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RADICALLY PRINCIPAL IDEAL RINGS

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Abstract. Let R be a commutative ring with identity, X be an indeterminate over R, and $R[X]$ be the polynomial ring over R. In this paper, we study when $R[X]$ is a radically principal ideal ring. We also study the t-operation analog of a radically principal ideal domain, which is said to be t-compactly packed. Among them, we show that if R is an integrally closed domain, then $R[X]$ is t-compactly packed if and only if R is t-compactly packed and every prime ideal Q of $R[X]$ with $Q \cap R = (0)$ is radically principal.

1. Introduction

All rings considered in this paper are commutative rings with identity. Let R be a ring, $\text{Spec}(R)$ be the set of prime ideals of R, X be an indeterminate, and $R[X]$ be the polynomial ring over R. An ideal I of R is said to be radically principal if $\sqrt{I} = \sqrt{aR}$ for some $a \in R$.

In [19, Theorem 1.1], Reis and Viswanathan showed that if R is a Noetherian ring, then every prime ideal of R is radically principal if and only if R is compactly packed, i.e., if an ideal I of R is contained in $\bigcup_{\alpha\in\mathcal{A}} P_\alpha$, where $\{P_\alpha \mid \alpha \in \mathcal{A}\} \subseteq \text{Spec}(R)$, then $I \subseteq P_\alpha$ for some $\alpha \in \mathcal{A}$. Then, in [20, Theorem], Smith completely generalized the result of [19, Theorem 1.1] to an arbitrary ring, i.e., he proved that R is compactly packed if and only if every prime ideal of R is radically principal. In [18], Oda studied Krull domains and Noetherian domains whose height-one prime ideals are radically principal. Oda also called an integral domain R a radically principal ideal domain (radically PID) if every nonzero ideal of R is radically principal. More generally, in [4], the authors called R a radically principal ring if every ideal of R is radically principal. Among other things, the authors of [4] showed that R is a radically principal ring if and only if every prime ideal of R is radically principal [4,

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Theorem 2.7], whence the radically principal ring is just the compactly packed ring by [20, Theorem]. They also showed that $R[X]$ is a radically principal ring if and only if R is a zero-dimensional radically principal ring [4, Theorem 4.3].

Let t be the so-called t-operation on an integral domain D. (The t-operation will be reviewed in the sequel.) In [7], the authors studied an integral domain whose prime t-ideals are radically principal. Among them, they showed that D is compactly packed if and only if every prime t-ideal of D is radically principal and every nonzero prime ideal of D is a t-ideal [7, Proposition 3.1]. In [5, 17], the authors also studied several types of integral domains in which every prime t-ideal is radically principal under the name of t-compactly packed.

A ring R is called a principal ideal ring (PIR) if each proper ideal of R is principal. Following [18] and [4], we will say that R is a radically principal ideal ring (radically PIR) if each ideal of R is radically principal. Hence, a PIR is a radically PIR, while a radically PIR need not be a PIR (for example, every finite-dimensional valuation domain is radically principal). In this paper, we study when $R[X]$ is a radically PIR. In Section 2, among them, we show that $R[X]$ is a radically PIR if and only if every maximal ideal of $R[X]$ is radically principal. An integral domain R is said to be t-compactly packed if every prime t-ideal of R is radically principal. In Section 3, we give a partial answer to the question of when $R[X]$ is t-compactly packed for an integral domain R. For example, we show that if R is an integrally closed domain, then $R[X]$ is t-compactly packed if and only if R is t-compactly packed and every prime ideal Q of $R[X]$ with $Q \cap R = (0)$ is radically principal. As a corollary, we have that a Krull domain R is t-compactly packed if and only if $R[X]$ is t-compactly packed.

2. Radically Principal Ideal Rings

Let R be a ring, X be an indeterminate over R, and $R[X]$ be the polynomial ring over R. It is well-known and easy to see that R is a PIR if and only if every prime ideal of R is principal. This is true of a radically PIR. That is, R is a radically PIR if and only if every prime ideal of R is radically principal [4, Theorem 2.7]. We first give a simple proof of [4, Theorem 2.7] for easy reference of the reader.

Lemma 1. *A ring R is a radically PIR if and only if every prime ideal of R is radically principal.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) Let I be an ideal of R. If every ideal of R is radically principal, then there are only finitely many minimal prime ideals of I [13, Theorem 1.6], say, $P_1, \ldots, P_n$. Hence, if $P_i = \sqrt{a_iR}$ for some $a_i \in R$, then $\sqrt{I} = P_1 \cap \cdots \cap P_n = \sqrt{a_1R} \cap \cdots \cap \sqrt{a_nR} = \sqrt{a_1 \cdots a_nR}$. Thus, I is radically principal. □

It is clear that if M is a maximal ideal of $R[X]$, then (i) $(M \cap R)[X] \subsetneq M$ and (ii) if M is principal, then $M \cap R$ is a minimal prime ideal of R [1, Theorem 9]. Recently, we generalized this result to a radically principal ideal of $R[X]$.

