we establish some common fixed point theorems satisfying weak $\psi-\varphi$ contraction on partially ordered non-Archimedean fuzzy metric spaces. With the help of the results established in the first section, we obtain some multidimensional fixed point results. By using this results we show the existence of fixed point on the domain of words and apply this approach to deduce the existence of solution for some recurrence equations associated to the analysis of Quicksort algorithms and divide and Conquer algorithms, respectively and also give an example to show the degree of validity of our hypothesis. We generalize, extend, improve and fuzzify the results of Gordji et al. [14] and several well-known results in the recent literature.

2. PRELIMINARIES

Definition 2.1 ([22]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

A few examples of continuous t-norm are

$$a * b = ab, \ a * b = \min\{a, \ b\} \text{ and } a * b = \max\{a + b - 1, \ 0\}.$$

Definition 2.2 ([13])**.** A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for each $x, y, z \in X$ and $t, s > 0$,
$(FM-1)$ $M(x, y, t) > 0$,
$(FM-2)$ $M(x, y, t) = 1$ iff $x = y$,
$(FM-3)$ $M(x, y, t) = M(y, x, t)$,
$(FM-4)$ $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$,
$(FM-5)$ $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

Remark 2.1. Assume that in the above definition (FM-4) is replaced by

$$(NAFM-4)\ M(x,\ z,\ \max\{t,\ s\}) \geq M(x,\ y,\ t) * M(y,\ z,\ s),$$

or equivalently,

$$(NAFM-4)\ M(x,\ z,\ t) \geq M(x,\ y,\ t) * M(y,\ z,\ t).$$

Then $(X, M, *)$ is called a non-Archimedean fuzzy metric space [18]. It is easy to check that (NAFM-4) implies (FM-4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 2.1 ([13])**.** Let (X, d) be a metric space. Define t-norm by $a * b = ab$ and

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)} \text{ for all } x,\ y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this *fuzzy metric* M induced by the metric d the standard fuzzy metric.

Remark 2.2 ([13])**.** In fuzzy metric space $(X, M, *)$, $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Definition 2.3 ([13])**.** Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_n$ in X is called *Cauchy* if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that

$$M(x_n,\ x_m,\ t) > 1 - \varepsilon \text{ whenever } n \geq m \geq n_0.$$

We say that $(X, M, *)$ is *complete* if every Cauchy sequence is convergent, that is, there exists $y \in X$ such that $\lim_{n\to\infty} M(x_n, y, t) = 1$, for all $t > 0$.

Definition 2.4 ([2])**.** Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.5 ([2])**.** Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$ be a given mapping. We say that F has the *mixed monotone property* if for all $x, y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),\\ y_1,\ y_2 &\in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

Definition 2.6 ([20])**.** Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *coupled coincidence point* of the mappings F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.7 ([20])**.** Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a *common coupled fixed point* of the mappings F and g if $x = F(x, y) = gx$ and $y = F(y, x) = gy$.

Definition 2.8 ([20])**.** The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be *commutative* if $gF(x, y) = F(gx, gy)$, for all $(x, y) \in X^2$.

Definition 2.9 ([20])**.** Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the *mixed* $g-$ *monotone property* if for all $x, y \in X$, we have

$$\begin{aligned} x_1,\ x_2 &\in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),\\ y_1,\ y_2 &\in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2). \end{aligned}$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.10 ([3])**.** Mappings $F : X^2 \to X$ and $g : X \to X$ are said to be *compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(gF(x_n,\ y_n),\ F(gx_n,\ gy_n)) &= 0,\\ \lim_{n\to\infty} d(gF(y_n,\ x_n),\ F(gy_n,\ gx_n)) &= 0, \end{aligned}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} gx_n = x \in X,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} gy_n = y \in X. \end{aligned}$$

Definition 2.11 ([17])**.** Suppose that $F, G : X^2 \to X$ are two mappings. F is said to be $G-$ *increasing*with $G(x, y) \preceq G(u, v)$ we have $F(x, y) \preceq F(u, v)$.

Definition 2.12 ([17])**.** Suppose that $F,\ G : X^2 \to X$ are two mappings. An element $(x,\ y) \in X^2$ is called a *coupled coincidence point* of mappings F and G if $F(x,\ y) = G(x,\ y)$ and $F(y,\ x) = G(y,\ x)$.

Definition 2.13 ([17])**.** Let $(X,\ \preceq)$ be a partially ordered set and $F : X^2 \to X$ and $g : X \to X$ be two mappings. We say that F is $g-$*increasing* with respect to $\preceq$ if for any $x,\ y \in X$,

$$\begin{aligned} gx_1 &\preceq gx_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \\ gy_1 &\preceq gy_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.14 ([17])**.** Let $(X,\ \preceq)$ be a partially ordered set and $F : X^2 \to X$ be a mapping. We say that F is *increasing* with respect to $\preceq$ if for any $x,\ y \in X$,

$$\begin{aligned} x_1 &\preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y), \\ y_1 &\preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2). \end{aligned}$$

Definition 2.15 ([17])**.** Let $F,\ G : X^2 \to X$ be two mappings. We say that the pair $\{F,\ G\}$ is *generalized compatible* if

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0, \end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x \in X, \\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y \in X. \end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Definition 2.16 ([2, 12])**.** An ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$. An ordered metric space $(X,\ d,\ \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X,\ d,\ \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

Definition 2.17 ([12])**.** Let $(X,\ \preceq)$ be a partially ordered set and $T,\ g : X \to X$ be two mappings. We say that T is $(g,\ \preceq)-$*non-decreasing* if $Tx \preceq Ty$ for all x,

$y \in X$ such that $gx \preceq gy$. If g is the identity mapping on X, we say that T is $\preceq -$*non-decreasing*. If T is $(g, \preceq)-$non-decreasing and $gx = gy$, then $Tx = Ty$.

Definition 2.18 ([12])**.** Let $(X, d, \preceq)$ be an ordered metric space. Two mappings $T, g : X \to X$ are said to be $O-$*compatible* if $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$, provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq -$monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Definition 2.19 ([6])**.** Let X be a non-empty set. Mappings $F, G : X^2 \to X$ are called *generalized weakly compatible mappings* if $F(x, y) = G(x, y)$ and $F(y, x) = G(y, x)$ imply that $G(F(x, y), F(y, x)) = F(G(x, y), G(y, x))$ and $G(F(y, x), F(x, y)) = F(G(y, x), G(x, y))$, for all $x, y \in X$.

Let Ψ be the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following:
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$.
Let Φ be the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following:
(i_φ) φ is lower semi-continuous and non-decreasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$.
and the class Θ of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying the following:
(i_θ) θ is continuous,
(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

3. Fixed Point Results

In the sequel, X is a non-empty set and $g : X \to X$ is a mapping. For simplicity, we denote $g(x)$ by gx where $x \in X$.

