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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DIFFERENCE EQUATION $x_{n+1} = \alpha + \beta x_{n-1}^p / x_n^p$

Zhaoshuang Liu and Zhenguo Zhang

Abstract. In this paper, we investigate asymptotic stability, oscillation, asymptotic behavior and existence of the period-2 solutions for difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}^p / x_n^p$$

where $\alpha \geq 0, \beta > 0, |p| \geq 1$, and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

1. introduction

Consider the following recursive equation

$$x_{n+1} = \alpha + \beta \frac{x_{n-1}^p}{x_n^p} \tag{1.1}$$

where $\alpha \geq 0, \beta > 0, |p| \geq 1$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

Recently, there has been an increasing interest in the study of the recursive sequences Amleh, Grove, Georgiou & Ladas [1], Gibbons, Kulenovic & Ladas [2], Kocić, Ladas & Rodrigues [3] and Kosmala, Kulenovic, Ladas & Teixeira [4]. In this paper, we study asymptotic stability, oscillation, asymptotic behavior and existence of the period-2 solutions for the difference equations (1.1).

We need the following definitions.

Definition 1. The *equilibrium point* $\overline{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots$$

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is the point that satisfies the condition:

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

Definition 2. A *positive semi-cycle* of $\{x_n\}$ of equation (1.1) consists of "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to the $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1 \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \overline{x},$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \overline{x}.$$

A *negative semi-cycle* of $\{x_n\}$ of equation (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than the $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1 \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \overline{x}$$

and

$$\text{either} \quad m = \infty \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \overline{x}.$$

Definition 3. A solution $\{x_n\}$ of equation (1.1) is called *oscillatory* if $x_n - \bar{x}$ is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*.

2. Main Results

First, we discuss asymptotic stability for equation (1.1).

Theorem 1. *f we assume $p \geq 1$, then following statements are true:*

(1) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is locally asymptotically stable if $\alpha > (2p-1)\beta$.*

(2) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is unstable if $0 \leq \alpha < (2p-1)\beta$.*

Proof. The linearized equation of the equation (1.1) about the equilibrium point $\overline{x} = \alpha + \beta$ is

$$y_{n+1} + \frac{p\beta}{\alpha+\beta} y_n - \frac{p\beta}{\alpha+\beta} y_{n-1} = 0. \tag{2.1}$$

The characteristic equation is given by

$$f(\lambda) = \lambda^2 + \frac{p\beta}{\alpha+\beta}\lambda - \frac{p\beta}{\alpha+\beta} = 0. \tag{2.2}$$

So by Linearized Stability Theorem Gibbons, Kulenovic Ladas [2] and Jury Criterion of Asymptotically Stable Kocić, Ladas & Rodrigues [3] $\overline{x} = \alpha + \beta$ is locally asymptotically stable if

$$f(-1) > 0, \quad f(1) > 0, \quad f(0) < 0$$

i. e.,

$$\alpha > (2p-1)\beta,$$

and the equilibrium point $\overline{x} = \alpha + \beta$ is unstable if $0 \le \alpha < (2p-1)\beta$.

This completes the proof. □

Remark. If $\beta = 1$, we have the same result as in Amleh, Grove, Georgiou & Ladas [1].

Corollary 2. *If we assume $p \le -1$, then following statements are true:*

(1) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is locally asymptotically stable if $\alpha > -(p+1)\beta$.*

(2) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is unstable if $0 \le \alpha < -(p+1)\beta$.*

The proof is the same method as in Theorem 1.

The following are some results of oscillation and asymptotic behavior for the equation (1.1).

Theorem 3. *Assume $p \ge 1$, and let $\{x_n\}$ be a positive solution of equation (1.1) which consists of at least two semi-cycles. Then $\{x_n\}$ is oscillatory. Moreover with the possible exception of the first semi-cycle, every semi-cycle has length 1 and every term of $\{x_n\}$ is strictly greater than α, and with the possible exception of the first two semi-cycles, no term of $\{x_n\}$ is ever equal to $\alpha + \beta$.*

Proof. Consider the following two cases.

Case 1. Let $x_{N-1} < \alpha + \beta \le x_N$ for some $N \ge 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_{N-1}^p}{x_N^p} < \alpha + \beta,$$

and

$$x_{N+2} = \alpha + \beta \frac{x_N^p}{x_{N+1}^p} > \alpha + \beta.$$

Thus

$$x_{N+1} < \alpha + \beta < x_{N+2}.$$

Case 2. Let $x_N < \alpha + \beta \leq x_{N-1}$ for some $N \geq 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_{N-1}^p}{x_N^p} > \alpha + \beta,$$

and

$$x_{N+2} = \alpha + \beta \frac{x_N^p}{x_{N+1}^p} < \alpha + \beta.$$

Thus

$$x_{N+2} < \alpha + \beta < x_{N+1}.$$

This completes the proof. □

Theorem 4. *Suppose $p = -1$, and let $\{x_n\}$ be a positive solution of equation (1.1). Then $\{x_n\}$ is oscillatory. Moreover, with the possible exception of the first semi-cycle, the length of every semi-cycle is equal to 2 or 3, and every term of $\{x_n\}$ is strictly greater than α.*

