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EXISTENCE OF A SOLUTION OF THE INTEGRAL EQUATIONS ON TRIPLED QUASI-METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper we study a tripled quasi-metric with new fixed point theorems around β-implicit contractions in tripled quasi-metric spaces. We give an example on a solution of a integral equations.

1. Introduction and Preliminaries

It is well known that passing from metric spaces to quasi-metric spaces, dropping the requirement that the metric function verifies $d(x, y) = d(y, x)$ carries with it immediate consequences to the general theory. For instance, the topological notions of quasi-metric spaces, such as, limit, continuity, completeness all should be re-considered under the left and right approaches since the quasi-metric is not symmetric. Furthermore, uniqueness of limit of a sequence should be examined carefully since one can easily consider a sequence which has a left limit and right limit which are not equal to each other. Thats why a few results on fixed points in such spaces are considered.

In this paper, we introduce tripled quasi-metric and prove many fixed point results in tripled quasi-metric. We come to the below of the definition of quasi metric space previously defined by a mathematician.

Definition 1.1. Let Y be a non-empty and let $d : Y \times Y \to [0, 1)$ be a function which satisfies:

($d1$) $d(u, v) = 0$ if and only if $u = v$;

($d2$) $d(u, v) \leq d(u, w) + d(w, v)$.

Then d is called a *quasi-metric* and the pair (Y, d) is called a *quasi-metric space.*

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Remark 1.2. Any metric space is a quasi-metric space, but the converse is not true in general.

Definition 1.3. Let (Y, d) be a quasi-metric space, $\{y_n\}$ be a sequence in Y, and $y \in Y$. The sequence $\{y_n\}$ converges to y if and only if

$$\lim_{n\to\infty} d(y_n, y) = \lim_{n\to\infty} d(y, y_n) = 0. \tag{1.1}$$

Remark 1.4. A quasi-metric space is Hausdorff, that is, we have the uniqueness of limit of a convergent sequence.

Definition 1.5. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *left-Cauchy* if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $n \geq m > N$.

Definition 1.6. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *right-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $m \geq n > N$.

Definition 1.7. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $m \geq n > N$.

Remark 1.8. A sequence $\{y_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 1.9. Let (Y, d) be a quasi-metric space. We say that

1) (Y, d) is *left-complete* if and only if each left-Cauchy sequence in Y is convergent.
2) (Y, d) is *right-complete* if and only if each right-Cauchy sequence in Y is convergent.
3) (Y, d) is *complete* if and only if each Cauchy sequence in Y is convergent.

Definition 1.10. Let (Y, d) be a quasi-metric space. We say $f : Y \to Y$ be *continuous* if for each sequence $\{y_n\}$ in Y converging to $y \in Y$, the sequence $\{fy_n\}$ converges to fy, that is,

$$\lim_{n\to\infty} d(fy_n, fy) = \lim_{n\to\infty} d(fy, fy_n) = 0. \tag{1.2}$$

On the other hand the study of fixed point for mappings satisfying on implicit relation in initiated and studies by Popa [21, 22]. It leads to interesting known fixed point results. Following Popa approach , many authors proved some fixed point, common fixed point and coincidence point results in various ambient spaces, see [14, 15, 16, 17, 19].

In the literature, there are several types of implicit contraction mappings, where many nice consequences of fixed point theorems could be derived.

First, denote the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ1) ψ is nondecreasing,

(ψ2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ. We show by Ψ, the set of all function ψ.

Remark 1.11. It is simple to see that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

2. Main Results

Definition 2.1. Let Y be a nonempty set and let $d : Y \times Y \times Y \to [0,\infty)$ be a function which satisfies

(d_1) $d(x,y,z) = 0$ if and only if $x = y = z$;

(d_2) $d(x,y,z) \leq d(x,a_1,a_2) + d(y,a_3,a_4) + d(z,a_2,a_3)$ for all $x,y,z \in Y$ and $a_i \in Y$ for $i = 1,2,3,4$.

Thus d is called a *tripled quasi-metric* and the pair (Y,d) is called a *tripled quasi-metric space.*

Example 2.2. Let $Y = [0,\infty)$ endowed with the tripled quasi metric, $d(x,y,z) = |x| + |y|$ if $x \neq y$, $x \neq z$, $y \neq z$ and $d(x,y,z) = 0$ whenever $x = y = z$.

Definition 2.3. Let (Y,d) de a tripled quasi-metric, $\{y_n\}$ be a sequence in Y, and $x \in Y$. The sequence $\{y_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(y_n,x,x) = \lim_{n\to\infty} d(x,x,y_n) = \lim_{n\to\infty} d(y_n,y_n,x) = \lim_{n\to\infty} d(x,y_n,y_n) = 0.$$

Definition 2.4. Let (Y,d) be a tripled quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *left-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $d(y_n,y_m,y_m) < \varepsilon$ for all $n \geq m > n$.

