J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
https://doi.org/10.7468/jksmeb.2023.30.4.417
Volume 30, Number 4 (November 2023), Pages 417–427

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

SOME RESULTS ON BEST PROXIMITY POINT FOR CYCLIC B-CONTRACTION AND S-WEAKLY CYCLIC B-CONTRACTION MAPPINGS

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Abstract. The purpose of this paper is establish the existence of proximity point for the cyclic B-contraction mapping on metric spaces and uniformly convex Banach spaces. Also, we prove the common proximity point for the S-weakly cyclic B-contraction mapping. In addition, a few examples are provided to demonstrate our findings.

1. Introduction

Results that support finding the best proximity points by using various cyclic contraction operators are among the popular topics in fixed point theory and have received considerable interest recently because of their numerous applications in astronomy, differential geometry, economics, and so on. The first result in this area was reported by Kirk et al. in [9]. Later, many authors continued investigation and more results have been obtained in [1], [5], [6], [7], [8], [10], [11]. Originally, Marudai et al. [12] found the B-contraction operator on metric spaces and proved various fixed-point results. In addition to that, which is also recent, Theivaraman et al. [4] demonstrated approximate fixed point results using B-contraction mapping on metric spaces which is not necessarily complete. Subsequently, fabulous results like proximity point results have extensively attracted much attention from more and more researchers, and they have proposed several methods to find proximity points (refer, [13], [14], [15], [16] & [17]). Successively, several researchers were carried out various contraction mappings in different spaces (see, [18], [19], [20]).

Received by the editors May 19, 2023. Revised August 19, 2023. Accepted August 22, 2023.
2020 *Mathematics Subject Classification.* 47H10, 54H26.
Key words and phrases. cyclic B-contraction, proximity point, S-weakly cyclic B-contraction, common proximity point.

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The purpose of this study is to investigate some innovative results that involve proximity point theorems for the cyclic B-contraction, the common best proximity point theorem for S-weakly cyclic B-contraction and the proximity point results for cyclic B-contraction on uniformly convex Banach spaces. We first recall the definition of cyclic map and best proximity point. The following notions are used subsequently:

$$\begin{aligned} d(A,B) :&= inf\{d(x,y) : x \in A \text{ and } y \in B\} \\ A_0 :&= \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\} \\ B_0 :&= \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\} \end{aligned}$$

Definition 1.1 ([3])**.** A selfmap $T : X \to X$ has a proximity point if there exists $x \in X$ such that $d(x,Tx) = d(A,B)$.

Definition 1.2 ([2])**.** Let (X,d) be a metric space X and $S,T : X \to X$ be a selfmap. Then, S and T have a common best proximity point, if there exists $x \in X$ such that $d(x,Tx) = d(x,Sx) = d(A,B)$.

Definition 1.3 ([3])**.** A subset K of a metric space $X$is boundedly compact if each bounded sequence in K has a subsequence converging to a point in K.

Suppose X is uniformly convex (and hence reflexive) Banach space with modulus of convexity δ. Then $\delta(\varepsilon > 0)$ for $\varepsilon > 0$, and $\delta(.)$ is strictly increasing. Moreover if $x,y,p \in X, R > 0$ and $r \in [0,2R]$,

$$\left.\begin{aligned} \|x-p\| \leq R \\ \|y-p\| \leq R \\ \|x-y\| \geq r \end{aligned}\right\} \implies \|\frac{x+y}{2} - p\| \leq \Big(1 - \delta\Big(\frac{r}{R}\Big)\Big)R.$$

Definition 1.4 ([3])**.** Let A,B be nonempty subsets of a metric space X, $T : A \cup B \to A \cup B$ is said to be *cyclic contraction*, if

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$; and
(ii) $d(Tx,Ty) \leq kd(x,y) + (1-k)d(A,B)$, for some $k \in (0,1)$ and for all $x \in A, y \in B$.

Note that from *(ii)*, T satisfies $d(Tx,Ty) \leq d(x,y)$, for all $x \in A, y \in B$. Also, *(ii)* can be rewritten as $d(Tx,Ty) - d(A,B) \leq k(d(Tx,Ty) - d(A,B))$, for all $x \in A$, $y \in B$.

Definition 1.5 ([3])**.** The set B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some x in A has a convergent subsequence.

It is obvious that any compact set is approximatively compact, and that any set is approximatively compact with respect to itself. Further, it is given in [5] that, if A is compact and B is approximatively compact with respect to A, then it is ensured that A_0 and B_0 are nonempty. The sections below show that our main findings in this manuscript.