Definition 3.1. Let $(X, M, *)$ be an ordered fuzzy metric space. Two mappings $T, g : X \to X$ are said to be $O-$*compatible* if

$$\lim_{n\to\infty} M(gTx_n,\ Tgx_n,\ t) = 1,$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq -$monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a non-Archimedean fuzzy metric space. Suppose $T, g : X \to X$ are two mappings satisfying* *(i) T is $(g, \preceq)-$non-decreasing and $T(X) \subseteq g(X)$,*

(ii) there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,

(iii) there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that

$$\begin{aligned} \psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \\ \leq\ \psi\left(\frac{1}{A(x,\ y)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(x,\ y)} - 1\right)\right) + \theta\left(\frac{1}{B(x,\ y)} - 1\right), \end{aligned} \tag{3.1}$$

where

$$A(x,\ y) = \min\{M(gx,\ gy,\ t),\ M(gx,\ Tx,\ t),\ M(gy,\ Ty,\ t)\}, \tag{3.2}$$

and

$$B(x,\ y) = \max\{M(gy,\ Tx,\ t),\ M(gy,\ Ty,\ t)\}, \tag{3.3}$$

for all x, $y \in X$ with $gx \preceq gy$. Also assume that, at least, one of the following conditions holds.

(a) (X, M) is complete, T and g are continuous and the pair (T, g) is $O-$compatible,

(b) $(g(X), M)$ is complete and $(X, M, \preceq)$ is non-decreasing-regular,

(c) (X, M) is complete, g is continuous and monotone-non-decreasing, the pair (T, g) is $O-$compatible and $(X, M, \preceq)$ is non-decreasing-regular.

Then T and g have a coincidence point. Moreover

(iv) for every x, $y \in X$ there exists $u \in X$ such that Tu is comparable to Tx and Ty, and also the pair (T, g) is weakly compatible.

Then T and g have a unique common fixed point.

Proof. Since $x_0 \in X$ and by (i), we have $Tx_0 \in T(X) \subseteq g(X)$, therefore there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Now, by (ii), we have $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g, \preceq)-$non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g, \preceq)-$non-decreasing, $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n \geq 0}$ such that $\{gx_n\}$ is $\preceq -$non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{3.4}$$

First we claim that $\{M(gx_n,\ gx_{n+1},\ t)\} \to 1$. Suppose that for each $n \in \mathbb{N}$, $M(gx_n,\ gx_{n+1},\ t) < 1$. It is clear that $B(x_n,\ x_{n+1}) = 1$ for all $n \in \mathbb{N}$. Now, by using contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have

$$\begin{aligned} & \psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right) \\ = & \ \psi\left(\frac{1}{M(Tx_n,\ Tx_{n+1},\ t)}-1\right) \\ \leq & \ \psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} & \psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right) \\ \leq & \ \psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)\right), \end{aligned} \tag{3.5}$$

which by the fact that $\varphi \geq 0$ implies

$$\psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right) \leq \psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right).$$

Since ψ is non-decreasing, we obtain

$$\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1 \leq \frac{1}{A(x_n,\ x_{n+1})}-1. \tag{3.6}$$

Again

$$\begin{aligned} & A(x_n,\ x_{n+1}) \\ = & \ \min\{M(gx_n,\ gx_{n+1},\ t),\ M(gx_n,\ Tx_n,\ t),\ M(gx_{n+1},\ Tx_{n+1},\ t)\} \\ = & \ \min\{M(gx_n,\ gx_{n+1},\ t),\ M(gx_{n+1},\ gx_{n+2},\ t)\}. \end{aligned}$$

Assume that $M(gx_{n+1}, gx_{n+2}, t) \leq M(gx_n, gx_{n+1}, t)$. Then

$$A(x_n,\ x_{n+1}) = M(gx_{n+1},\ gx_{n+2},\ t). \tag{3.7}$$

Thus, by (3.5) and (3.7), we have

$$\begin{aligned} & \psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right) \\ \leq & \ \psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right)-\varphi\left(\psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right)\right), \end{aligned}$$

which is only possible when $M(gx_{n+1}, gx_{n+2}, t) = 1$, it is a contradiction. Hence, $M(gx_n, gx_{n+1}, t) \leq M(gx_{n+1}, gx_{n+2}, t)$. Then

$$A(x_n,\ x_{n+1}) = M(gx_n,\ gx_{n+1},\ t). \tag{3.8}$$

Thus, by (3.6), we get

$$\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\leq\frac{1}{M(gx_n,\ gx_{n+1},\ t)}-1. \tag{3.9}$$

This shows that the sequence $\{R_n\}_{n\geq 0}$ defined by

$$R_n=\frac{1}{M(gx_n,\ gx_{n+1},\ t)}-1, \tag{3.10}$$

is a non-increasing sequence. Thus there exists $R\geq 0$ such that

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left(\frac{1}{M(gx_n,\ gx_{n+1},\ t)}-1\right)=R. \tag{3.11}$$

Then

$$\lim_{n\to\infty}\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)=R. \tag{3.12}$$

We shall prove that $R=0$. Assume to the contrary that $R>0$. Now, by using contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have

$$\begin{aligned}
&\psi\left(\frac{1}{M(gx_{n+1},\ gx_{n+2},\ t)}-1\right)\\
=\ &\psi\left(\frac{1}{M(Tx_n,\ Tx_{n+1},\ t)}-1\right)\\
\leq\ &\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)\right).
\end{aligned}$$

Thus

$$\psi\left(R_{n+1}\right)\leq\psi\left(R_n\right)-\varphi\left(\psi\left(R_n\right)\right).$$

Letting $n\to\infty$ in the above inequality, by using (i_ψ), (i_φ), (3.11) and (3.12), we get

$$\psi\left(R\right)\leq\psi\left(R\right)-\varphi\left(\psi\left(R\right)\right),$$

which is only possible when $R=0$. Thus

$$R=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left(\frac{1}{M(gx_n,\ gx_{n+1},\ t)}-1\right)=0,$$

or

$$\lim_{n\to\infty}M(gx_n,\ gx_{n+1},\ t)=1. \tag{3.13}$$

Now we claim that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon>0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for all positive integers k such that

$$M(gx_{n(k)},\ gx_{m(k)},\ t)\leq 1-\varepsilon \text{ for } n(k)>m(k)>k. \tag{3.14}$$

