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REFINEMENT OF HERMITE HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS

Muhammad Bilal [a,*] and Asif R. Khan [b]

Abstract. In this study, we would like to state two refined results related to Hermite Hadamard type inequality for convex functions with two distinct techniques. Hence our obtained results would be better than the results already established for the class of convex functions. Applications to trapezoidal rule and special means are also discussed.

> "Mathematics has been called the science of tautology; that is to say, mathematicians have been accused of spending their time proving that things are equal to themselves. This statement (appropriately by a philosopher) is rather inaccurate on two counts. In the first place, mathematics, although the language of science, is not a science. Rather, it is a creative art. Secondly, the fundamental results of mathematics are often *inequalities* rather than *equalities*."
>
> – Beckenbach and Bellman

1. Introduction and Preliminaries

In Mathematics, the branch of inequality is becoming more popular day by day due to its applications in almost every field of life. This branch includes many important fields of mathematics especially the theory of convex functions which is given valuable attention in literature from the last few decades. Convexity has its many applications in daily life like architecture, arts, business, economics, management science and many more. Among many applications related to the aforementioned

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*Corresponding author.

field, Hermite Hadamard type inequalities are one of the most important due to a number of applications especially in the field probability theory, approximation theory, numerical integration and many more. For further study related to the topic see [2] − [6], [11] − [13] and the cited therein.

Before we proceed further it is to be noted that, here we list down some notation which we would use in this article:

(1) X denotes the real interval $[\varpi_1, \varpi_2]$,
(2) X° denotes the interior of X
(3) and $\beta(\varpi_1, \varpi_2) = \int_0^1 u^{\varpi_1 - 1}(1-u)^{\varpi_2 - 1} du, \quad \varpi_1, \varpi_2 > 0$ is the famous Euler Beta function.

We shall begin with some important definitions and useful results:

Definition 1.1 ([1])**.** A function $\xi : X \to \mathbb{R}$ is called *convex*, if

$$\xi\left(u\zeta_1 + (1-u)\zeta_2\right) \leq u\xi(\zeta_1) + (1-u)\xi(\zeta_2),$$

$\forall\ \zeta_1, \zeta_2 \in X$ and $u \in [0, 1]$.

Theorem 1.2 ([8])**.** *If $\xi : X \to \mathbb{R}$ is a convex function, then ξ must holds*

$$\xi\left(\frac{\varpi_1 + \varpi_2}{2}\right) \leq \frac{1}{\varpi_2 - \varpi_1} \int\limits_{\varpi_1}^{\varpi_2} \xi(\zeta) d\zeta \leq \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2}. \tag{1.1}$$

The above stated result is well known as Hermite Hadamard dual inequality for convex function in literature.

Remark 1.3. It is to be noted that for concave function ξ, both inequalities would be in reverse order. Also, Hadamard's inequality may be regarded as a special case of refinement of Jensen's inequality.

In [9], İşcan obtained an integral inequality named as Hölder-İşcan integral inequality which gives better results than the classical Hölder's integral inequality [12] is defined as:

Theorem 1.4. *Let $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If ξ_1 and ξ_2 are real functions defined on $[\varpi_1, \varpi_2]$ and if $|\xi_1|^p$ and $|\xi_2|^q$ are integrable on $[\varpi_1, \varpi_2]$, then*

$$\int\limits_{\varpi_1}^{\varpi_2} |\xi_1(u)\xi_2(u)| du$$

$$\leq \frac{1}{\varpi_2-\varpi_1}\left[\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-u)|\xi_1(u)|^p du\right)^{\frac{1}{p}}\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-u)|\xi_2(u)|^q du\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\varpi_1}^{\varpi_2}(u-\varpi_1)|\xi_1(u)|^p du\right)^{\frac{1}{p}}\left(\int_{\varpi_1}^{\varpi_2}(u-\varpi_1)|\xi_2(u)|^q du\right)^{\frac{1}{q}}\right]. \tag{1.2}$$

Remark 1.5. Note that if we put $p=q=2$, the above inequality gives us improved Cauchy−Schwarz integral inequality.

