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# REAL HYPERSURFACES OF A QUATERNIONIC PROJECTIVE SPACE IN TERMS OF RICCI TENSOR

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Abstract. We obtain some characterizations of a pseudo Ricci-parallel real hypersurface in a quaternionic projective space $QP^n$ and find the condition that $M$ is locally congruent to a geodesic hypersphere of $QP^n$.

## 1. introduction

Let $M$ be a connected real hypersurface of quaternionic projective space $QP^n$, $n \geq 2$, endowed with the Fubini-Study metric $G$ of constant quaternionic sectional curvature 4. Let $N$ be a unit normal vector field to $M$. Then $U_i = -J_iN_{i=1,2,3}$ are structure vectors where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of $QP^n$ (Berndt [1], Hamada [2], Ishihara [3], Martínez & Pérez, [7], Pak [8], Pérez [9, 10]). We put $f_i(X) = g(X, U_i)$ for arbitrary $X \in TM, i = 1, 2, 3$. We denote by $A, R$ and $S$ the shape operator, the curvature tensor and the Ricci tensor of type (1,1) on $M$, respectively.

Kimura & Maeda [5, 6] showed to provide some characterizations of geodesic hyperspheres in $P_n(C)$ in terms of Ricci tensor $S$. $P_n(C)(n \geq 3)$ does not admit a real hypersurface $M$ with parallel Ricci tensor $S$ Ki [4]. They characterize geodesic hyperspheres in $P_n(C)$ in terms of the derivative of $S$. The statement is as follows:

**Theorem A** (Kimura & Maeda [5]). *Let $M$ be a real hypersurface of $P_n(C)$, $n \geq 3$. Then the following are equivalent:*

(i) *The Ricci tensor $S$ of $M$ satisfies*

$$(\nabla_X S)Y = \lambda\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$$

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*for any $X, Y \in TM$, where $\lambda$ is a non-zero constant on $M$.*

(ii) *$M$ is locally congruent to a geodesic hypersphere in $P_n(C)$.*

In the next year 1993, they Kimura & Maeda [6] generalized the above Theorem A by $\lambda$ which is a function. Moreover, Theorem A was extended by Pérez [9] in the quaternionic projective space $QP^n$ in 1996 (for details, see Theorem B).

The main purpose of this paper is to generalize Pérez's Theorem B.

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## 2. Preliminaries

A *quaternionic Kähler manifold* is a Riemannian manifold $(\bar{M}, G)$ on which there exists a 3-dimensional vector bundle $\bar{V}$ of tensors of type $(1,1)$ with a local basis $\{J_i\}_{i=1,2,3}$ of almost Hermitian structures satisfying the following conditions:

(1) ${J_i}^2 = -Id\ (i = 1,2,3)$, $J_iJ_j = J_k$, where $Id$ denotes the identity endomorphism and $(i,j,k)$ is a cyclic permutation of $(1,2,3)$.

(2) If $\bar{\nabla}$ denotes the Riemannian connection on $\bar{M}$, then there exist three local 1-forms $q$ on $M$ such that

$$\bar{\nabla}_X J_i = q_k(X)J_j - q_j(X)J_k,$$

for all vector field $X$ on $M$, where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$.

Let $W$ be a subspace of $T_p\bar{M}, p \in \bar{M}$.

(i) $W$ is called *quaternionic* if $JW \subset W$ for all $J \in \bar{V}_p$.

(ii) $W$ is called *totally complex* if there exists a 1-dimensional subspace $V$ of $\bar{V}_p$ such that $JW \subset W$ for all $J \in V$ and $JW \perp W$ for all $J \in V^\perp \subset \bar{V}_p$.

(iii) $W$ is called *totally real* if $JW \perp W$ for all $J \in \bar{V}_p$.

Let $Q(X)$ be the 4-subspace spanned by vectors $X, J_1X, J_2X$ and $J_3X$ for any $X \in T_p\bar{M}$, $p \in \bar{M}$. If the sectional curvature of any section for $Q(X)$ depends only on $X$, we call it *Q-sectional curvature. A quaternionic space form of Q-sectional curvature c* is a connected quaternionic Kähler manifold with constant Q-sectional curvature c. The standard model of a quaternionic space forms are the quaternionic projective space $QP^n(c)(c > 0)$, the quaternionic space $Q^n(c = 0)$ and the quaternionic hyperbolic space $QH^n(c)(c < 0)$.