Lemma 2. *Let R be a ring, $R[X]$ be the polynomial ring over R, and Q be a prime ideal of $R[X]$ such that $(Q \cap R)[X] \subsetneq Q$. If Q is a radically principal ideal, then $Q \cap R$ is a minimal prime ideal of R and $htQ = 1$.*

Proof. [6, Proposition 3]. □

The next result is a complete characterization of when $R[X]$ is a radically PIR.

Proposition 3. *The following statements are equivalent for a ring R.*

(1) R is a zero-dimensional radically PIR.

(2) R is a finite direct sum of zero dimensional local rings.

(3) $R[X]$ is a radically PIR.

(4) Every maximal ideal of $R[X]$ is radically principal.

(5) R has the following property: If a prime ideal P of R is contained in $\bigcup_{\alpha \in \mathcal{A}} P_\alpha$, where $\{P_\alpha \mid \alpha \in \mathcal{A}\} \subseteq Spec(R)$, then $P = P_\alpha$ for some $\alpha \in \mathcal{A}$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) See [4, Theorem 4.3].

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let P be a prime ideal of R. Then there is a maximal ideal M of $R[X]$ such that $P[X] \subseteq M$, so $P \subseteq M \cap R$. Hence, by Lemma 2, $P = M \cap R$ and P is minimimal. Thus, R is zero-dimensional. Moreover, $P[X] = \sqrt{fR[X]}$ for some $f \in R[X]$. It is clear that if a is the constant term of f, then $P[X] = \sqrt{aR[X]}$, and hence $P = \sqrt{aR}$. Thus, R is a zero-dimensional radically PIR.

(1) $\Leftrightarrow$ (5) [7, Proposition 2.2]. □

The following corollary is a special case of Proposition 3, which gives an answer to when the polynomial ring $D[X]$ over an integral domain D is a radically PIR.

Corollary 4. *The following statements are equivalent for an integral domain D.*

(1) D is a field.

(2) $D[X]$ is a PID.
(3) $D[X]$ is a radically PIR.
(4) Every maximal ideal of $D[X]$ is principal.
(5) Every maximal ideal of $D[X]$ is radically principal.

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.
(5) $\Rightarrow$ (3) This follows from Proposition 3.
(3) $\Rightarrow$ (2) [4, Corollary 4.2]. □

A PIR is a special primary ring (SPR) if it has exactly one prime ideal. We say that R is a unique factorization ring (UFR) if each nonunit element of R can be written as a finite product of prime elements [11, Theorem 4].

Lemma 5. *A ring R is a PIR if and only if R is a UFR and a radically PIR.*

Proof. This follows from the following observation: (i) R is a PIR (resp., UFR) if and only if R is a finite direct sum of PIDs (resp., UFDs) [22, Theorem 33, page 245] (resp., [10, Theorem 19]); (ii) if R is a finite direct sum of rings, then R is a radically PIR if and only if each direct summand of R is a radically PIR [4, Corollary 2.10], and (iii) a UFD D is a radically PIR if and only if each nonzero prime ideal of D is a maximal ideal, if and only if $\dim D \leq 1$, if and only if D is a PID. □

The polynomial ring $D[X]$ over an integral domain D is a PID if and only if D is a field. Hence, the following result is a simple corollary of an already known result on UFRs [2, Theorem 2.7(1)] that $R[X]$ is a UFR if and only if R is a finite direct sum of UFDs. This result was also proved by Chimal-Dzul and López-Andrade [9, Theorem 2.3] in a different way.

Corollary 6. *Let $R[X]$ be the polynomial ring over a ring R. Then $R[X]$ is a PIR if and only if R is a finite direct sum of fields.*

Proof. Suppose that $R[X]$ is a PIR. Then $R[X]$ is a UFR, and hence R is a finite direct sum of UFDs [2, Theorem 2.7(1)], say, $R = D_1 \oplus \cdots \oplus D_n$ for some UFDs $D_1, \ldots, D_n$. Also, by Lemma 5 and Proposition 3, R is a zero-dimensional radically PIR. Hence, $\dim D_i = 0$, and thus D_i is a field for $i = 1, \ldots, n$. Thus, R is a finite direct sum of fields. The converse is clear. □

Let V be a rank-one nondiscrete valuation domain with maximal ideal M. Then M is radically principal but not principal. We use the result of this section to give another example of maximal ideals that are radically principal but not principal.