A different representation of Hölder-İşcan integral inequality is stated as:

Theorem 1.6. *Let ξ_1, ξ_2 are real valued functions defined on $[\varpi_1,\varpi_2]$ and if $|\xi_1|$ and $|\xi_1||\xi_2|^q$ are integrable on $[\varpi_1,\varpi_2]$, then for $q\geq 1$ we have:*

$$\int_{\varpi_1}^{\varpi_2}|\xi_1(u)\xi_2(u)|du$$
$$\leq \frac{1}{\varpi_2-\varpi_1}\left[\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-u)|\xi_1(u)|du\right)^{1-\frac{1}{q}}\left(\int_{\varpi_1}^{\varpi_2}(\varpi_2-u)|\xi_1(u)||\xi_2(u)|^q du\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\varpi_1}^{\varpi_2}(u-\varpi_1)|\xi_1(u)|du\right)^{1-\frac{1}{q}}\left(\int_{\varpi_1}^{\varpi_2}(u-\varpi_1)|\xi_1(u)||\xi_2(u)|^q du\right)^{\frac{1}{q}}\right]. \tag{1.3}$$

The above inequality is known as Improved power mean integral inequality (see [10]), which is the refinement of Power mean integral inequality [14].

Now we are stating the following identity extracted from [3] which will be used to derive our main results of this article.

Lemma 1.7. *Let $\xi : X\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on X° with $\xi''\in L[\varpi_1,\varpi_2]$. Then the following identity holds:*

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(x)d\zeta$$
$$=\frac{(\varpi_2-\varpi_1)^2}{2}\int_0^1 u(1-u)\xi''(u\varpi_1+(1-u)\varpi_2)du.$$

In [3], Bhatti, Iqbal and Hussain et. al. established the results related to Hermite Hadamard type inequalities for convex function by using well-known Hölder and Power mean integral inequality as follows:

Theorem 1.8. *Let $\xi : X \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for $q > 1$ then following inequality holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left(\frac{|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}, \tag{1.4}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.9. *Let $\xi : X \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X^\circ$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for $q \geq 1$ then following inequality holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{12}\left(\frac{|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}} \tag{1.5}$$

This article is organized as: In the next section, we are going to refine the estimated bounds of the Hermite Hadamard type inequalities (right bound of (1.1)) for twice differentiable convex functions by using newly defined Hölder−İşcan and Improved power mean integral inequalities. These results would provide better bounds than already obtained in Theorem 1.8 and 1.9. Third and fourth sections deal with some applications related to Quadrature rules and Special means, respectively. Fifth section is purely devoted to concluding statements and the last section gives us some remarks and future ideas for interested researchers.

2. Various Improvements of Estimated Right Bound of Hermite Hadamard Inequality for Twice Differentiable Convex Function

In this section, we are going to state and prove two refined results related to Hermite Hadamard type inequalities for twice differentiable convex function using

Definition 1.1, Theorem 1.4, Theorem 1.6 and Lemma 1.7. Also, we show that our obtained results would be better than the results obtained in Theorem 1.8 and 1.9.

Theorem 2.1. *Let $\xi : X \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for $q > 1$ then following below stated inequality holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{2}\{\beta(p+1,p+2)\}^{\frac{1}{p}} \times$$
$$\left[\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}\right], \tag{2.1}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.7 and the definition of absolute value, we attain

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2}\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du. \tag{2.2}$$

Applying (1.2) to $\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du$ implies

$$\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du$$
$$\leq \left[\left(\int_0^1 (1-u)|u(1-u)|^p du\right)^{\frac{1}{p}}\left(\int_0^1 (1-u)|\xi''(u\varpi_1+(1-u)\varpi_2)|^q du\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 u|u(1-u)|^p du\right)^{\frac{1}{p}}\left(\int_0^1 u|\xi''(u\varpi_1+(1-u)\varpi_2)|^q du\right)^{\frac{1}{q}}\right].$$

As we have $|\xi''|^q$ is a convex function, so we can take

$$|\xi''(u\zeta_1+(1-u)\zeta_2)|^q \leq u|\xi''(\zeta_1)|^q + (1-u)|\xi''(\zeta_2)|^q.$$