2. Proximity Point Theorems for Cyclic B-contraction

Definition 2.1. Let A, B be nonempty closed subsets of a metric space X, $T : A \cup B \to A \cup B$ is said to be *cyclic B-contraction*, if

(i) T is cyclic;
(ii) there exists non-negative real numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ and for all $x, y \in A \cup B$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \\ + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B).$$

Proposition 2.2. *Let A, B be nonempty subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic B-contraction map. Then starting with any $x_0 \in A \cup B$, we have $d(x_n, x_{n+1}) \to d(A, B)$ where $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$.*

Proof. Let $x_0 \in A \cup B$. A sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N} \cup \{0\}$. Then, by Definition 2.1 , we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &\quad + \gamma[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \\ &\leq \alpha d(x_{n-1}, x_n) + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\quad + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \\ &\leq (\alpha + \beta + \gamma)d(x_{n-1}, x_n) + (\beta + \gamma)d(x_n, x_{n+1})] \\ &\quad + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \end{aligned}$$

That is,

$$[1-(\beta+\gamma)]d(x_n,x_{n+1}) \leq (\alpha+\beta+\gamma)d(x_{n-1},x_n)+[1-(\alpha+2\beta+2\gamma)]d(A,B)$$

Which gives as

$$d(x_n,x_{n+1}) \leq \frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}d(x_{n-1},x_n)+\left\{1-\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right\}d(A,B)$$

We note that $\dfrac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]} < 1$ and $\alpha+2\beta+2\gamma < \alpha+\beta+\gamma$. Then the above inequality becomes,

$$d(x_n,x_{n+1}) \leq \frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}d(x_{n-1},x_n)+\left\{1-\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right\}d(A,B)$$

Similarly,

$$d(x_n,x_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^2 d(x_{n-2},x_{n-1})+\left\{1-\left(\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right)^2\right\}d(A,B)$$

Continuing this process, we get

$$d(x_n,x_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^n d(x_0,x_1)+\left\{1-\left(\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right)^n\right\}d(A,B)$$

Letting limit as $n \to \infty$, we have

$$d(x_n,x_{n+1}) \to d(A,B).$$

□

Example 1. Let $X=[0,1]$ and consider the closed subsets $A=[0,5/6]$ and $B=[5/6,1]$ of a metric space (X,d) and $T: A\cup B \to A\cup B$ is defined by:

$$Tx = \begin{cases} \dfrac{5}{6}+q & \text{when } q\in[0,1] \\ 1-\dfrac{5}{6} & \text{when } q\in\left[\dfrac{5}{6},1\right] \end{cases}$$

This clearly shows that $T(A)\subseteq B$ and $T(B)\subseteq A$. Also for every $x,y\in A\cup B$ satisfies the Definition 2.1. Thus, T satisfies all the conditions of the Propositions 2.2.

Proposition 2.3. *Let A,B be nonempty closed subsets of a complete metric space X, $T: A\cup B\to A\cup B$ be a cyclic B-contraction map, let $x_0\in A$ and define $x_{n+1}=Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x\in A$ such that $d(x,Tx)=d(A,B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ and $\lim_{k\to\infty} x_{2n_k} = x$ for some $x \in A$. Now,

$$d(A,B) \le d(x, x_{2n_k-1}) \le d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(x, x_{2n_k-1}) \to d(A,B)$$

Since

$$d(A,B) \le d(x_{2n_k}, Tx) \le d(x_{2n_k-1}, x)$$

As $n \to \infty$, we have

$$d(x,Tx) = d(A,B).$$

□

Theorem 2.4. *Let A, B be nonempty subsets of a metric space X, $T : A\cup B \to A\cup B$ be a cyclic B-contraction map. Then for any $x_0 \in A$ and $x_{n+1} = Tx_n$, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose $x_0 \in A$ (the proof is similar when $x_0 \in B$), then since by Proposition(2.1), $d(x_{2n}, x_{2n+1}) \to d(A,B)$ as $n \to \infty$, it is enough to prove $\{x_{2n+1}\}$ is bounded. Suppose $\{x_{2n+1}\}$ is not bounded, then there exists $n_0 \in \mathbb{N}$ such that

$$d(x_2, x_{2n_0+1}) > M \text{ and } d(x_2, x_{2n_0-1}) \le M,$$

where $M > max\left\{\frac{2d(x_0,x_1)}{\frac{1}{k^2-1}} + d(A,B), d(x_1,x_2)\right\}$. Then, by the cyclic B-contraction property of T, we have

$$\begin{aligned}\frac{M - d(A,B)}{k^2} + d(A,B) &< d(x_0, x_{2n_0-1}) \\ &\le d(x_0,x_2) + d(x_2, x_{2n_0-1}) \\ &\le 2d(x_0,x_1) + M\end{aligned}$$