The curvature tensor $\bar{R}$ of $QP^n$ is given by

$$\bar{R}(X,Y)Z = \frac{c}{4}\,[G(Y,Z)X - G(X,Z)Y + \sum_{i=1}^{3}(G(J_iY,Z)J_iX - G(J_iX,Z)J_iY - 2G(J_iX,Y)J_iZ)], \quad (i=1,2,3)$$

for any vector fields $X, Y$ and $Z$ on $QP^n$ Ishihara [3].

Let $M$ be a real hypersurface of $QP^n$ and $i : M \to QP^n$ the isometric immersion. In a neighborhood of each point of $M$ we choose a unit normal vector field $N$ in $QP^n$. The Riemannian connections $\widetilde{\nabla}$ in $QP^n$ and $\nabla$ in $M$ are related by following formulas for any vector fields $X$ and $Y$ on $M$:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \tag{2.1}$$

$$\nabla_X N = -AX, \tag{2.2}$$

where $g$ denotes the Riemannian metric induced from the metric $G$ of $QP^n$ and $A$ is the second fundamental tensor of $M$ in $QP^n$. The mean curvature $H$ of $M$ in $QP^n$ is defined by $H = \frac{1}{4n-1}\,\mathrm{trace}\,A$.

Let $X$ be a tangent field to $M$. We write $J_iX = \phi_iX + f_i(X)N, i = 1,2,3$, where $\phi_iX$ is the tangent component of $J_iX$ and we get

$$\phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1,2,3 \tag{2.3}$$

for any $X$ tangent to $M$. We obtain

$$\phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k, \tag{2.4}$$

$$f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX), \tag{2.5}$$

where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$. It is also easy to see that for any $X, Y$ tangent to $M$,

$$g(\phi_iX,Y) + g(X,\phi_iY) = 0, \quad g(\phi_iX,\phi_iY) = g(X,Y) - f_i(X)f_i(Y), \tag{2.6}$$

$$\phi_iU_j = -\phi_jU_i = U_k. \tag{2.7}$$

From the expression of the curvature tensor of $QP^n$, $n \geq 2$, we have that the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z & \\ &= g(Y,Z)X - g(X,Z)Y \\ &+ \sum_{i=1}^{3} \Big\{ g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y + 2g(X, \phi_i Y)\phi_i Z \Big\} \\ &+ g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.8}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^{3} \Big\{ f_i(X)\phi_i Y - f_i(Y)\phi_i X + 2g(X, \phi_i Y)U_i \Big\} \tag{2.9}$$

for any $X$,$Y$,$Z$ tangent to $M$, where $R$ denotes the curvature tensor of $M$. From the equation of Gauss, if we denote by $S$ the (1,1)-type Ricci tensor of $M$ we get

$$SX = (4n+7)X - 3\sum_{i=1}^{3} f_i(X)U_i + hAX - A^2X \tag{2.10}$$

and

$$\begin{aligned} (\nabla_X S)Y = & -3\sum_{i=1}^{3} \Big\{ g(\phi_i X, Y)U_i + f_i(Y)\phi_i X \Big\} + (Xh)AY \\ & + h(\nabla_X A)Y - A(\nabla_X A)Y - (\nabla_X A)AY \end{aligned} \tag{2.11}$$

for any $X, Y$ tangent to $M$ and $h$ denotes the trace of $A$. Moreover, as we know how to derive $J_i$, $i = 1, 2, 3$, for any $X, Y$ tangent to $M$ we obtain

$$\nabla_X U_i = q_k(X)U_j - q_j(X)U_k + \phi_i AX, \tag{2.12}$$

$$(\nabla_X \phi_i)Y = q_k(X)\phi_j Y - q_j(X)\phi_k Y + f_i(Y)AX - g(AX, Y)U_i, \tag{2.13}$$

where $(i, j, k)$ denotes a cyclic permutation of $(1, 2, 3)$. These are the basic formulas for a real hypersurface of $QP^n$.