Example 7. Let R be an SPR with maximal ideal P, and assume that R is not a field. Then R is a zero-dimensional radically PIR, and hence $R[X]$ is a radically PIR by Proposition 3. In particular, each maximal ideal of $R[X]$ is radically principal. However, $R[X]$ is not a PIR by Corollary 6. Note that $P[X]$ is a unique non-maximal prime ideal of $R[X]$ and $P[X]$ is principal. Hence, $R[X]$ has a maximal ideal that is not principal.

Let D be an integral domain. It is easy to see that if P is a nonzero prime ideal of D that is radically principal, then P is minimal over a nonzero principal ideal. A nonzero principal ideal is a so-called t-ideal, and hence P is also a t-ideal. We end this section by recalling the notion of t-operation for the study of radically principal ideals of an integral domain in the next section.

Let K be the quotient field of D. A D-submodule A of K is said to be a fractional ideal of D if $dA \subseteq D$ for some $0 \neq d \in D$. For a nonzero fractional ideal A of D, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$, then A^{-1} is also a nonzero fractional ideal of D. Hence, $A_v = (A^{-1})^{-1}$ and $A_t = \bigcup\{J_v \mid J \subseteq A$ and J is a nonzero finitely generated fractional ideal of $D\}$ are well-defined. An ideal A of D is a t-ideal if $A_t = A$. A prime t-ideal is a prime ideal that is also a t-ideal. A maximal t-ideal is a t-ideal that is maximal among all proper integral t-ideals under inclusion. It is known that a maximal t-ideal is a prime ideal and if $\sqrt{aD}$ for $a \in D$ is a maximal t-ideal, then a is primary (cf. the proof of [3, Theorem 2.4]), i.e., aD is a primary ideal. Let t-Spec(D) be the set of prime t-ideals of D. It is well known that a nonzero principal ideal is a t-ideal and a prime ideal that is minimal over a t-ideal is a t-ideal, so t-Spec$(D) = \emptyset$ if and only if D is a field.

3. t-Compactly Packed Domains

In this section, we study the t-operation analog of radically PIDs. Let D be an integral domain. An integral t-ideal I of D is said to be t-compactly packed if for any set Λ of prime t-ideals of D with $I \subseteq \bigcup_{Q\in\Lambda} Q$, one has $I \subseteq P$ for some $P \in \Lambda$. A class $\mathcal{A}$ of integral t-ideals of D is said to be t-compactly packed if every element of $\mathcal{A}$ is t-compactly packed. Finally, D is said to be t-compactly packed if every integral t-ideal of D is t-compactly packed. The equivalence of (1) and (4) in the next proposition was noted in [5, Definition 2.1].

Proposition 8. *Let D be an integral domain and t-Spec(D) be the set of prime t-ideals of D. Then the following statements are equivalent.*

(1) D is t-compactly packed.
(2) t-Spec(D) is t-compactly packed.
(3) Every prime t-ideal of D is radically principal.
(4) Every integral t-ideal of D is radically principal.

Proof. (1) $\Leftrightarrow$ (2) [17, Theorem 2.1].

(2) $\Leftrightarrow$ (3) [7, Proposition 3.1].

(3) $\Rightarrow$ (4) Let I be an integral t-ideal of D. Then every minimal prime ideal P of I is a t-ideal, and hence $P = \sqrt{aD}$ for some $a \in D$. Hence, I has finitely many minimal prime ideals of D [13, Theorem 1.6], say, $P_1, \ldots, P_n$, and $P_i = \sqrt{a_i D}$ for some $a_i \in D$. Thus,

$$\sqrt{I} = P_1 \cap \cdots \cap P_n = \sqrt{a_1 D} \cap \cdots \cap \sqrt{a_n D} = \sqrt{a_1 \cdots a_n D}.$$

Therefore, I is radically principal.

(4) $\Rightarrow$ (3) Clear. □

A nonzero prime ideal Q of $D[X]$ is called an upper to zero in $D[X]$ if $Q \cap D = (0)$. It is useful to note that every upper to zero in $D[X]$ is a prime t-ideal, because it is minimal over a nonzero principal ideal. The next result is a special case of [5, Corollary 3.3] in which the authors studied when $D[X]$ is t-compactly packed.