Definition 2.5. Let (Y,d) be a tripled quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *right-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $d(y_n,y_m,y_m) < \varepsilon$ for all $m \geq n > N$.

Definition 2.6. Let (Y, d) be a tripled quasi-metric space. We say that $\{y_n\}$ is *Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N, such that $d(y_n, y_m, y_m) < \varepsilon$ for all $n, m > N$.

Definition 2.7. Let (Y, d) be a tripled quasi-metric space. We say that

(1) (Y, d) is *left-complete* if and only if each left-Cauchy sequence in Y is convergent;
(2) (Y, d) is *right-complete* if and only if each right-Cauchy sequence in Y is convergent;
(3) (Y, d) is *left-complete* if and only if each Cauchy sequence in Y is convergent.

Definition 2.8. Let (Y, d) be a tripled quasi metric space. The map $f : Y \to Y$ is continuous if for each sequence $\{y_n\}$ in Y converging to $y \in Y$, the sequence $\{fy_n\}$ converges to fy, such that

$$\begin{aligned}\lim_{n\to\infty} d(fy_n, fy, fy) = \lim_{n\to\infty} d(fy, fy, fy_n) &= \lim_{n\to\infty} d(fy_n, fy_n, fy)\\ &= \lim_{n\to\infty} d(fy, fy_n, fy_n) = 0.\end{aligned}$$

Definition 2.9. Let $T : Y \to Y$ and $d : Y \times Y \times Y \to [0, \infty)$ be mappings. We say that the self-mapping T on Y is β *admissible*, if for all $u, v, w \in Y$ we have

$$\beta(u, v, w) \geq 1 \Rightarrow \beta(Tu, Tv, Tw) \geq 1. \tag{2.1}$$

Definition 2.10. Let (Y, d) be a quasi-metric space and $f : Y \to Y$ be a given mapping. We say that f is an *β-implicit contractive mapping* if there exist two functions $\beta : Y \times Y \times Y \to [0, \infty)$ and $\phi \in \Psi$ such that

$$\begin{aligned}\phi\big(\beta(x, y, z)d(fx, fy, fz), d(x, y, z), d(x, fx, f^2x), d(y, fy, f^2y), d(z, fz, f^2z),\\ d(x, fx, z), d(y, fx, y), d(z, fy, z)\big) \leq 0\end{aligned}$$

for all $x, y, z \in Y$.

Definition 2.11. Let Φ be the set of all continuous functions $\phi(t_1, t_2, \ldots, t_8) : \mathbb{R}_+^8 \to \mathbb{R}$ such that

(Φ_1) ϕ is nondecreasing in variable t_1;
(Φ_2) There exists $f_1 \in \Psi$ such that for all $u, v, w \geq 0$, $\phi(u, v, v, u, w, v, 0, 0) \leq 0$ implies that $u \leq f_1(v)$;
(Φ_3) There exists $f_2 \in \Psi$ such that for all $t, t_1, t_2, t_3 > 0$ $\phi(t, t, 0, 0, 0, t_1, t_2, t_3) \leq 0$ implies that $t \leq f_2(t_3)$.

Example 2.12. Let

$$\phi(t_1,t_2,\ldots,t_8)=t_1-a_1t_2-a_2t_3-a_3t_4-a_4t_5-a_5t_6-a_6t_7-a_7t_8,$$

where $a_i\geq 0$ for $i=1,2,\ldots,7$ and $\sum_{i=1}^{7}a_i<1$.

Example 2.13. Let

$$\phi(t_1,t_2,\ldots,t_8)=t_1-k\max\{t_2,\ldots,t_8\},$$

where $k\in[0,1)$.

Theorem 2.14. *Let (Y,d) be a complete tripled quasi-metric space and $g:Y\to Y$ be an β-implicit contractive mapping. Let that*

(i) *g is β-admissible;*

(ii) *There exists $x_0\in Y$ such that $\beta(x_0,gx_0,g^2x_0)\geq 1$ and $\beta(g^2x_0,gx_0,x_0)\geq 1$;*

(iii) *g is continuous.*

Then there exists $\lambda\in Y$ such that $g\lambda=\lambda$.

