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ROBUST L_p-NORM ESTIMATORS OF MULTIVARIATE LOCATION IN MODELS WITH A BOUNDED VARIANCE

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Abstract. The least informative (favorable) distributions, minimizing Fisher information for a multivariate location parameter, are derived in the parametric class of the exponential-power spherically symmetric distributions under the following characterizing restrictions;

(i) a bounded variance,
(ii) a bounded value of a density at the center of symmetry, and
(iii) the intersection of these restrictions.

In the first two cases, (i) and (ii) respectively, the least informative distributions are the Gaussian and Laplace, respectively. In the latter case (iii) the optimal solution has three branches, with relatively small variances it is the Gaussian, with relatively large variances it is the Laplace, and it is the compromise between them with intermediate variances. The corresponding robust minimax M-estimators of location are given by the L_2-norm, the L_1-norm and the L_p-norm methods. The properties of the proposed estimators and their adaptive versions are studied in asymptotics and on finite samples by Monte Carlo.

1. Introduction

One of the main approaches to the synthesis of robust procedures is based on the minimax principle. In this case, in a given class of densities the least informative (favorable) one, minimizing Fisher information, is determined and the unknown parameters of a distribution model are estimated by applying the maximum likelihood method for this density (*cf.* Huber [4]). Such an approach makes it possible to construct robust statistical procedures which are stable with regard to the departures from assumptions about an underlying distribution model. The robust minimax procedures provide a guaranteed level of the estimator's accuracy (measured by the supremum of an asymptotic variance) for any density in a given class.

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The properties of minimax estimators essentially depend on the characteristics of a distribution class. Hence it seems rather important for many applications to consider the classes of distributions with available characteristics. Maronna [7] treated robust M-estimators of multivariate location and scatter, their consistency, asymptotical normality and qualitative robustness properties. Huber [4] considered the least informative distributions in the spherically symmetric ε-contaminated Gaussian models.

In this paper, we obtain the least informative distributions and the corresponding robust minimax L_p-norm estimators of a multivariate location parameter in the parametric classes with a bounded variance, qualitatively other than ε-contaminated models. The properties of these estimators and their adaptive versions are studied both asymptotically and on finite samples. As the proposed methods partly generalize the univariate minimax estimators of location, we briefly recall the basic stages of the Huber minimax approach in this case.

Let $x_1, \cdots, x_n$ be independent random variables with common density $f(x-\theta)$ in a convex class $\mathcal{F}$. Then the M-estimator $\widehat{\theta}$ of a location parameter is defined by Huber [4] as a zero of $\sum_1^n \psi(x_i - \cdot)$ with a suitable score function ψ. The minimax approach implies the determination of the least informative density f^* minimizing Fisher information $I(f)$ in the class $\mathcal{F}$: $f^* = \arg\min_{f\in\mathcal{F}} I(f)$, $I(f) = \int (f'/f)^2 f\, dx$, followed by designing the maximum likelihood estimator (MLE) with the score function $\psi^* = -f^{*\prime}/f^*$. Under rather general conditions (for details, see Huber [3, 4]), $\sqrt{n}(\widehat{\theta} - \theta)$ is asymptotically normally distributed and the asymptotic variance $V(\psi, f)$ has the saddle point (ψ^*, f^*) with the corresponding minimax property $V(\psi^*, f) \le V(\psi^*, f^*) \le V(\psi, f^*)$.

The shape of the least informative density f^* and the corresponding score function ψ^* depends on the structure of the distribution class $\mathcal{F}$. In the literature (for example, see Collins & Wiens [2], Huber [3, 4], Sacks & Ylvisaker [8]), there are many results on the least informative distributions, mainly in the ε-contaminated neighborhoods of a given distribution. The distribution classes with a bounded variance were considered in [5, 6, 9].