Assume that $n(k)$ is the smallest such positive integer. Then

$$M(gx_{n(k)-1},\ gx_{m(k)},\ t) > 1-\varepsilon. \tag{3.15}$$

Now, by (3.14), (3.15) and (NAFM-4), we have

$$\begin{aligned} 1-\varepsilon &\geq M(gx_{n(k)},\ gx_{m(k)},\ t) \\ &\geq M(gx_{n(k)},\ gx_{n(k)-1},\ t) * M(gx_{n(k)-1},\ gx_{m(k)},\ t) \\ &> M(gx_{n(k)},\ gx_{n(k)-1},\ t) * (1-\varepsilon). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (3.13), we have

$$\lim_{k\to\infty} M(gx_{n(k)},\ gx_{m(k)},\ t) = 1-\varepsilon. \tag{3.16}$$

By (NAFM-4), we have

$$\begin{aligned} &M(gx_{n(k)+1},\ gx_{m(k)+1},\ t) \\ \geq\ &M(gx_{n(k)+1},\ gx_{n(k)},\ t) * M(gx_{n(k)},\ gx_{m(k)},\ t) * M(gx_{m(k)},\ gx_{m(k)+1},\ t). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (3.13) and (3.16), we have

$$\lim_{k\to\infty} M(gx_{n(k)+1},\ gx_{m(k)+1},\ t) = 1-\varepsilon. \tag{3.17}$$

As $n(k) > m(k)$, so $gx_{n(k)} \succeq gx_{m(k)}$, by using contractive condition (3.1), we have

$$\begin{aligned} &\psi\left(\frac{1}{M(gx_{n(k)+1},\ gx_{m(k)+1},\ t)} - 1\right) \\ =\ &\psi\left(\frac{1}{M(Tx_{n(k)},\ Tx_{m(k)},\ t)} - 1\right) \\ \leq\ &\psi\left(\frac{1}{A(x_{n(k)},\ x_{m(k)})} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(x_{n(k)},\ x_{m(k)})} - 1\right)\right) \\ &+\theta\left(\frac{1}{B(x_{n(k)},\ x_{m(k)})} - 1\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (3.16), (3.17), we have

$$\psi\left(\frac{\varepsilon}{1-\varepsilon}\right) \leq \psi\left(\frac{\varepsilon}{1-\varepsilon}\right) - \varphi\left(\psi\left(\frac{\varepsilon}{1-\varepsilon}\right)\right) < \psi\left(\frac{\varepsilon}{1-\varepsilon}\right),$$

which is a contradiction due to $\varepsilon > 0$. It means that $\{gx_n\}_{n\geq 0}$ is a Cauchy sequence in X.

We claim that T and g have a coincidence point distinguishing between cases $(a) - (c)$.

Suppose now that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is $O-$compatible. Since $(X,\ d)$ is complete, there exists $z \in X$ such that $\{gx_n\} \to z$. By (3.4), we also have that $\{Tx_n\} \to z$. Since T and g are continuous, $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. By using the fact that the pair $(T,\ g)$ is $O-$compatible, we deduce that $\lim_{n\to\infty} M(gTx_n,\ Tgx_n,\ t) = 1$. In such a case, we conclude that

$$M(gz,\ Tz,\ t) = \lim_{n\to\infty} M(ggx_{n+1},\ Tgx_n,\ t) = \lim_{n\to\infty} M(gTx_n,\ Tgx_n,\ t) = 1,$$

that is, z is a coincidence point of T and g.

Suppose now that (b) holds, that is, $(g(X),\ M)$ is complete and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X),\ M)$, there exists $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$. In this case $\{gx_n\} \to gz$. Indeed, since $(X,\ M,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$ $-$non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (3.1) and by the monotonicity of ψ, we get

$$\begin{aligned}
(3.18)\qquad & \psi\left(\frac{1}{M(gx_{n+1},\ Tz,\ t)} - 1\right) \\
&= \psi\left(\frac{1}{M(Tx_n,\ Tz,\ t)} - 1\right) \\
&\leq \psi\left(\frac{1}{A(x_n,\ z)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(x_n,\ z)} - 1\right)\right) + \theta\left(\frac{1}{B(x_n,\ z)} - 1\right),
\end{aligned}$$

where

$$\begin{aligned}
A(x_n,\ z) &= \min\{M(gx_n,\ gz,\ t),\ M(gx_n,\ Tx_n,\ t),\ M(gz,\ Tz,\ t)\} \\
&= \min\{M(gx_n,\ gz,\ t),\ M(gx_n,\ gx_{n+1},\ t),\ M(gz,\ Tz,\ t)\},
\end{aligned}$$

and

$$\begin{aligned}
B(x_n,\ z) &= \max\{M(gz,\ Tx_n,\ t),\ M(gz,\ Tz,\ t)\} \\
&= \max\{M(gz,\ gx_{n+1},\ t),\ M(gz,\ Tz,\ t)\}.
\end{aligned}$$

Letting $n \to \infty$ in (3.18), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right) \leq \psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right) - \varphi\left(\psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right)\right),$$

which is possible only when $M(gz, Tz, t) = 1$, that is, z is a coincidence point of T and g.

Suppose now that (c) holds, that is, $(X,\ M)$ is complete, g is continuous and monotone non-decreasing, the pair $(T,\ g)$ is compatible and $(X,\ M,\ \preceq)$ is non-decreasing-regular. Since $(X,\ M)$ is complete, there exists $z \in X$ such that $\{gx_n\} \to z$. By (3.4), we also have that $\{Tx_n\} \to z$. Since g is continuous, $\{ggx_n\} \to gz$. Furthermore, since the pair $(T,\ g)$ is compatible, we have

$$\lim_{n\to\infty} M(ggx_{n+1},\ Tgx_n, t) = \lim_{n\to\infty} M(gTx_n,\ Tgx_n, t) = 1.$$

Since $\{ggx_n\} \to gz$, the previous property means that $\{Tgx_n\} \to gz$.

Indeed, since $(X,\ M,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$. It follows, from the monotonicity of g, that $ggx_n \preceq gz$. Applying the contractive condition (3.1), we get

$$\begin{aligned} &\psi\left(\frac{1}{M(Tgx_n,\ Tz,\ t)} - 1\right) \\ &\leq \psi\left(\frac{1}{A(gx_n,\ z)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(gx_n,\ z)} - 1\right)\right) + \theta\left(\frac{1}{B(gx_n,\ z)} - 1\right), \end{aligned} \tag{3.19}$$

where

$$A(gx_n,\ z) = \min\{M(ggx_n,\ gz,\ t),\ M(ggx_n,\ Tgx_n,\ t),\ M(gz,\ Tz,\ t)\},$$

and

$$B(gx_n,\ z) = \max\{M(gz,\ Tgx_n,\ t),\ M(gz,\ Tz,\ t)\}.$$

Letting $n \to \infty$ in (3.19), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right) \leq \psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right) - \varphi\left(\psi\left(\frac{1}{M(gz,\ Tz,\ t)} - 1\right)\right),$$

which is possible when $M(gz, Tz, t) = 1$, that is, z is a coincidence point of T and g.