Now we prepare the following without proof in order to prove our result:

**Theorem B** (Pérez [9]). *Let $M$ be a real hypersurface of $QP^n$, $n \geq 2$. Then the following are equivalent:*

(i) *The Ricci tensor $S$ of $M$ satisfies*

$$(\nabla_X S)Y = \lambda \sum_{i=1}^{3} \Big\{ g(\phi_i X, Y)U_i + f_i(Y)\phi_i X \Big\} \tag{2.14}$$

*for any $X, Y \in TM$, where $\lambda$ is a non-zero constant on $M$.*

(ii) *$M$ is locally congruent to a geodesic hypersphere in $QP^n$.*

A real hypersurface $M$ of $QP^n$ is said to be *pseudo Ricci-parallel* if it satisfies the equation (2.14).

## 3. Main Results

The purpose of this section is to prove the following

**Theorem 3.1.** *Let $M$ be a real hypersurface of $QP^n$, $n \geq 2$. Then the following are equivalent:*

(i) *The Ricci tensor $S$ of $M$ satisfies the equation* (2.14) *and*

$$-3\sum_{k=1}^{3}\sum_{i=1}^{3} f_i(A\phi_k AU_k)U_i = 2\sum_{k=1}^{3}\phi_k SU_k \tag{3.1}$$

*for any $X, Y \in TM$, where $\lambda$ is a function on $M$.*

(ii) *$M$ is locally congruent to a geodesic hypersphere of $QP^n$.*

*Proof.* Suppose that the condition (i) holds. From (2.12), (2.13) and (2.14), we have

$$\begin{aligned}
&(\nabla_W(\nabla_X S))Y - (\nabla_{\nabla_W X} S)Y \\
&\quad = \sum_{i=1}^{3}\Big[(W\lambda)\Big\{g(\phi_i X, Y)U_i + f_i(Y)\phi_i X\Big\} \\
&\qquad + \lambda\Big\{f_i(X)g(AW, Y)U_i + g(\phi_i X, Y)\phi_i AW \\
&\qquad + g(\phi_i AW, Y)\phi_i X + f_i(X)f_i(Y)AW - 2f_i(Y)g(AW, X)U_i\Big\}\Big]
\end{aligned} \tag{3.2}$$

for any $X$, $Y$,$W$ tangent to $M$.

Exchanging $X$ and $W$ in (3.2), we have the following

$$\begin{aligned}
&(R(W, X)S)Y \\
&\quad = \sum_{i=1}^{3}\Big[(W\lambda)\Big\{g(\phi_i X, Y)U_i + f_i(Y)\phi_i X\Big\} \\
&\qquad - (X\lambda)\Big\{g(\phi_i W, Y)U_i + f_i(Y)\phi_i W\Big\} \\
&\qquad + \lambda\Big\{f_i(X)g(AW, Y)U_i + g(\phi_i X, Y)\phi_i AW \\
&\qquad + g(\phi_i AW, Y)\phi_i X + f_i(X)f_i(Y)AW - f_i(W)g(AX, Y)U_i
\end{aligned}$$

$$- g(\phi_i W, Y)\phi_i AX - g(\phi_i AX, Y)\phi_i W - f_i(W)f_i(Y)AX\Big\}\Big]. \tag{3.3}$$

Let $e_1, \dots, e_{4n-1}$ be local fields of orthonormal vectors on $M$. From (3.3) and (2.3) we find

$$\begin{aligned}&\sum_{a=1}^{4n-1} g\big((R(e_a, X)S)Y, e_a\big)\\ &= \sum_{i=1}^{3}\Big[(U_i\lambda)g(\phi_i X, Y) + (\phi_i X\lambda)f_i(Y)\\ &\qquad + \lambda\Big\{f_i(X)f_i(AY) - g(\phi_i X, A\phi_i Y) + hf_i(X)f_i(Y) - 2f_i(Y)f_i(AX)\Big\}\Big]. \end{aligned}\tag{3.4}$$

Now note that the left hand side of (3.4) is symmetric with respect to $X$ and $Y$, then we have

$$\begin{aligned}\sum_{i=1}^{3}\Big[&2(U_i\lambda)g(\phi_i X, Y) + (\phi_i X\lambda)f_i(Y) - (\phi_i Y\lambda)f_i(X)\\ &+ 3\lambda\Big\{f_i(X)f_i(AY) - f_i(Y)f_i(AX)\Big\}\Big] = 0. \end{aligned}\tag{3.5}$$