Proof. By assumption (ii), exists $y_0\in Y$ such that

$$\beta(y_0,gy_0,g^2y_0)\geq 1 \text{ and } \beta(g^2y_0,gy_0,y_0)\geq 1.$$

We define a sequence $\{y_n\}$ in Y by $y_{n+1}=gy_n=g^{n+1}y_0$ for all $n\geq 0$. Let that $x_{n_0}=x_{n_0+1}$ for some n_0. So the proof is complete, because,

$$u=x_{n_0}=x_{n_0+1}=gx_{n_0}=gu.$$

Consequently, throughout the proof, we assume that $y_n\neq y_{n+1}$ for any n. Since g is β-admissible and $\beta(y_0,y_1,y_2)=\beta(y_0,gy_0,g^2y_0)\geq 1$, so observe that $\beta(gy_0,gy_1,gy_2)\geq 1$. By repeating the process above, we obtain that

$$\beta\left(y_n,y_{n+1},y_{n+2}\right)\geq 1 \tag{2.2}$$

for any $n\in\mathbb{N}\cup\{0\}$. Now, consider the case where $\beta(g^2y_0,gy_0,y_0)\geq 1$. By using the same way above, we get that

$$\beta\left(y_{n+2},y_{n+1},y_n\right)\geq 1 \tag{2.3}$$

for all $n\in\mathbb{N}\cup\{0\}$. By using (1.2) we get

$$\begin{aligned}\phi\big(\beta\left(y_{n-1},y_n,y_{n+1}\right)d\left(gy_{n-1},gy_n,gy_{n+1}\right),&\,d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_{n-1},gy_{n-1},g^2y_{n-1}\right),\\ &d\left(y_n,gy_n,g^2y_n\right),d\left(y_{n+1},gy_{n+1},g^2y_{n+1}\right),\\ &d\left(y_{n-1},gy_{n-1},y_{n+1}\right),d\left(y_n,gy_{n-1},y_n\right),\\ &d\left(y_{n+1},gy_n,y_{n+1}\right)\big)\leq 0,\end{aligned}$$

that is

$$\begin{aligned}\phi\big(\beta\left(y_{n-1},y_n,y_{n+1}\right)d\left(y_n,y_{n+1},y_{n+2}\right), d\left(y_{n-1},y_n,y_{n+1}\right), d\left(y_{n-1},y_n,y_{n+1}\right),\\ d\left(y_n,y_{n+1},y_{n+2}\right), d\left(y_{n+1},y_{n+2},y_{n+3}\right),\\ d\left(y_{n-1},y_n,y_{n+1}\right), d\left(y_n,y_n,y_n\right),\\ d\left(y_{n+1},y_{n+1},y_{n+1}\right)\big) \le 0,\end{aligned}$$

and

$$\begin{aligned}\phi\big(\beta\left(y_{n-1},y_n,y_{n+1}\right)d\left(y_n,y_{n+1},y_{n+2}\right), d\left(y_{n-1},y_n,y_{n+1}\right), d\left(y_{n-1},y_n,y_{n+1}\right),\\ d\left(y_n,y_{n+1},y_{n+2}\right), d\left(y_{n+1},y_{n+2},y_{n+3}\right),\\ d\left(y_{n-1},y_n,y_{n+1}\right), 0, 0\big) \le 0.\end{aligned}$$

By (2.2) and from (Φ_1) in the first variable, we have

$$\begin{aligned}\phi\big(d\left(y_n,y_{n+1},y_{n+2}\right), d\left(y_{n-1},y_n,y_{n+1}\right), d\left(y_{n-1},y_n,y_{n+1}\right), d\left(y_n,y_{n+1},y_{n+2}\right),\\ d\left(y_{n+1},y_{n+2},y_{n+3}\right), d\left(y_{n-1},y_n,y_{n+1}\right), 0, 0\big) \le 0.\end{aligned}$$

Due to (Φ_2), we obtain $d\left(y_n,y_{n+1},y_{n+2}\right) \le f_1\left(d\left(y_{n-1},y_n,y_{n+1}\right)\right)$. If we go on like this, we get

$$d\left(y_n,y_{n+1},y_{n+2}\right) \le f_1^n\left(d\left(y_0,y_1,y_2\right)\right). \tag{2.4}$$