Since the set of coincidence points of g and T is non-empty, assume that x and y are coincidence points of T and g, that is, $Tx = gx$ and $Ty = gy$. Now, we show that $gx = gy$. By the assumption, there exists $u \in X$ such that Tu is comparable with Tx and Ty. Put $u_0 = u$ and choose $u_1 \in X$ so that $gu_0 = Tu_1$. Then, we can inductively define a sequence $\{gu_n\}$ where $gu_{n+1} = Tu_n$ for all $n \geq 0$. Hence $Tx = gx$ and $Tu = Tu_0 = gu_1$ are comparable. Suppose that $gu_1 \preceq gx$ (the proof is similar to that in the other case). We claim that $gu_n \preceq gx$ for each $n \in \mathbb{N}$. In fact, we will use mathematical induction. Since $gu_1 \preceq gx$, our claim is true for $n = 1$.

We presume that $gu_n \preceq gx$ holds for some $n > 1$. Since T is $g-$non-decreasing with respect to $\preceq$, we get $gu_{n+1} = Tu_n \preceq Tx = gx$, and this proves our claim. Since

$gu_n \preceq gx$ and so by using contractive condition (3.1), we have

$$\begin{aligned}
&\psi\left(\frac{1}{M(gu_{n+1},\ gx,\ t)} - 1\right)\\
= \ & \psi\left(\frac{1}{M(Tu_n,\ Tx,\ t)} - 1\right)\\
\leq \ & \psi\left(\frac{1}{A(u_n,\ x)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(u_n,\ x)} - 1\right)\right) + \theta\left(\frac{1}{B(u_n,\ x)} - 1\right),
\end{aligned}$$

where

$$\begin{aligned}
A(u_n,\ x) &= \min\{M(gu_n,\ gx,\ t),\ M(gu_n,\ Tu_n,\ t),\ M(gx,\ Tx,\ t)\}\\
&= \min\{M(gu_n,\ gx,\ t),\ M(gu_n,\ gu_{n+1},\ t),\ M(gx,\ Tx,\ t)\},
\end{aligned}$$

and

$$B(u_n,\ x) = \max\{M(gx,\ Tu_n,\ t),\ M(gx,\ Tx,\ t)\} = 1.$$

Letting $n \to \infty$ in the above inequality and by using (ii_ψ) and (ii_φ), we get

$$\lim_{n\to\infty} M(gu_n,\ gx,\ t) = 1. \tag{3.20}$$

Similarly, one can prove that

$$\lim_{n\to\infty} M(gu_n,\ gy,\ t) = 0. \tag{3.21}$$

Hence, by (3.20) and (3.21), we get

$$gx = gy. \tag{3.22}$$

Since $gx = Tx$, by weak compatibility of g and T, we have $ggx = gTx = Tgx$. Let $z = gx$, then $gz = Tz$. Thus z is a coincidence point of g and T. Then from (3.22) with $y = z$, it follows that $gx = gz$, that is, $z = gz = Tz$. Therefore, z is a common fixed point of g and T. To prove the uniqueness, assume that w is another common fixed point of g and T. Then by (3.22) we have $w = gw = gz = z$. Hence the common fixed point of g and T is unique. □

If we put $\theta(t) = 0$ in Theorem 3.1, we get the following result:

Corollary 3.2. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a non-Archimedean fuzzy metric space. Suppose $T,\ g : X \to X$ are two mappings satisfying (i), (ii) of Theorem 3.1 and*

(iii) there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \leq \psi\left(\frac{1}{A(x,\ y)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(x,\ y)} - 1\right)\right),$$

where $A(x,\ y)$ is defined in (3.2), *for all $x,\ y \in X$ such that $gx \preceq gy$. Also assume that, at least, one of the conditions* $(a)-(c)$ *of Theorem 3.1 holds. Then T and g have a coincidence point. Moreover, assume that the condition* (iv) *of Theorem 3.1 holds. Then T and g have a unique common fixed point.*

If we put $\varphi(t) = t - t\varphi_1(t)$ for all $t \geq 0$ in Corollary 3.2, then we get the following result:

Corollary 3.3. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a non-Archimedean fuzzy metric space. Suppose $T,\ g : X \to X$ are two mappings satisfying* (i), (ii) *of Theorem 3.1 and*

(iii) *there exist $\varphi_1 \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \leq \varphi_1\left(\psi\left(\frac{1}{A(x,\ y)} - 1\right)\right)\psi\left(\frac{1}{A(x,\ y)} - 1\right),$$

where $A(x,\ y)$ is defined in (3.2), *for all $x,\ y \in X$ such that $gx \preceq gy$. Also assume that, at least, one of the conditions* $(a)-(c)$ *of Theorem 3.1 holds. Then T and g have a coincidence point. Moreover, assume that the condition* (iv) *of Theorem 3.1 holds. Then T and g have a unique common fixed point.*

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 3.3, then we get the following result:

Corollary 3.4. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a non-Archimedean fuzzy metric space. Suppose $T,\ g : X \to X$ are two mappings satisfying* (i), (ii) *of Theorem 3.1 and*

(iii) *there exists $\varphi_1 \in \Phi$ such that*

$$\frac{1}{M(Tx,\ Ty,\ t)} - 1 \leq \varphi_1\left(2\left(\frac{1}{A(x,\ y)} - 1\right)\right)\left(\frac{1}{A(x,\ y)} - 1\right),$$

where $A(x,\ y)$ is defined in (3.2), *for all $x,\ y \in X$ such that $gx \preceq gy$. Also assume that, at least, one of the conditions* $(a)-(c)$ *of Theorem 3.1 holds. Then T and g have a coincidence point. Moreover, assume that the condition* (iv) *of Theorem 3.1 holds. Then T and g have a unique common fixed point.*

If we put $\varphi_1(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 3.4, then we get the following result:

Corollary 3.5. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a non-Archimedean fuzzy metric space. Suppose $T,\ g : X \to X$ are two mappings satisfying* (i), (ii) *of Theorem 3.1 and* *(iii) there exists $k<1$ such that*

$$\frac{1}{M(Tx,\ Ty,\ t)}-1\leq k\left(\frac{1}{A(x,\ y)}-1\right),$$

where $A(x,\ y)$ is defined in (3.2), for all $x,\ y\in X$ such that $gx\preceq gy$ and $k<1$. Also assume that, at least, one of the conditions (a) – (c) of Theorem 3.1 holds. Then T and g have a coincidence point. Moreover, assume that the condition (iv) of Theorem 3.1 holds. Then T and g have a unique common fixed point.

If $g=I$ (the identity mapping) in Corollary 3.5, we get the following result:

Corollary 3.6. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Suppose $T:X\rightarrow X$ is a non-decreasing mapping satisfying*

$$\frac{1}{M(Tx,\ Ty,\ t)}-1\leq k\left(\frac{1}{\overline{A}(x,\ y)}-1\right),$$

where

$$\overline{A}(x,\ y)=\min\{M(x,\ y,\ t),\ M(x,\ Tx,\ t),\ M(y,\ Ty,\ t)\},$$

for all $x,\ y\in X$ such that $x\preceq y$ and $k<1$. Suppose that there exists $x_0\in X$ such that $x_0\preceq Tx_0$. Then T has a fixed point.