Utilizing the above two results (2.2) becomes,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \le \frac{(\varpi_2-\varpi_1)^2}{2}\times$$

$$\left[\left(\int_0^1 u^p(1-u)^{p+1}du\right)^{\frac{1}{p}}\left(|\xi''(\varpi_1)|^q\int_0^1 u(1-u)du+|\xi''(\varpi_2)|^q\int_0^1(1-u)^2du\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 u^{p+1}(1-u)^p du\right)^{\frac{1}{p}}\left(|\xi''(\varpi_1)|^q\int_0^1 u^2du+|\xi''(\varpi_2)|^q\int_0^1 u(1-u)du\right)^{\frac{1}{q}}\right].$$

After using the following below stated facts, the result of Theorem 2.1 is accomplished.

$$\int_0^1 u^{p+1}(1-u)^p dt=\int_0^1 u^p(1-u)^{p+1}du=\beta(p+1,p+2),$$

$$\int_0^1 u(1-u)du=\frac{1}{6}$$

and

$$\int_0^1 u^2du=\int_0^1(1-u)^2du=\frac{1}{3}.$$

□

Remark 2.2. Here we claim that inequality (2.1) of Theorem 2.1 is better than the inequality (1.4).

Proof. Since the function $h:[0,\infty)\to\mathbb{R}$, $h(\zeta)=\zeta^n$, $n\in(0,1]$ is concave, we can write:

$$\frac{{\varpi_1}^n+{\varpi_2}^n}{2}=\frac{h(\varpi_1)+h(\varpi_2)}{2}\le h\left(\frac{\varpi_1+\varpi_2}{2}\right)=\left(\frac{\varpi_1+\varpi_2}{2}\right)^n \tag{2.3}$$

$\forall\ \varpi_1,\varpi_2\ge 0$. For the above inequality if we choose

$$\varpi_1=\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6},\quad \varpi_2=\frac{2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}$$

and $n = \frac{1}{q}$. Applying the inequality (2.3) to the inequality (2.1), we have

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{2}\left\{\beta(p+1,p+2)\right\}^{\frac{1}{p}} \times$$
$$\left[\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}\right]$$
$$\leq (\varpi_2-\varpi_1)^2\left\{\beta(p+1,p+2)\right\}^{\frac{1}{p}} \times$$
$$\left[\frac{1}{2}\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q+2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}\right)\right]^{\frac{1}{q}}$$
$$= \frac{(\varpi_2-\varpi_1)^2}{2^{\frac{1}{q}}}\left\{\beta(p+1,p+2)\right\}^{\frac{1}{p}}\left[\frac{|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{2}\right]^{\frac{1}{q}}$$

By using

$$\beta(\zeta_1,\zeta_2+1) = \frac{\zeta_2}{\zeta_1+\zeta_2}\beta(\zeta_1,\zeta_2)$$

we get our required result i.e. inequality (1.4). □

Theorem 2.3. *Let $\xi : X \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X^\circ$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for $q \geq 1$ then following inequality holds:*

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24}\times$$
$$\left[\left(\frac{2|\xi''(\varpi_1)|^q+3|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}} + \left(\frac{3|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}\right]. \tag{2.4}$$

Proof. Using Lemma 1.7 and the definition of absolute value, we attain

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2}\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du. \tag{2.5}$$

Applying (1.3) to $\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du$ implies

$$\int_0^1 |u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|du$$

$$\leq \left[\left(\int_0^1 (1-u)|u(1-u)|du\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-u)|u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|^q du\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 u|u(1-u)|du\right)^{1-\frac{1}{q}}\left(\int_0^1 u|u(1-u)||\xi''(u\varpi_1+(1-u)\varpi_2)|^q du\right)^{\frac{1}{q}}\right].$$

As we have $|\xi''|^q$ is a convex function, so we can take

$$|\xi''\left(u\zeta_1+(1-u)\zeta_2\right)|^q \leq u|\xi''(\zeta_1)|^q+(1-u)|\xi''(\zeta_2)|^q.$$