We prove that $\{y_n\}$ is a Cauchy sequence in the tripled quasi-metric space (Y,d). Take $m > n$ from (d_2), we have

$$\begin{aligned}d\left(y_n,y_m,y_m\right) &\le d\left(y_n,y_{n+1},y_{n+2}\right) + d\left(y_m,y_m,y_m\right) + d\left(y_m,y_{n+2},y_m\right)\\ &\le f_1^n\big(d\left(y_0,y_1,y_2\right)\big) + d\left(y_m,y_{n+2},y_m\right)\\ &\le f_1^n\big(d\left(y_0,y_1,y_2\right)\big)\\ &\quad + \big[d\left(y_m,y_m,y_m\right) + d\left(y_{n+2},y_{n+3},y_{n+4}\right) + d\left(y_m,y_m,y_{n+3}\right)\big]\\ &\le f_1^n\big(d\left(y_0,y_1,y_2\right)\big) + f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big) + d\left(y_m,y_m,y_m\right)\\ &\quad + d\left(y_m,y_m,y_m\right) + d\left(y_{n+3},y_m,y_m\right)\\ &= f_1^n\big(d\left(y_0,y_1,y_2\right)\big) + f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big) + d\left(y_{n+3},y_m,y_m\right)\\ &\le f_1^n\big(d\left(y_0,y_1,y_2\right)\big) + f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big)\\ &\quad + \big[d\left(y_{n+3},y_{n+4},y_{n+5}\right) + d\left(y_m,y_m,y_m\right) + d\left(y_m,y_{n+4},y_m\right)\big]\end{aligned}$$

$$\leq f_1^n\big(d(y_0,y_1,y_2)\big) + f_1^{n+2}\big(d(y_0,y_1,y_2)\big)$$
$$+ f_1^{n+3}\big(d(y_0,y_1,y_2)\big) + d(y_m,y_{n+4},y_m)$$
$$\leq f_1^n\big(d(y_0,y_1,y_2)\big) + f_1^{n+2}\big(d(y_0,y_1,y_2)\big) + f_1^{n+3}\big(d(y_0,y_1,y_2)\big)$$
$$+ d(y_m,y_m,y_m) + d(y_{n+4},y_{n+5},y_{n+6}) + d(y_m,y_m,y_{n+5}).$$

Let $n+p=m$, then we have

(2.5)
$$\begin{aligned} d(y_n,y_m,y_m) &\leq f_1^n\big(d(y_0,y_1,y_2)\big) + f_1^{n+2}\big(d(y_0,y_1,y_2)\big) + f_1^{n+3}\big(d(y_0,y_1,y_2)\big) \\ &\quad + f_1^{n+4}\big(d(y_0,y_1,y_2)\big) + \ldots + f_1^{n+p}\big(d(y_0,y_1,y_2)\big) \\ &\leq \sum_{k=n}^{\infty} f_1^k\big(d(y_0,y_1,y_2)\big) \end{aligned}$$

which implies that $d(y_n,y_m,y_m) \to 0$, when $n,m \to \infty$, but $f_1 \in \Psi$. It follows that $\{y_n\}$ is a right-Cauchy sequence. By similarly way we can prove that, $\{y_n\}$ is a left-Cauchy sequence. There fore $\{y_n\}$ is a Cauchy sequence in (Y,d). Since, (Y,d) is tripled quasi-complete, then there exists a point λ in Y, such that $y_n \to \lambda$ as $n \to \infty$, that is

(2.6)
$$\lim_{n\to\infty} d(y_n,y,y) = \lim_{n\to\infty} d(y_n,y_n,y) = \lim_{n\to\infty} d(y,y,y_n) = \lim_{n\to\infty} d(y,y_n,y_n) = 0.$$

We shall prove that $g\lambda = \lambda$. Since g is continuous, we verify

$$\lim_{n\to\infty} d(y_{n+1},y_{n+1},g\lambda) = \lim_{n\to\infty} d(gy_n,gy_n,g\lambda) = 0, \tag{2.7}$$

and

$$\lim_{n\to\infty} d(g\lambda,y_{n+1},y_{n+1}) = \lim_{n\to\infty} d(g\lambda,gy_n,gy_n) = 0,$$

that is, $\lim_{n\to\infty} y_{n+1} = g\lambda$, by the uniqueness of limit, we conclude that $g\lambda = \lambda$, that is, λ is a fixed point of g. □

At present, we define a new condition.

(H) If $\{y_n\}$ is a sequence in Y, such that $\beta(y_n,y_{n+1},y_{n+2}) \geq 1$ for any n and $y_n \to y \in Y$, until $n \to \infty$, then there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\beta\left(y_{n(k)},y,y\right) \geq 1$ for all k.

Theorem 2.15. *Let (Y,d) be a tripled complete quasi-metric space and $g : Y \to Y$ be an β-implicit contractive mapping. Let that*

(i) *g is β-admissible;*

(ii) *there exists* $x_0 \in Y$ *such that* $\beta\left(x_0, gx_0, g^2x_0\right) \geq 1$ *and* $\beta\left(g^2x_0, gx_0, x_0\right) \geq 1$;

(iii) *(H) is verified.*

Thus there exists $a, \mu \in Y$ *such that* $g\mu = \mu$.