Example 3.1. Suppose that $X=[0,\ 1]$, equipped with the usual metric $d:X\times X\rightarrow[0,\ +\infty)$ with the natural ordering of real numbers $\leq$ and $*$ is defined by $a*b=ab$, for all $a,\ b\in[0,\ 1]$. Define

$$M(x,\ y,\ t)=\frac{t}{t+d(x,\ y)},\ \text{for all } x,\ y\in X \text{ and } t>0.$$

Clearly $(X,\ M,\ *)$ is a complete non-Archimedean fuzzy metric space. Let $T,\ g:X\rightarrow X$ be defined as

$$Tx=\frac{x^2}{3}\text{ and } gx=x^2 \text{ for all } x\in X.$$

Define $\psi:[0,\ +\infty)\rightarrow[0,\ +\infty)$ by

$$\psi(t)=t,\ \text{for all } t\geq 0,$$

and $\varphi:[0,\ +\infty)\rightarrow[0,\ +\infty)$ as follows

$$\varphi(t)=\frac{2t}{3},\ \text{for all } t\geq 0,$$

and $\theta:[0,\ +\infty)\rightarrow[0,\ +\infty)$ as follows

$$\theta(t)=\frac{t}{2},\ \text{for all } t\geq 0.$$

Now, for all $x,\ y \in X$ with $gx \preceq gy$, we have

$$\begin{aligned}
& \psi\left(\frac{1}{M(Tx,\ Ty,\ t)} - 1\right) \\
= \quad & \frac{1}{M(Tx,\ Ty,\ t)} - 1 \\
= \quad & \frac{1}{3}\left(\frac{1}{M(gx,\ gy,\ t)} - 1\right) \\
\leq \quad & \frac{1}{3}\left(\frac{1}{A(x,\ y)} - 1\right) \\
\leq \quad & \psi\left(\frac{1}{A(x,\ y)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(x,\ y)} - 1\right)\right) + \theta\left(\frac{1}{B(x,\ y)} - 1\right).
\end{aligned}$$

Thus the contractive condition of Theorem 3.1 is satisfied for all $x,\ y \in X$. In addition all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a unique common fixed point of T and g.

4. COUPLED FIXED POINT RESULTS

Next, we deduce the two dimensional version of Theorem 3.1. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the n^{th} Cartesian product $X \times X \times ... \times X$ (n times). For the ordered fuzzy metric space $(X,\ M,\ \preceq)$, let us consider the ordered fuzzy metric space $(X^2,\ M_\delta,\ \sqsubseteq)$, where $M_\delta : X^2 \times X^2 \times [0,\ \infty) \to [0,\ 1]$ is defined by

$$M_\delta(Y,\ V,\ t) = \min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\},\ \forall Y = (x,\ y),\ V = (u,\ v) \in X^2,$$

and $\sqsubseteq$ is introduced by

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v, \text{ for all } (u,\ v),\ (x,\ y) \in X^2.$$

It is easy to check that M_δ is a non-Archimedean fuzzy metric on X^2. Moreover $(X,\ M,\ *)$ is complete if and only if $(X^2,\ M_\delta,\ *)$ is complete. We define the mappings $T_F,\ T_G : X^2 \to X^2$, for all $(x,\ y) \in X^2$, by,

$$T_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } T_G(x,\ y) = (G(x,\ y),\ G(y,\ x)).$$

Lemma 4.1 ([16]). *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a non-Archimedean fuzzy metric space. Let $F,\ G : X^2 \to X$ and $T_F,\ T_G : X^2 \to X^2$ be mappings, then the following properties hold:*

(1) $(X,\ M,\ *)$ is complete if and only if $(X^2,\ M_\delta,\ *)$ is complete.

(2) If $(X,\ M,\ *,\ \preceq)$ is regular, then $(X^2,\ M_\delta,\ *,\ \sqsubseteq)$ is also regular.

(3) If F is $M-$continuous, then T_F is $M_\delta-$continuous.

(4) If F is $G-$increasing with respect to $\preceq$, then T_F is $(T_G, \sqsubseteq)-$non-decreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(6) For any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) If the pair (F, G) is generalized compatible, then the pair (T_F, T_G) is $O-$compatible in $(X^2, M_\delta, *, \sqsubseteq)$.

(8) If the pair (F, G) is weak compatible in $(X, M, \preceq)$, then the pair (T_F, T_G) is also weak compatible in $(X^2, M_\delta, \sqsubseteq)$.

(9) A point $(x, y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

(10) A point $(x, y) \in X^2$ is a common coupled fixed point of F and G if and only if it is a common fixed point of T_F and T_G.

(11) A point $(x, y) \in X^2$ is a coupled fixed point of F if and only if it is a fixed point of T_F.

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Assume $F, G : X^2 \to X$ are two generalized compatible mappings such that F is $G-$increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying

$$\begin{aligned} &\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\ \leq\ &\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right)\right) \\ &+\theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)} - 1\right), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &A_M^G(x,\ y,\ u,\ v) \\ =\ &\min\left\{\begin{array}{l} M(G(x,\ y),\ G(u,\ v),\ t),\ M(G(x,\ y),\ F(x,\ y),\ t), \\ M(G(u,\ v),\ F(u,\ v),\ t),\ M(G(y,\ x),\ G(v,\ u),\ t), \\ M(G(y,\ x),\ F(y,\ x),\ t),\ M(G(v,\ u),\ F(v,\ u),\ t) \end{array}\right\}, \end{aligned} \tag{4.2}$$

and

$$B_M^G(x,\ y,\ u,\ v) = \max\left\{ \begin{array}{c} M(G(u,\ v),\ F(x,\ y),\ t),\ M(G(u,\ v),\ F(u,\ v),\ t), \\ M(G(v,\ u),\ F(y,\ x),\ t),\ M(G(v,\ u),\ F(v,\ u),\ t) \end{array} \right\}, \tag{4.3}$$

for all $x,\ y,\ u,\ v \in X$, where $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$ and for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{4.4}$$

Also suppose that either

(*a*) *F is continuous or*

(*b*) *$(X,\ d,\ \preceq)$ is regular.*

Then F and G have a coupled coincidence point. In addition, suppose that for every $(x,\ y),\ (x^*, y^*) \in X^2$, there exists a point $(u,\ v) \in X^2$ such that $(F(u,\ v),\ F(v,\ u))$ is comparable to $(F(x,\ y),\ F(y,\ x))$ and $(F(x^*,\ y^*),\ F(y^*,\ x^*))$, and also the pair $(F,\ G)$ is weakly compatible. Then F and G have a unique common coupled fixed point.