Proof. Case 1. Let $x_{N-1} < \alpha + \beta$ and $x_N \leq \alpha + \beta$ for some $N \geq 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_N}{x_{N-1}} \tag{2.3}$$

from the above equality, we have

$$\frac{x_{N+1}}{x_N} = \frac{\alpha}{x_N} + \frac{\beta}{x_{N-1}} > \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1.$$

So,

$$x_{N+1} > x_N,$$

and thus,

$$x_{N+2} = \alpha + \beta \frac{x_{N+1}}{x_N} > \alpha + \beta.$$

Case 2. Let $x_{N-1} > \alpha + \beta$ and $x_N \geq \alpha + \beta$ for some $N \geq 0$. Then

$$\frac{x_{N+1}}{x_N} = \frac{\alpha}{x_N} + \frac{\beta}{x_{N-1}} < \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1.$$

So,

$$x_{N+1} < x_N$$

and so,

$$x_{N+2} = \alpha + \beta \frac{x_{N+1}}{x_N} < \alpha + \beta.$$

Case 3. Let $x_{N-1} < \alpha+\beta$ and $x_N \geq \alpha+\beta$ for some $N \geq 0$. Then $x_{N+1} > \alpha+\beta$;
Case 4. Let $x_{N-1} > \alpha+\beta$ and $x_N \leq \alpha+\beta$ for some $N \geq 0$. Then $x_{N+1} < \alpha+\beta$.

This completes the proof. □

Theorem 5. *Let $p \geq 1$, $0 \leq \alpha < 1 \leq \beta$, and $\{x_n\}$ be a solution of equation (1.1) such that*

$$0 < x_{-1} \leq \beta^{\frac{1}{p}} \quad and \quad x_0 \geq \left(\frac{\beta^2}{1-\alpha}\right)^{\frac{1}{p}}.$$

Then the following statements are true:

(1) $\lim\limits_{n\to\infty} x_{2n} = \infty$.
(2) $\lim\limits_{n\to\infty} x_{2n+1} = \alpha$.

Proof. Since $0 \leq \alpha < \beta$, so $\beta^2 - \alpha^2 < \beta^2$, and thus $\frac{\beta^2}{\beta-\alpha} > \alpha+\beta$.

Then

$$x_0^p \geq \frac{\beta^2}{1-\alpha} \geq \frac{\beta^2}{\beta-\alpha} > \alpha+\beta,$$

and we have

$$x_1 = \alpha + \beta\frac{x_{-1}^p}{x_0^p} \leq \alpha + \beta\frac{\beta}{x_0^p} \leq 1,$$

and

$$x_1 = \alpha + \beta\frac{x_{-1}^p}{x_0^p} > \alpha.$$

Thus

$$x_1 \in (\alpha, 1].$$

Similarly, we have

$$\begin{aligned} x_2 &= \alpha + \beta\frac{x_0^p}{x_1^p} \geq \alpha + \beta x_0^p, \\ x_3 &= \alpha + \beta\frac{x_1^p}{x_2^p} \leq \alpha + \beta\frac{1}{(\alpha + x_0^p)^p} \\ &\leq \alpha + \beta\frac{1}{\alpha + x_0^p} \leq \alpha + \frac{\beta^2}{x_0^p} \leq 1. \end{aligned}$$

Thus

$$x_3 \in (\alpha, 1].$$

Also

$$\begin{aligned} x_4 = \alpha + \beta\frac{x_2^p}{x_3^p} &\geq \alpha + \beta x_2^p \geq \alpha + \beta(\alpha + x_0^p)^p \\ &\geq \alpha + \beta(\alpha + x_0^p) = (1+\beta)\alpha + \beta x_0^p. \end{aligned}$$

Thus

$$x_4 \geq (1+\beta)\alpha + \beta x_0^p.$$

By induction, we have

$$x_{2n} \geq \alpha \sum_{i=0}^{n-1} \beta^i + \beta^{n-1} x_0^p$$

and

$$\alpha < x_{2n+1} \leq 1.$$

Thus

$$\lim_{n\to\infty} x_{2n} = \infty.$$

and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \left(\alpha + \beta \frac{x_{2n-1}^p}{x_{2n}^p} \right) = \alpha.$$

This completes the proof. □

Finally, we study the existence of the period-2 solutions for equation (1.1).