Proof. From the proof of Theorem 2.14, we hnow that the sequence $\{y_n\}$ defined by $y_{n+1} = gy_n$ for all $n \geq 0$ is Cauchy and converges to some $\mu \in Y$. From condition (iii), there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\beta\left(y_{n(k)}, \mu, \mu\right) \geq 1$ for all k. We must show that $g\mu = \mu$. By (1.2), we have

$$F\Big(\beta\left(y_{n(k)-1}, \mu, \mu\right) d\left(y_{n(k)-1}, g\mu, g\mu\right), d\left(y_{n(k)-1}, \mu, \mu\right), d\left(y_{n(k)-1}, gy_{n(k)-1}, g^2y_{n(k)-1}\right), d\left(\mu, g\mu, g^2\mu\right), d\left(y_{n(k)-1}, gy_{n(k)-1}, \mu\right), d\left(\mu, gy_{n(k)-1}, \mu\right), d\left(\mu, g\mu, \mu\right)\Big) \leq 0.$$

Using (ϕ_1) and $\beta\left(y_{n(k)-1}, \mu, \mu\right) \geq 1$, we get

$$\phi\Big(d\left(y_{n(k)-1}, g\mu, g\mu\right), d\left(y_{n(k)-1}, \mu, \mu\right), d\left(y_{n(k)-1}, y_{n(k)}, y_{n(k)+1}\right), d\left(\mu, g\mu, g^2\mu\right), d\left(\mu, g\mu, g^2\mu\right), d\left(y_{n(k)-1}, y_{n(k)}, \mu\right), d\left(\mu, y_{n(k)}, \mu\right), d\left(\mu, g\mu, \mu\right)\Big) \leq 0.$$

Letting $k \to \infty$ and by continuing of ϕ, we have

$$\phi\Big(d\left(\mu, g\mu, g\mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, g\mu, g^2\mu\right), d\left(\mu, g\mu, g^2\mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, g\mu, \mu\right)\Big) \leq 0,$$

and $\phi(t_1, 0, 0, t_2, t_2, 0, 0, t_3) \leq 0$. By (ϕ_2), $t_1 \leq 0$, that is $d\left(\mu, g\mu, g\mu\right) \leq 0$, which implies $d\left(\mu, g\mu, g\mu\right) = 0$, that is, $\mu = g\mu$. □

For the uniqueness, we need additional condition.

(U) For all $x, y, z \in Fix(g)$, we have $\beta(x, y, z) \geq 1$ where $Fix(g)$ denotes the set of fixed points of g.

Theorem 2.16. *Adding condition (U) to the hypothesis of Theorem 2.14 (resp., Theorem 2.15), we obtain that* μ *is the unique fixed point of* g.

Proof. We obtain by contradiction, that is, there exist $u, v, w \in Y$ such that $u = gu$, $v = gv$ and $w = gu$ with $u \neq v$, $v \neq w$ and $u \neq w$. By (1.2) we get

$$\phi\big(\beta(u,v,w)\,d(gu,gv,gw), d(u,v,w), d(u,u,u), d(v,v,v), d(w,w,w), d(u,u,w), d(v,u,v), d(w,v,w)\big) \le 0,$$

and

$$\phi\big(\beta(u,v,w)\,d(u,v,w), d(u,v,w), 0,0,0, d(u,u,w), d(v,u,v), d(w,v,w)\big) \le 0.$$

Due to the fact that $\beta(u,v,w) \ge 1$, so by (Φ_1), we argue

$$\phi\big(d(u,v,w), d(u,v,w), 0,0,0, d(u,u,w), d(v,w,v), d(w,v,w)\big) \le 0.$$

Since ϕ satisfies property (Φ_3), so there exists $h_2 \in \Psi$, such that

$$\begin{aligned} d(u,v,w) &\le h_2(d(w,v,w)) \\ &\le h_2^2(d(w,v,w)) \\ &\le \cdots \\ &\le h_2^n(d(w,v,w)). \end{aligned} \tag{2.8}$$

Since $\sum_{n=1}^{\infty} h_2^n(t) < \infty$, for each $t \in \mathbb{R}^+$, then as $n \to \infty$, we have

$$\lim_{n\to\infty} h_2^n(d(u,v,w)) = 0.$$

Thus $d(w,v,w) \le 0$, implies that $d(w,v,w) = 0$, that is, $u = v = w$ a contradiction. □

In the sequel we present the following corollaries consequences of Theorem 2.14 (resp. Theorem 2.15).