Proof. Let $(x,\ y),\ (u,\ v) \in X^2$ be such that $T_G(x,\ y) \sqsubseteq T_G(u,\ v)$. Therefore $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$. Using (4.1), we have

$$\begin{aligned} & \psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\ \leq\ & \psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right)\right) \\ & + \theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)} - 1\right). \end{aligned}$$

Furthermore taking into account that $G(y,\ x) \succeq G(v,\ u)$ and $G(x,\ y) \preceq G(u,\ v)$, the contractive condition (4.1) also guarantees that

$$\begin{aligned} & \psi\left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)} - 1\right) \\ \leq\ & \psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)} - 1\right)\right) \\ & + \theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)} - 1\right). \end{aligned}$$

Combining them, we get

$$\max\left\{\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right),\ \psi\left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)}-1\right)\right\}$$
$$\leq \psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)\right)$$
$$+\,\theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)}-1\right).$$

Since ψ is non-decreasing,

(4.5)
$$\psi\left(\max\left\{\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right),\ \left(\frac{1}{M(F(y,\ x),\ F(v,\ u),\ t)}-1\right)\right\}\right)$$
$$\leq \psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)\right)$$
$$+\,\theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)}-1\right).$$

Thus, it follows from (4.5) that

$$\begin{aligned}
&\psi\left(\frac{1}{M_\delta(T_F(x,\ y),\ T_F(u,\ v),\ t)}-1\right)\\
=\ &\psi\left(\frac{1}{\min\{M(F(x,\ y),\ F(u,\ v),\ t),\ M(F(y,\ x),\ F(v,\ u),\ t)\}}-1\right)\\
=\ &\psi\left(\max\left\{\begin{array}{c}\left(\dfrac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right),\\ \left(\dfrac{1}{M(F(y,\ x),\ F(v,\ u),\ t)}-1\right)\end{array}\right\}\right)\\
\leq\ &\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M^G(x,\ y,\ u,\ v)}-1\right)\right)\\
&+\theta\left(\frac{1}{B_M^G(x,\ y,\ u,\ v)}-1\right)\\
\leq\ &\psi\left(\frac{1}{A_{M_\delta}((x,\ y),\ (u,\ v))}-1\right)-\varphi\left(\psi\left(\frac{1}{A_{M_\delta}((x,\ y),\ (u,\ v))}-1\right)\right)\\
&+\theta\left(\frac{1}{B_{M_\delta}((x,\ y),\ (u,\ v))}-1\right),
\end{aligned}$$

where

$$A_{M_\delta}((x,\ y),\ (u,\ v))=\min\left\{\begin{array}{c}M_\delta(T_G(x,\ y),\ T_G(u,\ v),\ t),\\ M_\delta(T_G(x,\ y),\ T_F(x,\ y),\ t),\\ M_\delta(T_G(u,\ v),\ T_F(u,\ v),\ t)\end{array}\right\},$$

and

$$B_{M_\delta}((x,\ y),\ (u,\ v)) = \max\left\{ \begin{array}{c} M_\delta(T_G(u,\ v),\ T_F(x,\ y),\ t), \\ M_\delta(T_G(u,\ v),\ T_F(u,\ v),\ t) \end{array} \right\}.$$

It is only necessary to apply Theorem 3.1 to the mappings $T = T_F$ and $g = T_G$ in the ordered metric space $(X^2,\ M_\delta,\ \sqsubseteq)$ and using Lemma 4.1. □

Now we deduce the results without mixed $g-$monotone property of F.

Corollary 4.2. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F is $g-$increasing with respect to $\preceq$ and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} &\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)} - 1\right) \\ \leq\ &\psi\left(\frac{1}{A_M^g(x,\ y,\ u,\ v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^g(x,\ y,\ u,\ v)} - 1\right)\right) \\ &+\theta\left(\frac{1}{B_M^g(x,\ y,\ u,\ v)} - 1\right), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} &A_M^g(x,\ y,\ u,\ v) \\ =\ &\max\left\{ \begin{array}{c} M(gx,\ gu,\ t),\ M(gx,\ F(x,\ y),\ t),\ M(gu,\ F(u,\ v),\ t), \\ M(gy,\ gv,\ t),\ M(gy,\ F(y,\ x),\ t),\ M(gv,\ F(v,\ u),\ t) \end{array} \right\}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &B_M^g(x,\ y,\ u,\ v) \\ =\ &\min\left\{ \begin{array}{c} M(gu,\ F(x,\ y),\ t),\ M(gu,\ F(u,\ v),\ t), \\ M(gv,\ F(y,\ x),\ t),\ M(gv,\ F(v,\ u),\ t) \end{array} \right\}, \end{aligned} \tag{4.8}$$

for all $x,\ y,\ u,\ v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F,\ g\}$ is compatible. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

If there exist two elements $x_0,\ y_0 \in X$ with

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F and g have a coupled coincidence point. In addition, suppose that for every $(x,\ y)$, $(x^*, y^*) \in X^2$, there exists a point $(u,\ v) \in X^2$ such that $(F(u,\ v),\ F(v,\ u))$ is comparable to $(F(x,\ y),\ F(y,\ x))$ and $(F(x^*,\ y^*),\ F(y^*,\ x^*))$, and also the pair $(F,\ g)$ is weakly compatible. Then F and g have a unique common coupled fixed point.

Now, we deduce the result without mixed monotone property of F.

Corollary 4.3. *Let $(X,\ \preceq)$ be a partially ordered set and $(X,\ M,\ *)$ be a complete non-Archimedean fuzzy metric space. Assume $F: X^2 \to X$ is an increasing mapping with respect to $\preceq$ and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\begin{aligned}
&\psi\left(\frac{1}{M(F(x,\ y),\ F(u,\ v),\ t)}-1\right)\\
\leq\ &\psi\left(\frac{1}{A_M(x,\ y,\ u,\ v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M(x,\ y,\ u,\ v)}-1\right)\right)\\
&+\theta\left(\frac{1}{B_M(x,\ y,\ u,\ v)}-1\right),
\end{aligned}$$

where

$$\begin{aligned}
&A_M(x,\ y,\ u,\ v)\\
=\ &\min\left\{\begin{array}{c} M(x,\ u,\ t),\ M(x,\ F(x,\ y),\ t),\ M(u,\ F(u,\ v),\ t),\\ M(y,\ v,\ t),\ M(y,\ F(y,\ x),\ t),\ M(v,\ F(v,\ u),\ t)\end{array}\right\},
\end{aligned}$$

and

$$B_M(x,\ y,\ u,\ v)=\max\left\{\begin{array}{c} M(u,\ F(x,\ y),\ t),\ M(u,\ F(u,\ v),\ t),\\ M(v,\ F(y,\ x),\ t),\ M(v,\ F(v,\ u),\ t)\end{array}\right\},$$

for all $x,\ y,\ u,\ v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Assume that there exist two elements $x_0,\ y_0 \in X$ with

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F has a coupled fixed point.