Utilizing the above two results (2.5) becomes,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{2}\times$$
$$\left[\left(\int_0^1 u(1-u)^2du\right)^{1-\frac{1}{q}}\left(|\xi''(\varpi_1)|^q\int_0^1 u^2(1-u)^2du+|\xi''(\varpi_2)|^q\int_0^1 u(1-u)^3du\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 u^2(1-u)du\right)^{1-\frac{1}{q}}\left(|\xi''(\varpi_1)|^q\int_0^1 u^3(1-u)du+|\xi''(\varpi_2)|^q\int_0^1 u^2(1-u)^2du\right)^{\frac{1}{q}}\right].$$

After using the following below stated facts, the result of Theorem 2.3 is accomplished.

$$\int_0^1 u^2(1-u)du=\int_0^1 u(1-u)^2du=\frac{1}{12},$$

$$\int_0^1 u^2(1-u)^2du=\frac{1}{30}$$

and

$$\int_0^1 u(1-u)^3 du = \int_0^1 u^3(1-u) du = \frac{1}{20}.$$

□

Remark 2.4. Also we claim that inequality (2.4) of Theorem 2.3 is better than the inequality (1.5).

Proof. We can prove this claim in a similar manner as we done in Remark 2.2. So, we replace

$$\varpi_1 = \frac{2|\xi''(\varpi_1)|^q + 3|\xi''(\varpi_2)|^q}{5}, \quad \varpi_2 = \frac{3|\xi''(\varpi_1)|^q + 2|\xi''(\varpi_2)|^q}{5}$$

and $n = \frac{1}{q}$ in the inequality (2.3) and apply it to the inequality (2.4), we have

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24}\times$$

$$\left[\left(\frac{2|\xi''(\varpi_1)|^q + 3|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}} + \left(\frac{3|\xi''(\varpi_1)|^q + 2|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{12}\left[\frac{1}{2}\left(\frac{2|\xi''(\varpi_1)|^q + 3|\xi''(\varpi_2)|^q + 3|\xi''(\varpi_1)|^q + 2|\xi''(\varpi_2)|^q}{5}\right)\right]^{\frac{1}{q}}$$

$$= \frac{(\varpi_2-\varpi_1)^2}{12}\left[\frac{|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{2}\right]^{\frac{1}{q}}$$

which is our required result i.e., inequality (1.5). □

In the next two sections we are going to give some applications of our obtained results to trapezoidal rule and special means.

3. Application to Trapezoidal Rule

Let J be a division of the interval $[\varpi_1, \varpi_2]$, i.e., $J : \varpi_1 = y_0 < y_1 < ... < y_{n-1} < y_n = \varpi_2$ and consider the quadrature formula

$$I = \int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta = T(\xi, J) + R(\xi, J)$$

where

$$T(\xi,J)=\sum_{k=0}^{n-1}\frac{\xi(y_k)+\xi(y_{k+1})}{2}(y_{k+1}-y_k)$$

is the trapezoidal formula and $R(\xi,J)$ denotes the associated approximation error of the integral I.

Theorem 3.1. *Let $\xi : X \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for every partition J of $[\varpi_1,\varpi_2]$ with $q \geq 1$ then following inequality holds:*

$$|R(\xi,J)| \leq \frac{1}{2}\sum_{k=0}^{n-1}(y_{k+1}-y_k)^3\left\{\beta(p+1,p+2)\right\}^{\frac{1}{p}}\times$$
$$\left[\left(\frac{|\xi''(y_k)|^q+2|\xi''(y_{k+1})|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\xi''(y_k)|^q+|\xi''(y_{k+1})|^q}{6}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2}\sum_{k=0}^{n-1}(y_{k+1}-y_k)^3\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|\xi''(y_k)|^q+|\xi''(y_{k+1})|^q}{2}\right]^{\frac{1}{q}}, \tag{3.1}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying inequality (2.1) on $[y_k,y_{k+1}]$ and summing over k from 0 to $n-1$ and then by using triangular inequality we get (3.1). □