Corollary 2.17. *Let (Y,d) be a complete tripled quasi-metric space and $g : Y \to Y$ be such that*

$$\begin{aligned} \beta(x,y,z)\,d(gx,gy,gz) \le{}& a_1 d(x,y,z) + a_2 d\left(x,gx,g^2x\right) + a_3 d\left(y,gy,g^2y\right) \\ &+ a_4 d\left(z,gz,g^2z\right) + a_5 d(x,gx,z) + a_6 d(y,gx,y) \\ &+ a_7 d(z,gy,z), \end{aligned}$$

for all $x,y,z \in Y$, where $a_i \ge 0$ for $i = 1,2,\ldots,7$ and $\sum_{i=1}^{7} a_i < 1$. Let that

(i) *g is β-admissible;*
(ii) *there exists $y_0 \in Y$ such that $\beta\left(y_0, gy_0, g^2y_0\right) \ge 1$ and $\beta\left(g^2y_0, gy_0, y_0\right) \ge 1$;*
(iii) *g is continuous or (H) is verified.*

Then there exists $\lambda \in Y$ such that $g\lambda = \lambda$.

Proof. It suffices to put ϕ in Theorem 2.14 (resp. Theorem 2.15) as given in Example 2.12. □

Corollary 2.18. *Let (Y,d) be a tripled complete quasi-metric space and $g : Y \to Y$ be such that*

$$\beta(x,y,z)\, d(gx,gy,gz) \le k \max\Big\{ d(x,y,z), \big(x,gx,g^2x\big), d\big(y,gy,g^2y\big), d\big(z,gz,g^2z\big), d(x,gx,z), d(y,gx,y), d(z,gy,z) \Big\},$$

for any $x,y,z \in Y$, where $k \in [0,1)$. Let that

(i) *g is β-admissible;*
(ii) *there exists $x_0 \in Y$ such that $\beta\big(x_0,gx_0,g^2x_0\big) \ge 1$ and $\beta\big(g^2x_0,gx_0,x_0\big) \ge 1$;*
(iii) *g is continuous or (H) is verified.*

Then there exists a $\lambda \in Y$, such that $g\lambda = \lambda$.

Proof. It suffices to take ϕ in Theorem 2.14 (resp. Theorem 2.15) as given in Example 2.12, that is $\phi(t_1,t_2,\cdots,t_8) = t_1 - k\max\{t_2,\ldots,t_8\}$ where $k \in [0,1)$. □

Corollary 2.19. *Let (Y,d) be a complete tripled quasi-metric space and $g : (Y,d) \to (Y,d)$ be a given mapping. Let that*

$$\phi\big(d(gx,gy,gz) \le d(x,y,z), \big(x,gx,g^2x\big), d\big(y,gy,g^2y\big), d\big(z,gz,g^2z\big), d(x,gx,z), d(y,gx,y), d(z,gy,z)\big) \le 0,$$

for all $x,y,z \in Y$, where $\phi \in \Gamma$. Then g has a unique fixed point.

Proof. It is enough to take $\beta(x,y,z) = 1$ for all $x,y,z \in Y$ in Theorem 2.15. Notice that the hypotheses (U) is satisfied, so we use Theorem 2.14. □

Corollary 2.20. *Let (Y,d) be a complete tripled quasi-metric space and $g : (Y,d) \to (Y,d)$ be a given mapping such that*

$$d(gx,gy,gz) \le k\max\big\{ d(x,y,z), \big(x,gx,g^2x\big), d\big(y,gy,g^2y\big), d\big(z,gz,g^2z\big), d(x,gx,z), d(y,gx,y), d(z,gy,z) \big\} \le 0,$$

for all $x,y,z \in X$, where $k \in [0,1)$. Then g has a unique fixed point.

Proof. It suffices to take ϕ as given in Example 2.12. Then we apply Corollary 2.17. □

Now we show the following example establishing Corollary 2.18.

Example 2.21. Let $Y = [0, \infty)$ endowed with the ripled quasi-metric $d(x, y, z) = |x| + |y|$, if $x \neq y$, $y \neq z$ and $x \neq z$, also $d(x, y, z) = 0$ whenever $x = y = z$. It is obvious that (Y, d) is a complete tripled quasi-metric space. Let the mapping $S : Y \to Y$ defined by

$$Sx = \begin{cases} x^2 - 5x + 6, & x > 2, \\ \dfrac{x}{3}, & x \in [0, 2]. \end{cases}$$