Proposition 9. *Let D be an integrally closed domain. Then $D[X]$ is t-compactly packed if and only if D is t-compactly packed and every upper to zero in $D[X]$ is radically principal.*

Proof. ($\Rightarrow$) Let P be a prime t-ideal of D. Then $P[X]$ is a prime t-ideal of $D[X]$ [16, Corollary 2.3], so $P[X]$ is radically principal by assumption, whence P is radically principal. Thus, D is t-compactly packed. Next, let Q be an upper to zero in $D[X]$. Then Q is a prime t-ideal of $D[X]$, so Q is radically principal by assumption.

($\Leftarrow$) Let Q be a prime t-ideal of $D[X]$. Then either $Q \cap D = (0)$ or $Q \cap D \neq (0)$ and $Q = (Q \cap D)[X]$ [14, Lemma 4.5] because D is integrally closed. Hence, if $Q \cap D = (0)$, then Q is radically principal by assumption. Next, assume that $Q \cap D \neq (0)$ and $Q = (Q \cap D)[X]$. Then $Q = Q_t = (Q \cap D)_t[X]$ [16, Corollary 2.3], so $(Q \cap D)_t = Q \cap D$, and hence $Q \cap D$ is radically principal by assumption. Thus, $Q = (Q \cap D)[X]$ is radically principal. □

We next give a partial answer to the question of when a prime ideal of $D[X]$ is radically principal. We first recall that an integral domain D is an almost GCD-domain (AGCD-domain) if for any $0 \neq a, b \in D$, there is a positive integer $n = n(a,b)$ such that $a^nD \cap b^nD$ is principal. Clearly, a GCD domain is an AGCD-domain. For more on AGCD-domains, see [21].

Proposition 10. *Let D be an integrally closed AGCD domain and Q be a nonzero prime ideal of $D[X]$. Then Q is radically principal if and only if Q satisfies one of the following conditions:*

(1) $Q \cap D = (0)$.

(2) $Q \cap D \neq (0)$, $Q = (Q \cap D)[X]$, and $Q \cap D$ is radically principal.

Proof. ($\Rightarrow$) Suppose that $Q = \sqrt{fD[X]}$ for some $f \in D[X]$. Then Q is minimal over $fD[X]$, so Q is a prime t-ideal of $D[X]$. Hence, either $Q \cap D = (0)$ or $Q \cap D \neq (0)$ and $Q = (Q \cap D)[X]$ [14, Lemma 4.5]. In particular, if $Q = (Q \cap D)[X]$, then $f \in D$, and thus $Q \cap D = \sqrt{fD}$.

($\Leftarrow$) If $Q \cap D = (0)$, then Q contains a primary element [3, Corollary 2.5]. Note that Q is a nonzero minimal prime ideal of $D[X]$, so if $f \in Q$ is a primary element, then $Q = \sqrt{fD[X]}$. Next, assume that $Q = (Q \cap D)[X]$ and $Q \cap D$ is radically principal. Then $Q \cap D = \sqrt{aD}$ for some $a \in D$, and hence $(Q \cap D)[X] = \sqrt{aD[X]}$. Thus, Q is radically principal. □

An integral domain D is a Prüfer v-multiplication domain (PvMD) if each nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$. Let $T(D)$ be the abelian group of t-invertible fractional t-ideals of D under $I * J = (IJ)_t$ and $P(D)$ be its subgroup of nonzero principal fractional ideals. The t-class group of D is defined by the factor group $\mathrm{Cl}(D) := T(D)/P(D)$ of $T(D)$ modulo $P(D)$. It is known that D is an integrally closed AGCD domain if and only if D is a PvMD with $\mathrm{Cl}(D)$ torsion [21, Theorem 3.9], i.e., if I is a nonzero finitely generated ideal, then $(I^n)_t$ is principal for some integer $n \geq 1$.

Corollary 11. [8, Corollary 1.2] *Let D be a PvMD. Then $D[X]$ is t-compactly packed if and only if D is a t-compactly packed AGCD domain.*

Proof. ($\Rightarrow$) Let Q be an upper to zero in $D[X]$. Then $Q = \sqrt{fD[X]}$ for some $f \in D[X]$, and since D is a PvMD, Q is a maximal t-ideal [15, Proposition 3.2], so $fR[X]$ is a primary ideal of $D[X]$. Thus, each upper to zero in $D[X]$ contains a primary element, and hence D is an AGCD domain [3, Corollary 2.3]. ($\Leftarrow$) A PvMD is integrally closed, so D is an integrally closed AGCD domain, and hence every upper to zero in $D[X]$ is radically principal by Proposition 10. Thus, the result follows from Proposition 9. □

An integral domain D is a Krull domain if every nonzero ideal of D is t-invertible [14, Theorem 2.3]. A Krull domain D is called an almost factorial domain if $\mathrm{Cl}(D)$ is torsion [12].