In the class of nondegenerate densities $\mathcal{F}1 = \{f : f(0) \ge 1/(2a) > 0\}$ (*cf.* Vil'chevskiy & Shevlyakov [9]), the least informative density is the Laplace: $f^*(x) = L(x; 0, a) = (2a)^{-1} \exp(-|x|/a)$. The optimal score function $\psi^*(u) = \operatorname{sgn} u$ gives the L_1-norm estimator: the sample median. Note that the value of the parameter a characterizes the dispersion of a distribution in the central zone.

In the class with a bounded variance $\mathcal{F}_2 = \left\{f : \sigma^2(f) = \int x^2 f\, dx \le \overline{\sigma}^2\right\}$, the Gaussian density $f^*(x) = N(x; 0, \overline{\sigma}) = (2\pi)^{-1/2}\overline{\sigma}^{-1}\exp(-x^2/(2\overline{\sigma}^2))$ is optimal with the score function $\psi^*(u) = u$ and the L_2-norm estimator; the sample mean (*cf.* Kagan, Linnik & Rao [5]).

In the intersection of these classes, the least informative distribution has three branches (*cf.* Vil'chevskiy & Shevlyakov [9]);

(i) with relatively large variances — it is the Laplace,
(ii) with relatively small variances — it is the Gaussian, and
(iii) with the intermediate zone — it is described by the Weber-Hermite functions or the functions of the parabolic cylinder.

The corresponding estimators are the sample median, sample mean and compromise between them, respectively. The latter can be efficiently approximated by the L_p-norm estimators with $1 < p < 2$ (*cf.* Vil'chevskiy & Shevlyakov [9]). Below, these results are extended on the multivariate case.

2. Problem Statement and Main Result

Let $\mathbf{x}_1, \cdots, \mathbf{x}_n$ be a sample from an m-variate spherically symmetric density

$$f(\mathbf{x} - \theta) = f(|\mathbf{x} - \theta|), \quad \mathbf{x}, \theta \in \mathbb{R}^m,$$

with f belonging to the parametric class of exponential-power distributions

$$\mathcal{F}_q = \left\{ f \,:\, f_q(r; \beta) = \frac{q\Gamma(m/2)}{2\pi^{m/2}\beta^m\Gamma(m/q)} \exp\left(-\frac{r^q}{\beta^q}\right)\right\}, \tag{2.1}$$

where $q \ge 1$ and

$$r = |\mathbf{x} - \theta| = \left(\sum_{j=1}^{m}(x_j - \theta_j)^2\right)^{1/2}$$

and β is a scale parameter.

The L_p-norm estimator of a location parameter $\theta = (\theta_1, \cdots, \theta_m)$ is defined as

$$\widehat{\theta}L_p = \arg\min_\theta \sum_{i=1}^{n} r_i^p, \quad p \ge 1, \quad r_i = \left(\sum_{j=1}^{m}(x_{ij} - \theta_j)^2\right)^{1/2}. \tag{2.2}$$

Note that we use the L_p-norm estimators as they are maximum likelihood estimators of location for densities (2.1) when $p = q$.

We shall now obtain the minimax variance L_p-norm estimators of multivariate location in the class $\mathcal{F}_q$. In the general nonparameteric case, Huber [4] showed that the minimax variance estimation problem is reduced to the variational problem of minimizing Fisher information for multivariate location

$$f^*(r) = \arg\min_{f\in\mathcal{F}} \int_0^\infty \left[\frac{f'(r)}{f(r)}\right]^2 f(r)\, r^{m-1}\, dr. \tag{2.3}$$

From spherical symmetry, it follows that the saddle point (p^*, q^*) of the covariance matrix $\mathbf{V}(p,q)$ of the L_p-norm estimator $\widehat{\theta}L_p$

$$\mathbf{V}(p^*, q) \le \mathbf{V}(p^*, q^*) = \mathbf{V}(q^*, q^*) = \mathbf{I}^{-1}(q^*)\,,$$

where $\mathbf{I}$ is the Fisher information matrix, is determined from the solution of the variational problem

$$f^*(r) = \arg\min_{f\in\mathcal{F}} \int_0^\infty \left[\frac{f'(r)}{f(r)}\right]^2 f(r)\, r^{m-1} dr\,.$$