Putting $Y = \phi_k Y$ and contracting with respect to $X, Y$ in (3.5), we find

$$\begin{aligned}\sum_{a=1}^{4n-1}\sum_{i=1}^{3}\Big[&2(U_i\lambda)g(\phi_i e_a, \phi_k e_a) + (\phi_i e_a\lambda)f_i(\phi_k e_a) - (\phi_i\phi_k e_a\lambda)f_i(e_a)\\ &+ 3\lambda\Big\{f_i(e_a)f_i(A\phi_k e_a) - f_i(\phi_k e_a)f_i(Ae_a)\Big\}\Big] = 0,\end{aligned}$$

therefore

$$U_i\lambda = 0 = f_j(AU_k), \tag{3.6}$$

where $(i, j, k)$ denotes a cyclic permutation of $(1, 2, 3)$.

On the other hand, setting $Y = U_k$ and $X = \phi_k W$ in (3.5), we see

$$(\phi_k^2 W\lambda) - 3\lambda f_k(A\phi_k W) = 0, \quad k = 1, 2, 3.$$

This, together with (2.3) and (3.6), shows

$$W\lambda = 3\lambda\phi_k AU_k, \quad k = 1, 2, 3$$

for any $W \in TM$, therefore,

$$\operatorname{grad}\lambda = 3\lambda\phi_k AU_k, \quad k = 1, 2, 3. \tag{3.7}$$

Hence Equation (3.3) asserts that

$$\begin{aligned}(R(W,X)S)Y&\\ =\lambda\sum_{i=1}^{3}\Big[&g\big(\phi_k AU_k,W\big)\big\{g(\phi_i X,Y)U_i+f_i(Y)\phi_i X\big\}\\ &-g\big(\phi_k AU_k,X\big)\big\{g(\phi_i W,Y)U_i+f_i(Y)\phi_i W\big\}+f_i(X)g(AW,Y)U_i\\ &+g\big(\phi_i X,Y\big)\phi_i AW+g\big(\phi_i AW,Y\big)\phi_i X+f_i(X)f_i(Y)AW\\ &-f_i(W)g(AX,Y)U_i-g\big(\phi_i W,Y\big)\phi_i AX,\\ &-g\big(\phi_i AX,Y\big)\phi_i W-f_i(W)f_i(Y)AX\Big].\end{aligned}\tag{3.8}$$

It follows from (2.3) and (3.8) that

$$\sum_{k=1}^{3}\sum_{a=1}^{4n-1}g\Big(\big(R(e_a,X)S\big)U_k,\phi_k e_a\Big)=\big(-12n+17\big)\lambda\sum_{k=1}^{3}g\big(\phi_k AU_k,X\big).\tag{3.9}$$

On the other hand we have, where $k=1,2,3$,

$$\begin{aligned}&\sum_{a=1}^{4n-1}g\Big(\big(R(e_a,X)S\big)U_k,\phi_k e_a\Big)\\ &\quad=\sum_{a=1}^{4n-1}g\Big(R(e_a,X)(SU_k),\phi_k e_a\Big)-\sum_{a=1}^{4n-1}g\Big(R(e_a,X)U_k,S\phi_k e_a\Big).\end{aligned}\tag{3.10}$$

Equation (2.10) shows that

$$\operatorname{trace}AS\phi_k=0,\quad k=1,2,3.\tag{3.11}$$

From (2.3), (2.8), (3.10) and (3.11) we see that

$$\begin{aligned}&\sum_{k=1}^{3}\sum_{a=1}^{4n-1}g\Big(\big(R(e_a,X)S\big)U_k,\phi_k e_a\Big)\\ &\qquad=\sum_{k=1}^{3}\Big[g\Big(AX,\big(S\phi_k A-\phi_k AS\big)U_k\Big)+4ng\big(\phi_k X,SU_k\big)\Big].\end{aligned}\tag{3.12}$$

By virtue of (3.9) and (3.12) we get

$$(-12n+17)\lambda\sum_{k=1}^{3}\phi_k AU_k=\sum_{k=1}^{3}\big(AS\phi_k AU_k-A\phi_k ASU_k-4n\phi_k SU_k\big).\tag{3.13}$$