Thus,

$$M < \frac{2d(x_0,x_1)}{\frac{1}{k^2-1}} + d(A,B)$$

Which is a contradiction. Hence $\{x_{2n+1}\}$ is bounded. □

Theorem 2.5. *Let A, B be nonempty closed subsets of a metric space X and $T : A \cup B \to A \cup B$ is a cyclic $B-$contraction. If either A or B is boundedly compact, then there exists $x \in A \cup B$ such that $d(x,Tx) = d(A,B)$.*

Proof. It follows directly from Proposition 2.2 and 2.3. □

3. Common Best Proximity Point Theorem for S-weakly Cyclic B-contractions

Definition 3.1. Let A, B be nonempty closed subsets of a metric space X and self maps $T, S : A \cup B \to A \cup B$. Then, T is said to be *S-weakly cyclic B-contraction*, if

(i) T and S be cyclic; and
(ii) there exists nonnegative real numbers α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ and for all $x, y \in A \cup B$ such that

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta[d(Sx, Tx) + d(Sy, Ty)] + \gamma[d(Sx, Ty) + d(Sy, Tx)] \\ &\quad + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B). \end{aligned}$$

Theorem 3.2. *Let A, B be nonempty closed subsets of a complete metric space X and $T, S : A \cup B \to A \cup B$. If*

(i) T is S-weakly cyclic B-contraction;
(ii) $T(A) \subseteq S(A)$;
(iii) S and T are commute;
(iv) S and T are continuous.

Then, there exists $x \in A$ such that $d(x, Tx) = d(x, Sx) = d(A, B)$.

Proof. Let $x_0 \in A$, then by condition (ii) there exists $x_1 \in A$ such that $Tx_0 = Sx_1$. Now, $x_1 \in A$, then there exists $x_2 \in A$ such that $Tx_1 = Sx_2$. Continuing this process, we have a sequence

$$Tx_n = Sx_{n+1}, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Since T is S- weakly cyclic B-contraction, which implies

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \alpha d(Sx_n, Sx_{n+1}) + \beta[d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})] \\ &\quad + \gamma[d(Sx_n, Tx_{n+1}) + d(Sx_{n+1}, Tx_n)] + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \\ &\leq \alpha d(Tx_{n-1}, Tx_n) + \beta[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\ &\quad + \gamma[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_n, Tx_n)] \\ &\quad + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \\ &\leq (\alpha + \beta + \gamma)d(Tx_{n-1}, Tx_n) + (\beta + \gamma)d(Tx_n, Tx_{n+1})] \\ &\quad + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B) \end{aligned}$$

Finally, we have

$$[1 - (\beta + \gamma)]d(Tx_n, Tx_{n+1}) \leq (\alpha + \beta + \gamma)d(Tx_{n-1}, Tx_n) + [1 - (\alpha + 2\beta + 2\gamma)]d(A, B)$$

Which gives as

$$d(Tx_n, Tx_{n+1}) \leq \frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]} d(Tx_{n-1}, Tx_n) + \left\{1 - \frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right\} d(A, B)$$

We note that $\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]} < 1$ and $\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]} < 1$. Then the above inequality becomes,

$$d(Tx_n, Tx_{n+1}) \leq \frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]} d(Tx_{n-1}, Tx_n) + \left\{1 - \frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right\} d(A, B)$$

Now,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\quad + \left\{1 - \left(\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right)^2\right\} d(A, B) \end{aligned}$$

Continuing this process, we get

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^n d(x_0, x_1) + \left\{1 - \left(\frac{\alpha+2\beta+2\gamma}{[1-(\beta+\gamma)]}\right)^n\right\} d(A, B)$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) \to d(A, B).$$

Since $\{x_n\}$ be a sequence in A, and it converges to some point $x^* \in A$. Now by the commutativity of S and T, we have $S(Tx_n) \to S(x^*) = x^*$ and $T(Sx_n) \to Tx^* = x^*$ for all $n \in \mathbb{N}$. Hence x^* is a common proximity point for S and T. □

4. Proximity Pint Results for Cyclic B-contraction on Uniformly Convex Banach Spaces

Theorem 4.1. *Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and $\{x_n\}, \{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $\|z_n - y_n\| \to d(A, B)$;

(ii) for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$

$$\implies \|x_m - y_n\| \leq d(A, B) + \varepsilon$$

Then, for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for all $m > n \geq n_1$, $\|x_m - z_n\| \leq \varepsilon$.