Hence, in the parametric class $\mathcal{F}_q$, problem (2.3) takes the form of the simple parametric minimization

$$(q^*, \beta^*) = \arg\min_{q,\beta} \frac{q^2\Gamma\left(\frac{m-2}{q}+2\right)}{\beta^2\Gamma(m/q)}. \tag{2.4}$$

Using additional restrictions put on densities (2.1), we now obtain the multivariate analogues of the univariate least informative densities reviewed in Section 1.

Proposition 1. *In the class of nondegenerate densities*

$$\mathcal{F}_{1q} = \{f_q \,:\, f_q(0;\beta) \ge 1/(2a^m) > 0,\ q \ge 1\}\,, \tag{2.5}$$

the least informative distribution is the multivariate analogue of the Laplace

$$f_1^*(r) = L(r;\beta^*) = (2a^m)^{-1} \exp\left(-r/\beta^*\right), \tag{2.6}$$

where

$$\beta^* = \frac{a}{[2^{(m-1)/m}\ \pi^{(m-1)/(2m)}\ \Gamma^{1/m}(\frac{m+1}{2})]}.$$

Proof. Minimization problem (2.4) with the side condition (2.5) of the inequality type can be rewritten as follows; first, minimize Fisher information (2.4) subject to the side condition of the equality type $f_q(0;\beta) = 1/(2a_1^m)$; secondly, minimize (2.4) by the auxiliary parameter a_1 subject to $a_1 \le a$.

The first problem is directly solved by eliminating the parameter β from the equation $f_q(0;\beta) = 1/(2a_1^m)$ and substituting it into the expression for Fisher information. It can be shown that $dI/dq > 0$, therefore we have $q^* = 1$, and the following minimization by $a1$ implies $a_1^* = a$. □

Proposition 2. *In the class with bounded variances*

$$\mathcal{F}_{2q} = \left\{ f_q \,:\, \sigma_k^2(f_q) = \int \cdots \int x_k^2 f_q(r)\, dx_1 \cdots dx_m \le \overline{\sigma}^2, \quad k = 1, \cdots, \ m \right\}, \tag{2.7}$$

the least informative density is the Gaussian

$$f_2^*(r) = g(r;\overline{\sigma}) = \frac{1}{(2\pi\overline{\sigma})^{-m/2}} \exp\left(-\frac{r^2}{2\overline{\sigma}^2}\right). \tag{2.8}$$

Proof. This assertion also can be obtained by using the above approach, but it directly follows from the general result of Luneva [6]; the multivariate Gaussian density $g(\mathbf{x};\theta,\overline{\mathbf{V}})$ is the least informative in the class of multivariate distributions with a bounded covariance matrix, $\mathbf{V}(f) \le \overline{\mathbf{V}}$. □

Theorem 2.1. *In the intersection of the above two classes of distribution*

$$\mathcal{F}_{12q} = \left\{ f_q \,:\, f(0;\beta) \ge 1/(2a^m) > 0, \quad \sigma_k^2(f_q) \le \overline{\sigma}^2, \quad k = 1, \cdots, \ m \right\},$$

the least informative density is of the form

$$f_q^*(r) = \begin{cases} g(r;\overline{\sigma}), & \text{for } \overline{\sigma}^2/a^2 \le \phi_1(m), \\ f_{\alpha^*}(r;\beta^*), & \text{for } \phi_1(m) < \overline{\sigma}^2/a^2 \le \phi_2(m), \\ L(r;\beta^*), & \text{for } \overline{\sigma}^2/a^2 > \phi_2(m), \end{cases} \tag{2.9}$$

where

$$\phi_1(m) = \frac{2^{2/m}}{2\pi}, \qquad \phi_2(m) = \frac{m+1}{(4\pi)^{(m-1)/m}\,\Gamma^{2/m}(\frac{m+1}{2})},$$