Gauss equation (2.8) tells us that

$$\sum_{a=1}^{4n-1}g\Big(\big(R(e_a,\phi_k e_a)S\big)U_k,X\Big)$$

$$= \Big[ 2g\big(A\phi_k AX, SU_k\big) + 2g\big(A\phi_k AU_k, SX\big) - 4(2n-1)g\big(\phi_k SU_k, X\big) \Big], \qquad (3.14)$$

for $k = 1, 2, 3$. On the other hand, from (3.8), we obtain

$$\sum_{a=1}^{4n-1} g\Big(\big(R(e_a, \phi_k e_a)S\big)U_k, X\Big) = 6\lambda g\big(AU_k, \phi_k X\big), \quad k = 1, 2, 3. \qquad (3.15)$$

In view of (3.14) and (3.15) we have

$$3\lambda \sum_{k=1}^{3} \phi_k AU_k = \sum_{k=1}^{3} \Big[ A\phi_k ASU_k - SA\phi_k ASU_k + 2(2n-1)\phi_k SU_k \Big]. \qquad (3.16)$$

Equation (2.10) implies that

$$SA\phi_k AU_k - AS\phi_k AU_k = 3\sum_{i=1}^{3} f_i(A\phi_k AU_k)U_i, \quad k = 1, 2, 3. \qquad (3.17)$$

From (3.13), (3.16) and (3.17) we find

$$(-12n + 20)\lambda \sum_{k=1}^{3} \phi_k AU_k = \sum_{k=1}^{3} \big(AS\phi_k AU_k - SA\phi_k AU_k - 2\phi_k SU_k\big). \qquad (3.18)$$

By virtue of (3.1) we get

$$\lambda \phi_k AU_k = 0, \quad k = 1, 2, 3. \qquad (3.19)$$

Consequently, from (3.7) and (3.19) we can conclude that $\lambda$ is locally constant. Hence this Theorem 3.1 is proved by Theorem B. □

*Remark.* As illustrated by Theorem 3.1, without any additional condition it is impossible to generalize Theorem B under the condition that $\lambda$ is a fuction.

Motivated by Theorem 3.1, we prove the following

**Proposition 1.** *Let $M$ is a real hypersurface of $QP^n$, $n \geq 2$. Then the following inequality holds:*

$$\Big\|\nabla S\Big\|^2 \geq \frac{1}{3(2n-1)} \left( \sum_{i=1}^{3} \sum_{a=1}^{4n-1} g\Big((\nabla_a S)U_i, \phi_i e_a\Big) \right)^2 \qquad (3.20)$$

*where $S$ is the Ricci tensor of $M$ and $e_1, \ldots, e_{4n-1}$ are local fields of orthonormal frames of $M$. Moreover, the equality of (3.20) holds if and only if $M$ is locally congruent to a geodesic hypersphere of $QP^n$.*

*Proof.* We define the following tensor $T$ on $M$ as:

$$T(X,Y) = (\nabla_X S)Y - \lambda \sum_{i=1}^{3} \Big\{ g\big(\phi_i X, Y\big)U_i + f_i(Y)\phi_i X \Big\}, \tag{3.21}$$

where $\lambda$ is a function on $M$. Calculating the length of $T$, we obtain

$$\|T\|^2 = \|\nabla S\|^2 - 4\lambda \sum_{i=1}^{3} \sum_{a=1}^{4n-1} g((\nabla_a S)U_i, \phi_i e_a) + 12\lambda^2(2n-1)$$

for any real number $\lambda$ at any point $p \in M$, we obtain the following inequality

$$12\lambda^2(2n-1) - 4\lambda \sum_{i=1}^{3} \sum_{a=1}^{4n-1} g((\nabla_a S)U_i, \phi_i e_a) + \|\nabla S\|^2 \geq 0. \tag{3.22}$$

Hence the discriminant of (3.22) shows (3.20). From to this discussion, we find that the equality of (3.20) implies $T = 0$, that is to say, $M$ is locally congruent to a geodesic hypersphere in $QP^n$ (*cf.* Theorem 3.1). □

*Remark.* The right hand side of (3.20) can be expressed in terms of the shape operator $A$ as:

$$\frac{1}{3(2n-1)} \left\{ \sum_{i=1}^{3} \Big( 4n\big(h - f_i(AU_i)\big) + \phi_i AU_i(h) + \operatorname{trace}\big((\nabla_{U_i} A)A\phi_i\big)\Big)\right\}^2.$$

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