Theorem 3.2. *Let $\xi : X \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on X° such that $\xi'' \in L[\varpi_1,\varpi_2]$, where $\varpi_1,\varpi_2 \in X^\circ$ and $\varpi_1 < \varpi_2$. If $|\xi''|^q$ is convex function on $[\varpi_1,\varpi_2]$ for every division J of $[\varpi_1,\varpi_2]$ with $q \geq 1$ then following inequality holds:*

$$|R(\xi,J)| \leq \frac{1}{24}\sum_{k=0}^{n-1}(y_{k+1}-y_k)^3\times$$
$$\left[\left(\frac{2|\xi''(y_k)|^q+3|\xi''(y_{k+1})|^q}{5}\right)^{\frac{1}{q}}+\left(\frac{3|\xi''(y_k)|^q+2|\xi''(y_{k+1})|^q}{5}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{12}\sum_{k=0}^{n-1}(y_{k+1}-y_k)^3\left[\frac{|\xi''(y_k)|^q+|\xi''(y_{k+1})|^q}{2}\right]^{\frac{1}{q}}. \tag{3.2}$$

Proof. The proof of the above result is followed with a similar technique applying on inequality (2.4) instead of inequality (2.1) as we done in Theorem 3.1. □

4. Application to Special Means

We start with the definition of following special means extracted from [7]:
(1) The Arithmetic mean:

$$A = A(\varpi_1, \varpi_2) = \frac{\varpi_1 + \varpi_2}{2}; \ \varpi_1, \varpi_2 \in [0, \infty).$$

(2) The Harmonic mean:

$$H = H(\varpi_1, \varpi_2) = \frac{2\varpi_1\varpi_2}{\varpi_1 + \varpi_2}; \quad \varpi_1, \varpi_2 \in (0, \infty).$$

(3) The Geometric mean:

$$G = G(\varpi_1, \varpi_2) = \sqrt{\varpi_1\varpi_2}; \quad \varpi_1, \varpi_2 \in [0, \infty).$$

(4) The Logarithmic mean:

$$L = L(\varpi_1, \varpi_2) = \frac{\varpi_2 - \varpi_1}{\ln \varpi_2 - \ln \varpi_1}; \ \varpi_1 \neq \varpi_2 \quad \varpi_1, \varpi_2 \in (0, \infty).$$

(5) The p-Logarithmic mean:

$$L_p = L_p(\varpi_1, \varpi_2) = \left[\frac{{\varpi_2}^{p+1} - {\varpi_1}^{p+1}}{(p+1)(\varpi_2 - \varpi_1)}\right]^{\frac{1}{p}}; \ \varpi_1 \neq \varpi_2, \ \varpi_1, \varpi_2 \in (0, \infty),$$

where $p \in \mathbb{R} - \{-1, 0\}$.
(6) The Identric mean:

$$I = I(\varpi_1, \varpi_2) = \frac{1}{e}\left[\frac{{\varpi_2}^{\varpi_2}}{{\varpi_1}^{\varpi_1}}\right]^{\frac{1}{\varpi_2 - \varpi_1}}; \ \varpi_1 \neq \varpi_2 \quad \varpi_1, \varpi_2 \in (0, \infty).$$

Now we are going to give some relations among different means by using results obtained in previous section.

Example 4.1. Let the function ξ be defined by $\xi(\zeta) = \frac{1}{\zeta}$ and $0 < \varpi_1 < \varpi_2$ then we have:

$$\frac{1}{\varpi_2 - \varpi_1}\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta = L^{-1}(\varpi_1, \varpi_2) = L^{-1},$$

$$\frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} = H^{-1}(\varpi_1, \varpi_2) = H^{-1},$$

$$\left(\frac{2|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}} = \frac{2}{G^6}\left(\frac{{\varpi_1}^{3q} + 2{\varpi_2}^{3q}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}=\frac{2}{G^6}\left(\frac{2{\varpi_1}^{3q}+{\varpi_2}^{3q}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{3|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=\frac{2}{G^6}\left(\frac{2{\varpi_1}^{3q}+3{\varpi_2}^{3q}}{5}\right)^{\frac{1}{q}}$$

and

$$\left(\frac{2|\xi''(\varpi_1)|^q+3|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=\frac{2}{G^6}\left(\frac{3{\varpi_1}^{3q}+2{\varpi_2}^{3q}}{5}\right)^{\frac{1}{q}}.$$