the parameters α^* $(1 < \alpha^* < 2)$ and β^* are determined from the equations

$$\frac{\overline{\sigma}}{a} = \frac{(\alpha^*)^{1/m}\Gamma^{1/m}(m/2)\Gamma^{1/2}(\frac{m+2}{\alpha^*})}{\sqrt{\pi m}\,\Gamma^{1/2}(m/\alpha^*)}, \quad \beta^* = m^{1/2}\,\overline{\sigma}\,\Gamma^{1/2}(m/\alpha^*)\Gamma^{1/2}\left(\frac{m+2}{\alpha^*}\right).$$

Proof. The three branches of solution (2.9) appear due to the degree in which the restrictions are taken into account. In the first domain $\overline{\sigma}^2/a^2 \le \phi_1(m)$, it is just the restriction on a variance that matters: $\sigma_k^2(\widetilde{f}_2) = \overline{\sigma}^2, k = 1, \cdots, \ m$; the restriction on the value of a density at the center of symmetry has the form of the strict inequality

$f_2(0) > 1/(2a^m)$. In the third domain $\overline{\sigma}^2/a^2 > \phi_2(m)$, the restriction on a density value is essential:

$$f_1(0) = 1/(2a^m),\ \sigma_k^2(f_1) < \overline{\sigma}^2,\ k = 1, \cdots,\ m.$$

In the middle domain, the both restrictions are the equalities

$$f_{\alpha^*}(0) = 1/(2a^m),\ \sigma_k^2(f_{\alpha^*}) = \overline{\sigma}^2,\ k = 1, \cdots,\ m,$$

thus they determine the unknown parameters α and β. □

Corollary 2.1. *The minimax variance estimator of location is the multivariate L_p-norm estimator with $p = \alpha^*$. Thus, in the first domain with relatively small variances, the L_2-norm method is optimal; in the third domain with relatively large variances, the L_1-norm method is optimal; in the middle domain, the L_p-norm estimators with $1 < p < 2$ are the best.*

It can be seen, from Theorem 2.1, that the optimal value of q^* is determined independently on β^* because of the scale equivariancy of L_2-norm estimators. The thresholds of switching of the minimax algorithm from the L_1-norm estimator to the L_p-norm with $1 < p < 2$, and to the L_2-norm estimator, are given by the functions $\phi_1(m)$ and $\phi_2(m)$. The values of these thresholds are displayed in Table 1.

Table 1. The values of the switching thresholds of the multivariate L_p-norm estimators

m	1	2	3	4	5	∞
$\phi_1(m)$	$2/\pi$	$1/\pi$	$1/(2^{1/3}\pi)$	$1/(2^{1/2}\pi)$	$1/(2^{2/3}\pi)$	$1/(2\pi)$
$\phi_2(m)$	2	$3/\pi$	$(2/\pi)^{2/3}$	$5/(6^{1/2}\pi)$	$3/(2\pi^{4/5})$	$e/(2\pi)$

It can be seen from Table 1 that, first, the asymptotic values of the thresholds are reached rather fast as $m \to \infty$, and second, with increasing m these values become smaller in approximately three times than $m = 1$.

We now consider the behavior of the minimax multivariate L_p-norm estimators under the conventional ε-contaminated Gaussian distributions:

$$f(r) = (1-\varepsilon)g(r; \sqrt{2}) + \varepsilon g(r; k\sqrt{2}), \qquad 0 \le \varepsilon < 1, \qquad k > 1 \tag{2.10}$$

where

$$g(r; k\sqrt{2}) = \frac{1}{(2\pi)^{m/2}k^m} \exp\left(-\frac{r^2}{2k^2}\right), \qquad k > 1.$$