In a similar way, we may state the results analog of Corollary 3.2, Corollary 3.3, Corollary 3.4, Corollary 3.5 and Corollary 3.6 for Theorem 4.1, Corollary 4.2 and Corollary 4.3.

5. Application to Domain of Words

Let Σ be a non-empty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞. The symbol $\sqsubseteq$ denotes the prefix order on Σ^∞, that is, $x \sqsubseteq y \iff x$ is a prefix of y.

Now for each $x \in \Sigma^\infty$ denote by $l(x)$ the length of x. Then $l(x) \in [1, \infty)$ whenever $x \neq \phi$ and $l(\phi) = 0$. For each $x, y \in \Sigma^\infty$, let $x \sqcap y$ be the common prefix of x and y. Thus the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_\sqsubseteq(x, y) = \begin{cases} 0, & \text{if } x \sqsubseteq y, \\ 2^{-l(x \sqcap y)}, & \text{otherwise,} \end{cases}$$

is a quasi-metric on Σ^∞. Actually $d_\sqsubseteq$ is a non-Archimedean quasi-metric space on Σ^∞. We also observe that the non-Archimedean metric space $(d_\sqsubseteq)^s$ is the Baire metric on Σ^∞, that is,

$$(d_\sqsubseteq)^s(x,\ x) = 0 \text{ and } (d_\sqsubseteq)^s(x,\ y) = 2^{-l(x \sqcap y)},\ \text{for all } x,\ y \in \Sigma^\infty \text{ such that } x \neq y.$$

It is well known that $(d_\sqsubseteq\ )^s$ is complete.

Example 5.1. Let $(\Sigma^\infty, (d_\sqsubseteq\ )^s)$ be a non-Archimedean metric space and let $M_{(d_\sqsubseteq)^s}$ be a fuzzy set in $X \times X \times [0, \infty)$ given by

$$M_{(d_\sqsubseteq)^s}(x,\ y,\ t) = \frac{t}{t + (d_\sqsubseteq)^s(x,\ y)},\ \text{for all } x,\ y \in \Sigma^\infty \text{ and } t > 0.$$

Then $(M_{(d_\sqsubseteq)^s}, T)$ is a complete non-Archimedean fuzzy metric on X, where T denotes the continuous t-norm given by $T(a, b) = \min\{a, b\}$.

Example 5.2. Let M be a fuzzy set in $X \times X \times [0, \infty)$ given by

$$\begin{aligned} M(x,\ y,\ 0) &= 0,\ \text{for all } x,\ y \in \Sigma^\infty. \\ M(x,\ x,\ t) &= 1 \text{ and } M(x,\ y,\ t) = 1 - 2^{-l(x \sqcap y)}\ \forall x,\ y \in \Sigma^\infty \text{ such that } x \neq y. \end{aligned}$$

Then (M, Δ) is a complete non-Archimedean fuzzy metric on X, where Δ denotes the continuous t-norm.

Now, we apply Corollary 3.6 to the complexity analysis of quicksort algorithm, to show, in direct way, the existence of solution for the following recurrence equation:

$$T(1) = 0 \text{ and } T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1),\ n \geq 2.$$

The average case analysis of Quicksort was discussed in [21] (see also [4]), where the above recurrence equation is obtained.

Consider as an alphabet Σ the set of non-negative real numbers, that is, $\Sigma = [0, \infty)$. We associate to T the functional $\Phi : \Sigma^\infty \to \Sigma^\infty$ given by

$$(\Phi(x))_1 = T(1) \text{ and } (\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n} x_{n-1},\ \text{for all } n \geq 2.$$

If $x \in \Sigma^{\infty}$ has length $n < \infty$, we write $x = x_1 x_2 x_3 ... x_n$, and if x is an infinite word we write $x = x_1 x_2 x_3 ...$ Next we show that Φ satisfies the contractive condition of Corollary 3.6 on $(\Sigma^{\infty}, M, \Delta)$ with $k = 1/2$. To this end, we first note that, by construction, we have $l(\Phi(x)) = l(x) + 1$ for all $x \in \Sigma^{\infty}$ (in particular $l(\Phi(x)) = \infty$ whenever $l(x) = \infty$). Furthermore, it is clear that

$$x \neq y \Longleftrightarrow \Phi(x) \neq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y), \text{ for all } x, \ y \in \Sigma^{\infty}.$$

Hence

$$l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y)), \text{ for all } x, \ y \in \Sigma^{\infty}.$$

Indeed, if $x = y$, then

$$M(\Phi(x), \ \Phi(y), \ t) = M(x, \ y, \ t) = 1.$$

If $x \neq y$, then

$$M(\Phi(x), \ \Phi(y), \ t) = 1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}. \tag{5.1}$$

Now, since $l(x \sqcap y) \leq l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y))$,

$$2^{-l(\Phi(x) \sqcap \Phi(y))} \leq 2^{-l(\Phi(x \sqcap y))} \text{ and } \frac{2^{-l(\Phi(x \sqcap y))}}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}} \leq \frac{2^{-l(\Phi(x \sqcap y))}}{1 - 2^{-l(x \sqcap y)}}. \tag{5.2}$$

Now, by (5.1) and (5.2), we have

$$\begin{aligned} \frac{1}{M(\Phi(x), \ \Phi(y), \ t)} - 1 &= \frac{1 - M(\Phi(x), \ \Phi(y), \ t)}{M(\Phi(x), \ \Phi(y), \ t)} \\ &= \frac{2^{-l(\Phi(x) \sqcap \Phi(y))}}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\ &\leq \frac{2^{-l(\Phi(x \sqcap y))}}{1 - 2^{-l(\Phi(x) \sqcap \Phi(y))}} \\ &\leq \frac{2^{-l(\Phi(x \sqcap y))}}{1 - 2^{-l(x \sqcap y)}}. \end{aligned}$$

Thus

$$\frac{1}{M(\Phi(x), \ \Phi(y), \ t)} - 1 \leq \frac{2^{-l(\Phi(x \sqcap y))}}{1 - 2^{-l(x \sqcap y)}}. \tag{5.3}$$

Now

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right) &= \frac{1}{2}\left(\frac{1-M(x,\ y,\ t)}{M(x,\ y,\ t)}\right)\\ &= \frac{1}{2}\left(\frac{2^{-l(x\sqcap y)}}{1-2^{-l(x\sqcap y)}}\right)\\ &= \frac{2^{-l(x\sqcap y)-1}}{1-2^{-l(x\sqcap y)}}\\ &= \frac{2^{-l(\Phi(x\sqcap y))}}{1-2^{-l(x\sqcap y)}}.\end{aligned}$$

Thus

$$\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right)=\frac{2^{-l(\Phi(x\sqcap y))}}{1-2^{-l(x\sqcap y)}}. \tag{5.4}$$

Hence, by (5.3) and (5.4), we get

$$\frac{1}{M(\Phi(x),\ \Phi(y),\ t)}-1\le\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right)\le\frac{1}{2}\left(\frac{1}{\overline{A}(x,\ y)}-1\right),$$

for all $x,\ y\in\Sigma^{\infty}$ and $t>0$. Thus the contractive condition of Corollary 3.6 is satisfied with $k=1/2\in(0,\ 1)$. So, by Corollary 3.6, Φ has a unique fixed point $z=z_1z_2z_3\ldots$, which is obviously the unique solution to the recurrence equation T, that is, $z_1=0$ and $z_n=\dfrac{2(n-1)}{n}+\dfrac{n+1}{n}z_{n-1}$ for all $n\ge 2$.