Corollary 12. *The following statements are equivalent for a Krull domain D.*

(1) D is an almost factorial domain.
(2) D is t-compactly packed.
(3) $D[X]$ is t-compactly packed.

Proof. (1) ⇔ (2) See [18, Proposition 7] or [5, Proposition 3.1].

(2) ⇒ (3) A Krull domain is a PvMD, so a Krull domain is an AGCD domain if and only if it is an almost factorial domain. Hence, if D is t-compactly packed, then D is a t-compactly packed AGCD domain by the equivalence of (1) and (2), and hence $D[X]$ is t-compactly packed by Corollary 11.

(3) ⇒ (2) A Krull domain is integrally closed, so if $D[X]$ is t-compactly packed, then D is t-compactly packed by Proposition 9. □

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References

1. D.D. Anderson & S. Chun: Some remarks on principal prime ideals. Bull. Aust. Math. Soc. **83** (2011), 130-137. https://doi.org/10.1017/S000497271000170X
2. D.D. Anderson & R. Markanda: Unique factorization rings with zero divisors. Houston J. Math. **11** (1985), 15-30.
3. D.F. Anderson & G.W. Chang: Almost splitting sets in integral domains. II. J. Pure Appl. Algebra **208** (2007), 351-359. https://doi.org/10.1016/j.jpaa.2006.01.006
4. M. Aqalmoun & M. El Ouarrachi: Radically principal rings. Khayyam J. Math. **6** (2020), 243-249.

5. A. Benobaid & A. Minouni: Compact and coprime packedness with respect to star operations. Houston J. Math. **37** (2011), 1043-1061.
6. G.W. Chang & S. Chun: How many prime polynomials are there in a polynomial ring ?. submitted.
7. G.W. Chang & C.J. Hwang: Covering and intersection conditions for prime ideals. Korean J. Math. **17** (2009), 15-23.
8. G.W. Chang & H. Kim: Radical perfectness of prime ideals in certain integral domains. J. Commutative Algebra **9** (2017), 31-48. `https://doi.org/10.1216/JCA-2017-9-1-31`
9. H. Chimal-Dzul & C.A. López-Andrade: When is $R[x]$ a principal ideal ring ?. Rev. Integr. Temas Mat. **35** (2017), 143-148. `https://dx.doi.org/10.18273/revint.v35n2-2017001`
10. C.R. Fletcher: The structure of unique factorization rings. Proc. Cambridge Philos. Soc. **67** (1970), 535-540. `https://doi.org/10.1017/S0305004100045825`
11. ______: Equivalent conditions for unique factorization. Publ. Dép. Math. (Lyon) **8** (1971), 13-22.
12. R. Fossum: The Divisor Class Group of a Krull Domain. Springer-Verlag, New York/ Berlin, 1973.
13. R. Gilmer & W. Heinzer: Primary ideals with finitely generated radical in a commutative ring. Manuscripta Math. **78** (1993), 201-221. `https://doi.org/10.1007/BF02599309`
14. E. Houston & M. Zafrullah: Integral domains in which each t-ideal is divisorial. Michigan Math. J. **35** (1988), 291-300. `https://doi.org/10.1307/mmj/1029003756`
15. ______: On t-invertibility II. Comm. Algebra **17** (1989), 1955-1969. `https://doi.org/10.1080/00927878908823829`
16. B.G. Kang: Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra **123** (1989), 151-170. `https://doi.org/10.1016/0021-8693(89)90040-9`
17. H. Kim: Overrings of t-comprimely packed domains. J. Korean Math. Soc. **48** (2011), 191-205. `https://doi.org/10.4134/JKMS.2011.48.1.191`
18. S. Oda: Radically principal and almost factorial. Bull. Fac. Sci. Ibaraki Univ. Ser. A **26** (1994), 17-24. `https://doi.org/10.5036/bfsiu1968.26.17`
19. C.M. Reis & T.M. Viswanathan: A compactness property for prime ideals in Noetherian rings. Proc. Amer. Math. Soc. **25** (1970), 353-356. `https://doi.org/10.1090/S0002-9939-1970-0254031-6`
20. W. Smith: A covering condition for prime ideals. Proc. Amer. Math. Soc. **30** (1971), 451-452.
21. M. Zafrullah: A general theory of almost factoriality. Manuscripta Math. **51** (1985), 29-62.
22. O. Zariski & P. Samuel: Commutative Algebra. vol. I, Van Nostrand, Princeton, 1960.

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