It suffices to check the robust L_1-branch of the minimax solution. Under densities (2.10), the asymptotic efficiency of the L_1-norm estimator relative to the L_2-norm estimator is given by

$$ARE(L_1, L_2) = \phi(m)(1-\varepsilon+\varepsilon k^2)(1-\varepsilon+\varepsilon/k)^{-2},$$

where

$$\phi(m) = \frac{(m-1)^2\Gamma^2(\frac{m-1}{2})}{2m\Gamma^2(m/2)}, \quad m \geq 2.$$

Now we display several values of $\phi(m)$:

$$\phi(1) = 2/\pi,\ \phi(2) = \pi/4,\ \phi(3) = 8/(3\pi),\ \phi(\infty) = 1.$$

For example, under the Gaussian distribution with $k = 1$ or $\varepsilon = 0$ the superiority of the L_2-estimator vanishes fast as $m \to \infty$. In other words, all estimators become equally bad with high dimensionality.

3. Adaptive robust procedure

Designing robust estimators, we have supposed the availability of the information about the characteristics of a distribution class. However, in real-life problems, these characteristics are usually unknown and can be determined while data processing.

In applications, the approximate value of the upper bound of a variance can be obtained from the restrictions of a physical, technical or any other data measuring procedure. A statistician may estimate this value analyzing the extreme values in the data or using the upper confident bounds for a variance. We propose another way in an adaptive procedure.

As observations are coming in successively, it is feasible to develop estimators that are capable of adapting to the ever increasing volume of data and correcting the characteristics of a class $\mathcal{F}$ for improving the accuracy of estimation. With not large samples, such an approach is heuristic and the simplest for the examination by Monte Carlo technique.

Consider the following adaptive algorithm for robust estimation of a multivariate location parameter, called ARML-estimator.

(i) Choose the initial L_1-norm estimate for θ:

$$\widehat{\theta}_{L_1} = \arg\min_{\theta} \sum_{i=1}^{n} r_i, \quad p \geq 1, \quad r_i = \left(\sum_{j=1}^{m}(x_{ij}-\theta_j)^2\right)^{1/2}.$$

(ii) Evaluate the residuals:

$$\widehat{\mathbf{e}}_i = \mathbf{x}_i - \widehat{\theta}_{L_1}, \qquad i = 1, \cdots, n.$$

(iii) Evaluate the estimates of the characteristics $\overline{\sigma}^2$ and a of the class $\mathcal{F}_{12q}$:

$$\widehat{\overline{\sigma}}^2 = \frac{1}{nm}\sum_{i=1}^{n}\widehat{r}_i^2, \quad r_i = |\mathbf{e}_i|,\ i = 1, \cdots, n; \quad \widehat{a} = \frac{\pi^{1/2}(n+1)^{1/m}\widehat{r}_{(1)}}{\Gamma^{1/m}(m/2)},$$

where $r_{(1)}$ is the minimal order statistic of the sample $r_1, \cdots, r_n$.

(iv) Use the robust minimax L_p-norm estimator ($p = q^*$) of Section 2 with the estimates $\widehat{a}$ and $\widehat{\overline{\sigma}}^2$ as the characteristics of the class $\mathcal{F}_{12q}$.

Then we obtain the estimate $\widehat{a}$ from the following relations:

$$P(r \le R) = F(R), \quad F(R) = 2\pi^{m/2}(\Gamma(m/2))^{-1}\int_0^R f(t)t^{m-1}\,dt,$$
$$\widehat{F}(r_{(1)}) = 2\pi^{m/2}(m\Gamma(m/2))^{-1}\,r_{(1)}^m\,\widehat{f}(0),$$
$$\widehat{F}(r_{(1)}) = 1/(n+1), \quad \widehat{f}(0) = 1/(2\widehat{a}^m). \tag{3.1}$$

Under the ε-contaminated bivariate Gaussian distributions (2.10), the behavior of the ARML-algorithm was studied by Monte Carlo on samples $n = 20$ and $n = 100$. The number of replications was 1000. The L_1-, L_2- and ML-estimators were also evaluated. The relative efficiency of estimators was defined as the ratio of the absolute values of the determinants of their sample covariance matrices. The results of simulation are presented in Table 2. The mean values of the optimal parameter $p = \alpha^*$ used in the ARML-estimate are displayed in the third line of Table 2 as well.