(1) Then (2.1) becomes,

$$\left|H^{-1}-L^{-1}\right|$$

$$\leq\frac{(\varpi_2-\varpi_1)^2}{G^6}\{\beta(p+1,p+2)\}^{\frac{1}{p}}\left[\left(\frac{{\varpi_1}^{3q}+2{\varpi_2}^{3q}}{6}\right)^{\frac{1}{q}}+\left(\frac{2{\varpi_1}^{3q}+{\varpi_2}^{3q}}{6}\right)^{\frac{1}{q}}\right]. \tag{4.1}$$

(2) Then (2.4) becomes,

$$\left|H^{-1}-L^{-1}\right|\leq\frac{(\varpi_2-\varpi_1)^2}{12G^6}\left[\left(\frac{2{\varpi_1}^{3q}+3{\varpi_2}^{3q}}{5}\right)^{\frac{1}{q}}+\left(\frac{3{\varpi_1}^{3q}+2{\varpi_2}^{3q}}{5}\right)^{\frac{1}{q}}\right]. \tag{4.2}$$

Example 4.2. Let the function ξ be defined by $\xi(\zeta)=\zeta^n$ with $0<\varpi_1<\varpi_2$ and $n\in\mathbb{R}\setminus\{-1,0\}$ then we have:

$$\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta=L_n^n(\varpi_1,\varpi_2)=L_n^n,$$

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}=A\left({\varpi_1}^n,{\varpi_2}^n\right),$$

$$\left(\frac{2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}=n(n-1)\left(\frac{2{\varpi_1}^{q(n-2)}+{\varpi_2}^{q(n-2)}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}=n(n-1)\left(\frac{{\varpi_1}^{q(n-2)}+2{\varpi_2}^{q(n-2)}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{3|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=n(n-1)\left(\frac{3{\varpi_1}^{q(n-2)}+2{\varpi_2}^{q(n-2)}}{5}\right)^{\frac{1}{q}}$$

and

$$\left(\frac{2|\xi''(\varpi_1)|^q+3|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=n(n-1)\left(\frac{2{\varpi_1}^{q(n-2)}+3{\varpi_2}^{q(n-2)}}{5}\right)^{\frac{1}{q}}.$$

(1) Then (2.1) becomes,

$$|L_n^n-A({\varpi_1}^n,{\varpi_2}^n)|\le\frac{(\varpi_2-\varpi_1)^2}{2}\{\beta(p+1,p+2)\}^{\frac{1}{p}}\times$$

$$n(n-1)\left[\left(\frac{2{\varpi_1}^{q(n-2)}+{\varpi_2}^{q(n-2)}}{6}\right)^{\frac{1}{q}}+\left(\frac{{\varpi_1}^{q(n-2)}+2{\varpi_2}^{q(n-2)}}{6}\right)^{\frac{1}{q}}\right]. \tag{4.3}$$

(2) Then (2.4) becomes,

$$|L_n^n-A({\varpi_1}^n,{\varpi_2}^n)|\le\frac{(\varpi_2-\varpi_1)^2}{24}[n(n-1)]\times$$

$$\left[\left(\frac{3{\varpi_1}^{q(n-2)}+2{\varpi_2}^{q(n-2)}}{5}\right)^{\frac{1}{q}}+\left(\frac{2{\varpi_1}^{q(n-2)}+3{\varpi_2}^{q(n-2)}}{5}\right)^{\frac{1}{q}}\right]. \tag{4.4}$$

Example 4.3. Let the function ξ be defined by $\xi(\zeta)=-\ln\zeta$ and $0<\varpi_1<\varpi_2$ then we have:

$$\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta=-\ln I(\varpi_1,\varpi_2)=-\ln I,$$