We can also apply our results to the complexity analysis of Divide and Conquer algorithm. Recall [4, 21] that Divide and Conquer algorithms solve a problem by recursively splitting it into sub problems each of which is solved separately by the same algorithm, after which the results are combined into a solution of the original problem. Thus the complexity of a Divide and Conquer algorithm typically is the solution to the recurrence equation given by

$$T(1)=c \text{ and } T(n)=aT(\frac{n}{b})+h(n),$$

where $a,\ b,\ c\in\mathbb{N}$ with $a,\ b\ge 2$, n range over the set $\{b^p : p=0,\ 1,\ 2,\ \ldots\}$, and $h(n)\ge 0$ for all $n\in\mathbb{N}$.

As in the case of Quicksort algorithm, take $\Sigma=[0,\infty)$ and put $\Sigma^{\mathbb{N}}=\{\ x\in\Sigma^{\infty} : l(x)=\infty\}$. Clearly $\Sigma^{\mathbb{N}}$ is a closed subset of Σ^{∞}, $(\Sigma^{\mathbb{N}},\ M,\ \Delta)$ is a non-Archimedean fuzzy metric space. Now we associate to T the functional $\Phi:\Sigma^{\mathbb{N}}\to\Sigma^{\mathbb{N}}$ given by

$$\begin{aligned}(\Phi(x))_1 &= T(1) \text{ and } (\Phi(x))_n=\frac{ax_n}{b}+h(n) \text{ if } n\in\{b^p : p=1,\ 2,\ \ldots\}\\ \text{and } (\Phi(x))_n &= 0 \text{ otherwise, for all } x\in\Sigma^{\mathbb{N}}.\end{aligned}$$

For our purposes here it suffices to observe that for each $x,\ y \in \Sigma^{\mathbb{N}}$, the following inequality holds:

$$l(\Phi(x) \sqcap \Phi(y)) \geq 1 + l(x \sqcap y).$$

In fact, if $l(x \sqcap y) = 0$, then $l(\Phi(x) \sqcap \Phi(y)) \geq 1$; and if $b^p > l(x \sqcap y) \geq b^{p-1}$, $p \geq 1$, then $b^{p+1} > l(\Phi(x) \sqcap \Phi(\ y)) \geq b^p$.

If $x \in \Sigma^{\infty}$ has length $n < \infty$, we write $x = x_1x_2x_3.....x_n$, and if x is an infinite word we write $x = x_1x_2x_3...$ Φ satisfies the contractive condition of Corollary 3.6 on $(\Sigma^{\mathbb{N}},\ M,\ \Delta)$ with $k = 1/2$. Indeed, if $x = y$, then

$$M(\Phi(x),\ \Phi(y),\ t) = M(x,\ y,\ t) = 1.$$

If $x \neq y$, then

$$M(\Phi(x),\ \Phi(y),\ t) = 1 - 2^{-l(\Phi(x)\sqcap\Phi(y))}. \tag{5.5}$$

Since $l(x \sqcap y) \leq l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y))$,

$$2^{-l(\Phi(x)\sqcap\Phi(y))} \leq 2^{-l(\Phi(x\sqcap y))} \text{ and } \frac{2^{-l(\Phi(x\sqcap y))}}{1 - 2^{-l(\Phi(x)\sqcap\Phi(y))}} \leq \frac{2^{-l(\Phi(x\sqcap y))}}{1 - 2^{-l(x\sqcap y)}}. \tag{5.6}$$

Now, by (5.5) and (5.6), we have

$$\begin{aligned}
\frac{1}{M(\Phi(x),\ \Phi(y),\ t)} - 1 &= \frac{1 - M(\Phi(x),\ \Phi(y),\ t)}{M(\Phi(x),\ \Phi(y),\ t)} \\
&= \frac{2^{-l(\Phi(x)\sqcap\Phi(y))}}{1 - 2^{-l(\Phi(x)\sqcap\Phi(y))}} \\
&\leq \frac{2^{-l(\Phi(x\sqcap y))}}{1 - 2^{-l(\Phi(x)\sqcap\Phi(y))}} \\
&\leq \frac{2^{-l(\Phi(x\sqcap y))}}{1 - 2^{-l(x\sqcap y)}}.
\end{aligned}$$

Thus

$$\frac{1}{M(\Phi(x),\ \Phi(y),\ t)} - 1 \leq \frac{2^{-l(\Phi(x\sqcap y))}}{1 - 2^{-l(x\sqcap y)}}. \tag{5.7}$$

Again

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right) &= \frac{1}{2}\left(\frac{1-M(x,\ y,\ t)}{M(x,\ y,\ t)}\right)\\
&= \frac{1}{2}\left(\frac{2^{-l(x\sqcap y)}}{1-2^{-l(x\sqcap y)}}\right)\\
&= \frac{2^{-l(x\sqcap y)-1}}{1-2^{-l(x\sqcap y)}}\\
&= \frac{2^{-l(\Phi(x\sqcap y))}}{1-2^{-l(x\sqcap y)}}.
\end{aligned}$$

Thus

$$\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right)=\frac{2^{-l(\Phi(x\sqcap y))}}{1-2^{-l(x\sqcap y)}}. \tag{5.8}$$

Hence, by (5.7) and (5.8), we get

$$\frac{1}{M(\Phi(x),\ \Phi(y),\ t)}-1\leq\frac{1}{2}\left(\frac{1}{M(x,\ y,\ t)}-1\right)\leq\frac{1}{2}\left(\frac{1}{\overline{A}(x,\ y)}-1\right),$$

for all $x,\ y\in\Sigma^{\mathbb{N}}$ and $t>0$. Thus the contractive condition of Corollary 3.6 is satisfied with $k=1/2\in(0,\,1)$. So, by Corollary 3.6, Φ has a fixed point $z=z_1z_2z_3...$ Consequently, the function F defined on $\{b^p : p=0,\ 1,\ 2,...\}$ by $F(b^p)=z_{b^p}$, for all $p\geq 0$, is the solution to the recurrence equation of the given Divide and Conquer algorithm.

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