At first we observe that the Banach contraction principle for $d_0(x, y, z) = |x - y| + |x - z| + |y - z|$ can not be used in this case because we have

$$d_0(S0, S4, S8) = d_0(0, 2, 30) = 60 > d_0(0, 4, 8) = 16.$$

We define the mapping $\beta : Y \times Y \times Y \to [0, \infty)$ by $\beta(x, y, z) = 1$, if $x, y, z \in [0, 1]$, otherwise $\beta(x, y, z) = 0$. If $x, y, z \in [0, 1]$ and $x \neq y$, $y \neq z$ and $z \neq z$, we have

$$\begin{aligned} \beta(x, y, z)d(Sx, Sy, Sz) &= d(Sx, Sy, Sz) \\ &\leq |Sx| + |Sy| \\ &= \frac{x}{3} + \frac{y}{3} \\ &= \frac{1}{3} d(x, y, z) \\ &\leq k \max \big\{ d(x, y, z), d(x, Sx, S^2x), d(y, Sy, S^2y), \\ &\quad d(z, Sz, S2^z), d(x, Sx, z), d(y, Sx, y), d(z, Sy, z) \big\}, \end{aligned}$$

where $k = \frac{1}{3}$. Now, we shall prove that the hypotheses (H) is satisfied. Let $\{x_n\}$ be a sequence in Y, such that $\beta(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all n and $x_n \to x \in Y$ as $n \to \infty$. Then by definition of β, we get $(x_n, x_{n+1}, x_{n+2}) \in [0, 1] \times [0, 1] \times [0, 1]$ for any n. Let that $x > 1$, then $x_n \neq x$ for any n. Since $x_n \to x \in Y$, so $d(x, x, x_n) = 2|x| \to 0$, which is a contradiction. Thus $x \in [0, 1]$. We obtain that $(x_n, x, x) \in [0, 1] \times [0, 1] \times [0, 1]$ for all n, that is $\beta(x_n, x, x) = 1$, (H) is verified. Put $x_0 = 1$, we have $\beta(x_0, Sx_0, S^2x_0) = \beta(1, \frac{1}{3}, \frac{1}{9})$ and $\beta(S^2x_0, Sx_0, x_0) = \beta(\frac{1}{9}, \frac{1}{3}, 1) = 1$. The mapping T is β-admissible. Let $x, y, z \in Y$ such that $\beta(x, y, z) \geq 1$, so $x, y, z \in [0, 1]$. Then

$$\beta(Sx, Sy, Sz) = \beta\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = 1.$$

All hypotheses of Corollary 2.18 hold and the mapping S has a fixed point in Y. Note that in this case, we obtain two fixed points of S, that are $\lambda = 0$ and $\lambda = 3 + \sqrt{3}$.

Definition 2.22. Let $(Y, \preceq)$ be a partially ordered set and $g : Y \to Y$ be a given mapping. We say that f is *nondecreasing* with respect to $\preceq$ if $x \preceq y$ then $gx \preceq gy$ for all $x, y, \in Y$.

Definition 2.23. Let $(Y, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset Y$ is said to be *nondecreasing* with respect to $\preceq$, if $x_n \preceq x_{n+1}$ for all n.

Definition 2.24. Let $(Y, \preceq)$ be a partially ordered set and d be a tripled quasi-metric on Y. We say that $(Y, \preceq, d)$ is *regular* if for every nondecreasing sequence $\{x_n\} \subset Y$ such that $x_n \to x \in Y$ as $n \to \infty$, there exists a subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We state the following result.

Theorem 2.25. *Let $(Y, \preceq)$ be a partially ordered set and d be a tripled quasi-metric on Y, such that (Y, d) is complete. Let $g : Y \to Y$ be a nondecreasing mapping with respect to $\preceq$. Let that there exists a function $\phi \in \Gamma$ such that*

$$\phi\big(d\left(gx, gy, gz\right), d\left(x, y, z\right), d\left(x, gx, g^2x\right), d\left(y, gy, g^2y\right), d\left(z, gz, g^2z\right), \\ d\left(x, gx, z\right), d\left(y, gx, y\right), d\left(z, gy, z\right)\big) \leq 0,$$

for all $x, y, z \in Y$ with $x \succeq y \succeq z$ or $x \preceq y \preceq z$. Let that the following conditions hold.

(i) *There exists $x_0 \in Y$ such that $x_0 \preceq gx_0 \preceq g^2x_0$ or $g_2x_0 \preceq gx_0 \preceq x_0$;*

(ii) *g is continuous or $(Y, \preceq, d)$ is regular.*

Then g has a fixed point. Moreover, if $Fix(g)$ is well-ordered, we have uniqueness of the fixed point.