Theorem 6. *Let $p = 1, \alpha > 0$. The following statements are true.*

(1) *Equation (1.1) has solutions of prime period 2 if and only if $\alpha = \beta$.*

(2) *Assume that $\alpha = \beta$ and $\{x_n\}$ be a solution of equation (1.1). Then x_n is periodic with period 2 if and only if $x_{-1} > \alpha, x_0 = \frac{\alpha x_{-1}}{x_{-1}-\alpha}$.*

Proof. (i) Let $\{x_n\}$ be a periodic solution of (1.1) with period 2. Then

$$x_{-1} = \alpha + \beta \frac{x_{-1}}{x_0}, \quad x_0 = \alpha + \beta \frac{x_0}{x_{-1}}.$$

Since $\alpha > 0, \beta > 0$, from the above equality, it implies $x_{-1} - \alpha \neq 0$ and $x_{-1} - \beta \neq 0$. Thus,

$$x_0 = \frac{\beta x_{-1}}{x_{-1} - \alpha}, \quad x_0 = \frac{\alpha x_{-1}}{x_{-1} - \beta}.$$

We have,

$$\frac{\beta x_{-1}}{x_{-1} - \alpha} = \frac{\alpha x_{-1}}{x_{-1} - \beta}.$$

Therefore,

$$(\alpha - \beta) x_{-1} - (\alpha^2 - \beta^2) = 0.$$

If $\alpha \neq \beta$, then $x_{-1} = \alpha + \beta$, we have

$$x_0 = \alpha + \beta, \quad \text{and} \quad x_n = \alpha + \beta,$$

which contradicts $\{x_n\}$ is periodic with period 2.

If $\alpha = \beta$, for any $x_{-1} > \alpha$, set

$$x_0 = \frac{\alpha x_{-1}}{x_{-1} - \alpha}.$$

Then

$$x_1 = \alpha + \alpha \frac{x_{-1}}{x_0} = \alpha + \alpha \frac{x_{-1}(x_{-1} - \alpha)}{\alpha x_{-1}} = x_{-1}.$$

Similarly, we have $x_2 = x_0$. So $\{x_n\}$ is periodic with period 2.

(ii). From the proof of (i), for any $x_{-1} > \alpha$, set $x_0 = \frac{\alpha x_{-1}}{x_{-1} - \alpha}$, the solution $\{x_n\}$ is periodic with period 2; contrarily, if $\{x_n\}$ is the solution periodic with period 2 of (1.1), we have $x_1 = \alpha + \alpha \frac{x_{-1}}{x_0} = x_{-1}$, so $x_{-1} > \alpha, x_0 = \frac{\alpha x_{-1}}{x_{-1} - \alpha}$.

This completes the proof. □

Theorem 7. *Suppose $p = -1$. Then for any $\alpha \geq 0$, $\beta > 0$ equation (1.1) has no solution of prime period 2.*

Proof. If not, let $\{x_n\}$ be a solution of (1.1) which is periodic with period 2. From equation (1.1),
we have

$$x_{-1} = \alpha + \beta \frac{x_0}{x_{-1}}, \quad x_0 = \alpha + \beta \frac{x_{-1}}{x_0}.$$

It is evident that $x_{-1} > \alpha, x_0 > \alpha$.

So

$$x_0 = \frac{1}{\beta}({x_{-1}}^2 - \alpha x_{-1}) \text{ and } x_{-1} = \frac{1}{\beta}({x_0}^2 - \alpha x_0)$$

and we have

$$\begin{gathered} {x_{-1}}^4 - 2\alpha {x_{-1}}^3 + \alpha(\alpha - \beta){x_{-1}}^2 + \beta(\alpha^2 - \beta^2)x_{-1} = 0, \\ x_{-1}(x_{-1} - \alpha - \beta)\left({x_{-1}}^2 + (\beta - \alpha)x_{-1} + \beta(\beta - \alpha)\right) = 0. \end{gathered}$$

We obtain

$$x_{-1} = \alpha + \beta,$$

or

$$f(x_{-1}) = {x_{-1}}^2 + (\beta - \alpha)x_{-1} + \beta(\beta - \alpha) = 0. \tag{2.4}$$

If $x_{-1} = \alpha + \beta$, then

$$x_0 = \frac{1}{\beta}\left((\alpha + \beta)^2 - \alpha(\alpha + \beta)\right) = \alpha + \beta.$$

It is easy to see $x_n \equiv \alpha + \beta$, which have a contradiction.

Therefore $f(x_{-1}) = 0$.

(i) When $\alpha = \beta$, from (2.4), we has $x_{-1} = 0$, which contradicts $x_{-1} > \alpha$.

(ii) When $\alpha < \beta$, $\Delta = -(\beta - \alpha)(\alpha + 3\beta) < 0$, equation (2.4) have no real roots.

(iii) When $\alpha > \beta$, $f(0) = \beta(\beta - \alpha) < 0$, $f(\alpha) = \beta^2 > 0$.

Then, since the roots of (2.4) satisfy $x_{-1} < \alpha$, we has a contradiction.

Therefore for any $\alpha \geq 0$, $\beta > 0$, equation (1.1) has no solution of prime period 2. This completes the proof. □

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(Z. LIU) COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HEBEI NORMAL UNIVERSITY, SHIJIAZHUANG, 050016, P. R. CHINA

(Z. ZHANG) COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HEBEI NORMAL UNIVERSITY, SHIJIAZHUANG, 050016, P. R. CHINA
Email address: zhaozhao1962@sina.com