Table 2. The relative efficiency of the ARML, L_2- and L_1-norm estimators under contamination: $\varepsilon = 0.1$

$n = 20$						$n = 100$					
k	1	2	3	4	5	k	1	2	3	4	5
ARML	0.80	0.84	0.89	0.92	0.95	ARML	0.96	0.94	0.90	0.90	0.93
p	1.70	1.58	1.42	1.23	1.13	p	1.92	1.70	1.40	1.15	1.05
L_2	1.00	0.92	0.67	0.40	0.30	L_2	1.00	0.94	0.70	0.42	0.31
L_1	0.70	0.76	0.83	0.88	0.93	L_1	0.73	0.78	0.86	0.89	0.93

The ARML-estimator proved to be better than the L_1- and L_2-norm estimators both on small and large samples, especially under heavy contamination.

On small samples, the ARML-estimator is similar to the L_1-norm estimator. The same effect was observed in the univariate case for the adaptive robust estimators of a location parameter (*cf.* Vil'chevskiy & Shevlyakov [9]), and it is explained by the positive bias of the sample distribution of the switching threshold statistic $\widehat{\sigma}^2/\widehat{a}^2$ on small samples.

4. Concluding remarks

Regarding the robustness properties of the obtained parametric solution (2.9), we now consider the general nonparametric case of spherically symmetric distributions.

For variational problem (2.3), the Euler equation is given by

$$u'' + [(m-1)/r]u' - \lambda u = 0, \tag{4.1}$$

where $u(r) = \sqrt{f(r)}$, λ is a Lagrange multiplier corresponding to the condition of norming (*cf.* Huber [4]).

Setting $w(r) = r^\nu u(r)$, $\nu = m/2 - 1$ and $z = \sqrt{|\lambda|}\, r$, from (4.1) we obtain the Bessel equation

$$z^2 w''(z) + z w'(z) - (z^2 \operatorname{sgn} \lambda + \nu^2) w(z) = 0.$$

Its solutions can be written as

$$w(z) = \begin{cases} J_\nu(z) \quad \text{or} \quad N_\nu(z), & \text{for} \quad \lambda < 0, \\ I_\nu(z) \quad \text{or} \quad K_\nu(z), & \text{for} \quad \lambda \geq 0, \end{cases} \tag{4.2}$$

where $J_\nu(z)$ and $N_\nu(z)$ are the Bessel and Neyman functions of the ν-th order, $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel and Macdonald functions (*cf.* Abramowitz & Stegun [1]). Using (4.2), we can describe the multivariate analogs of the univariate least informative densities, that is, the first is the Bokk's generalization for Huber's least informative density under ε-contaminated distribution and the second is the generalization of the cosine-type density minimizing Fisher information over the class of finite distributions.

As the Gaussian distribution is optimal both in the parametric and nonparametric classes with a bounded variance, it is sufficient to consider only the nonparametric class of the nondegenerate spherically symmetric densities $\mathcal{F} = \{f : f(0) \geq 1/(2a^m) > 0\}$. As the robustness properties of any minimax variance estimation procedure are mainly defined by the shape of the tails of the least informative density, now we obtain their asymptotic structure. From the above it follows that these tails should be as $Ar^{-2\nu}K_\nu^2(Br)$ for sufficiently large values of r, where A and B are some constants. Hence the asymptotic behavior of these tails can be described by $f^*(r) \sim r^{1-m}\exp(-Cr)$ as $r \to \infty$ for some positive constant C. Thus the tails of the least informative distribution in the nonparametric class of nondegenerate distributions are shorter than for its parametric analogue (2.6), the Laplace density $L(r) \sim \exp(-Dr)$ where D is also constant. So, we can state that the L_1-norm estimator provides the resistance to outliers in the general nonparametric family of distributions but certainly with some loss of efficiency.

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