Proof. Assume the contrary, then there exists $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$, for which $\|x_{m_k} - z_{n_k}\| \geq \varepsilon_0$.
Let $0 < \zeta < 1$ such that $\varepsilon_0/\zeta > d(A,B)$ and choose ε such that

$$0 < \varepsilon < \min\Big(\frac{\varepsilon_0}{\zeta} - d(A,B), \frac{d(A,B)\delta(\zeta)}{1-\delta(\zeta)}\Big)$$

For this $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m_k > n_k \geq n_0$ implies

$$\|x_{m_k} - y_{n_k}\| \leq d(A,B) + \varepsilon$$

Also, there exists $n_2 \in \mathbb{N}$ such that

$$\|z_{n_k} - y_{n_k}\| \leq d(A,B) + \varepsilon$$

for all $n_k \geq n_2$. Choose $N_1 = \max\{n_0, n_2\}$. By uniform convexity, for all $m_k > n_k \geq N_1$ implies that

$$\Big\|\frac{x_{m_k} + z_{n_k}}{2} - y\Big\| \leq \Big(1 - \delta\Big(\frac{\varepsilon_0}{d(A,B)+\varepsilon}\Big)\Big)(d(A,B)+\varepsilon)$$

Using the fact that δ is strictly increasing and by the choice of ε, we have

$$\Big\|\frac{z_{n_k} + x_{m_k}}{2} - y\Big\| < d(A,B)$$

for all $m_k > n_k \geq N_1$, which is a contradiction. □

In a similar way we can prove the following lemma.

Lemma 4.2. *Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space and $\{x_n\}, \{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $\|x_n - y_n\| \to d(A,B)$*; and*
(ii) $\|z_n - y_n\| \to d(A,B)$

Then $\|x_n - z_n\| \to 0$.

Corollary 4.3. *Let A be a nonempty closed and convex subset, B be a nonempty closed subset of a uniformly convex Banach space X. Let $\{x_n\}$ be a sequence in A and $y_0 \in B$ such that*

$$\|x_n - y_0\| \to d(A,B)$$

Then, $\{x_n\}$ converges to $P_A(y_0)$.

Proof. Since $d(A,B) \leq \|y_0 - P_A(y_0)\| \leq \|y_0 - x_n\|$, we have $\|y_0 - P_A(y_0)| = d(A,B)$. Now put $y_n = y_0$ and $z_n = P_A(y_0)$ in Lemma 4.2. □

Theorem 4.4. *Let A, B be nonempty closed and convex subsets of a uniformly convex Banach space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic B-contraction map, then there exists a unique best proximity point $x \in A$. Further if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Proof. Since $\|x_{2n} - Tx_{2n}\| \to d(A,B)$ and $\|T^2x_{2n} - Tx_{2n}\| \to d(A,B)$. By Lemma 4.2, $\|x_{2n} - Tx_{2n}\| \to 0$. Similarly, we can show that $\|Tx_{2n} - Tx_{2n+1}\| \to 0$. Now, we show that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$, $\|x_{2m} - Tx_{2n}\| \leq d(A,B) + \varepsilon$. Suppose not, then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$ for which

$$\|x_{2m_k} - Tx_{2n_k}\| \geq d(A,B) + \varepsilon$$

That is,

$$\begin{aligned} d(A,B) + \varepsilon &\leq \|x_{2m_k} - Tx_{2n_k}\| \\ &\leq \|x_{2m_k} - x_2m_k - 1\| + \|x_2m_k - 1 - Tx_{2n_k}\|. \end{aligned}$$

Hence,

$$\begin{aligned} d(A,B) + \varepsilon &\leq \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^2 + \left\{1 - \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^2\right\} d(A,B) \\ &\leq d(A,B) + \left(\frac{\alpha+\beta+\gamma}{[1-(\beta+\gamma)]}\right)^2 \varepsilon \end{aligned}$$

This is a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence by Lemma 4.1 and hence converges to some $x \in A$. Also, from Theorem 3.2, we have $\|x - Tx\| = d(A,B)$. Now, we have to prove the uniqueness. Suppose x and y are the proximity points for T and $x \neq y$, that is $\|x - Tx\| = d(A,B)$ and $\|y - Ty\| = d(A,B)$ where necessarily, $T^2x = x$ and $T^2y = y$. Therefore,

$$\begin{aligned} \|Tx - y\| &= \|Tx - T^2y\| \leq \|x - Ty\|; \text{ and} \\ \|Ty - x\| &= \|Ty - T^2x\| \leq \|y - Tx\| \end{aligned}$$

This implies $\|Ty - x\| = \|y - Tx\|$. But $\|y - Tx\| > d(A,B)$, it follows that

$$\|Ty - x\| < \|y - Tx\|,$$

which is a contradiction. Hence $x = y$. □

Conclusion

In this paper, we conclude the existence of the results of proximity point theorems for the cyclic B-contraction, the common best proximity point theorem for S-weakly cyclic B-contraction and the proximity point results for cyclic B-contraction on uniformly convex Banach spaces. As various future results can be demonstrated in a smaller setting to ensure the existence of the proximity point results.

Acknowledgment

All the authors are highly grateful to the editor and refree(s) for their valuable comments and suggestions for improving the paper.

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