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}=-\ln G(\varpi_1,\varpi_2)=-\ln G,$$

$$\left(\frac{2|\xi''(\varpi_1)|^q+|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}=\frac{1}{G^4}\left(\frac{{\varpi_1}^{2q}+2{\varpi_2}^{2q}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{6}\right)^{\frac{1}{q}}=\frac{1}{G^4}\left(\frac{2{\varpi_1}^{2q}+{\varpi_2}^{2q}}{6}\right)^{\frac{1}{q}},$$

$$\left(\frac{3|\xi''(\varpi_1)|^q+2|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=\frac{1}{G^4}\left(\frac{2{\varpi_1}^{2q}+3{\varpi_2}^{2q}}{5}\right)^{\frac{1}{q}}$$

and

$$\left(\frac{2|\xi''(\varpi_1)|^q+3|\xi''(\varpi_2)|^q}{5}\right)^{\frac{1}{q}}=\frac{1}{G^4}\left(\frac{3{\varpi_1}^{2q}+2{\varpi_2}^{2q}}{5}\right)^{\frac{1}{q}}.$$

(1) Then (2.1) becomes,

$$\left|\ln \frac{I}{G}\right| \leq \frac{(\varpi_2-\varpi_1)^2}{2G^4}\{\beta(p+1,p+2)\}^{\frac{1}{p}}\times$$
$$\left[\left(\frac{{\varpi_1}^{2q}+2{\varpi_2}^{2q}}{6}\right)^{\frac{1}{q}}+\left(\frac{2{\varpi_1}^{2q}+{\varpi_2}^{2q}}{6}\right)^{\frac{1}{q}}\right]. \tag{4.5}$$

(2) Then (2.4) becomes,

$$\left|\ln \frac{I}{G}\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24G^4}\left[\left(\frac{2{\varpi_1}^{2q}+3{\varpi_2}^{2q}}{5}\right)^{\frac{1}{q}}+\left(\frac{3{\varpi_1}^{2q}+2{\varpi_2}^{2q}}{5}\right)^{\frac{1}{q}}\right]. \tag{4.6}$$

5. CONCLUSION

Hermite Hadamard dual inequality is one of the most recognized inequalities. We can find its various refinements and variants in literature. We have given its refinement for ordinary convex functions. In Section 2, we have stated two distinct results related to refinement of estimated right bound of Hermite Hadamard dual inequality in the absolute sense for the aforementioned class of twice differentiable functions. Here we used two distinct techniques including Hölder−İşcan and Improved power mean integral inequalities. These results are better than the results obtained in the article [3]. In Section 3 and 4, we have obtained some relations between our derived results with well known trapezoidal rule and special means, respectively. Last section is devoted to some remarks and future ideas for readers.

Now, we are going to give some remarks and future ideas related to our stated results.

6. REMARKS AND FUTURE IDEAS

(1) All the inequalities given in this article can be stated in reverse direction for concave function using simple relation ξ is concave if and only if $-\xi$ is convex.

(2) From [15, p. 140] for convex function ξ we have that

$$\frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta-\xi\left(\frac{\varpi_1+\varpi_2}{2}\right)$$

$$\leq \frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta \tag{6.1}$$

In all our results stated in Section 2, we have found refined bounds only for

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$

by using relation given in (6.1) we automatically get refined bounds for

$$\left|\xi\left(\frac{\varpi_1+\varpi_2}{2}\right) - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$

(3) One may also work on Fejer Hermite Hadamard type inequality by introducing symmetric weights to our obtained results.
(4) One may also work on Weighted Hermite Hadamard type inequality by introducing non symmetric weights to our obtained results.
(5) One may do similar work by using various distinct classes of convex functions.
(6) One may try to state all results stated in this article in discrete case.
(7) One may also state all results stated in this article in Multi-dimensions.
(8) One can extend this work to time scale domain or Quantum Calculus.
(9) One can try to attain this work for Fuzzy set theory.

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[a]PROFESSOR: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KARACHI, UNIVERSITY ROAD, KARACHI-75270, PAKISTAN
Email address: mbilalfawad@gmail.com

[b]PROFESSOR: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KARACHI, UNIVERSITY ROAD, KARACHI-75270, PAKISTAN
Email address: asifrk@uok.edu.pk