Proof. Define the mapping $\beta : Y \times Y \times Y \to [0, \infty)$ by $\beta(x, y, z) = 1$, if $x \preceq y \preceq z$ or $z \preceq y \preceq x$, otherwise $\beta(x, y, z) = 0$. Obviously, g is an β-implicit contractive mapping, that is

$$\phi\big(\beta(x, y, z)d\left(gx, gy, gz\right), d\left(x, y, z\right), d\left(x, gx, g^2x\right), d\left(y, gy, g^2y\right), d\left(z, gz, g^2z\right), \\ d\left(x, gx, z\right), d\left(y, gx, y\right), d\left(z, gy, z\right)\big) \leq 0.$$

From condition (i) we have $\beta\left(x_0, gx_0, g^2x_0\right) \geq 1$ or $\beta\left(g^2x_0, gx_0, x_0\right) \geq 1$. Moreover, for all $x, y, z \in Y$, from the monotone property of g, we have $\beta(x, y, z) \geq 1$, then $x \succeq y \succeq z$ or $x \preceq y \preceq z$, so $gx \succeq gy \succeq gz$ or $gx \preceq gy \preceq gz$, hence $\beta(gx, gy, gz) \geq 1$. Thus g is β-admissible. Now, if g is continuous the existence of a fixed point follows from Theorem 2.14. Consider now that $(Y, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in Y such that $(x_n, x_{n+1}, x_{n+2}) \geq 1$ for any n and $x_n \to x \in Y$ as $n \to \infty$. From the regularity hypotheses, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of β that $\beta(x_{n(k)}, x, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2.15. To show the uniqueness. Let $x, y \in Y$, ($x \preceq y$ or $y \preceq x$). By hypotheses, there exists $z \in Y$ such that $x \preceq y \preceq z$ or $z \preceq y \preceq x$, which implies $\beta(x, y, z) \geq 1$ or $\beta(z, y, x) \geq 1$. This, we deduce the uniqueness of the fixed point by Theorem 2.16. □

3. Application

Now, we provide an application on the research of a solution of an integral equation. For instance by Corollary 2.20 , we will prove the existence of a solution of the following integral equation, where $E : [0, 1] \times \mathbb{R} \to [0, \infty)$ is a continuous function

$$\begin{aligned} x(t) &= \int_0^1 G(s,t)E(s,x(s))\,ds, \\ y(t) &= \int_0^1 G(s,t)E(s,y(s))\,ds, \\ z(t) &= \int_0^1 G(s,t)E(s,z(s))\,ds. \end{aligned} \tag{3.1}$$

Let $Y = C([0, 1], [0, \infty))$ be the set of nonnegative continuous functions defined on $[0, 1]$, Take the tripled quasi-metric $d : Y \times Y \times Y \to [0, \infty)$ defined by $d(x, y, z) = \|x\|_\infty + \|y\|_\infty$, if $x \neq y$, $x \neq z$ and $y \neq z$, $d(x, y, z) = 0$ whenever $x = y = z$, where $\|x\|_\infty = \sup_{t \in [0,1]} x(t)$. It is easy to show that (Y, d) is a complete tripled quasi-metric. Now, we define the mapping $S : Y \to Y$ as follows

$$Sx(t) = \int_0^1 G(s,t)E(s,x(s))\,ds.$$

Theorem 3.1. *Let the following condition hold. Assume that there exist $\mu_1, \mu_2, \mu_3 \in [0, 1)$ such that $\mu_1 + \mu_2 + \mu_3 < 1$ and for any $s \in [0, 1]$ and $x, y, z \in Y$, ($x \neq y$, $x \neq z$ and $y \neq z$), we have $E(s, x(s)) \leq \mu_1 \|x\|_\infty$, $E(s, y(s)) \leq \mu_2 \|y\|_\infty$, and $E(s, z(s)) \leq \mu_3 \|z\|_\infty$, where*

$$\begin{aligned} \int_0^1 G(s,t)E(s,x(s))\,ds &\neq \int_0^1 G(s,t)E(s,y(s))\,ds, \\ \int_0^1 G(s,t)E(s,x(s))\,ds &\neq \int_0^1 G(s,t)E(s,z(s))\,ds, \end{aligned}$$

$$\int_0^1 G(s,t)E(s,y(s))\,ds \neq \int_0^1 G(s,t)E(s,z(s))\,ds.$$

Then the integral equation (3.1) *has a unique solution* $x \in C([0,1],[0,\infty))$.

Proof. For all $x, y, z \in Y$, ($x \neq y$, $x \neq z$ and $y \neq z$), we have

$$\|Sx\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,x(s))\,ds \leq \frac{1}{8}\mu_1\|x\|_\infty,$$

$$\|Sy\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,y(s))\,ds \leq \frac{1}{8}\mu_2\|y\|_\infty,$$

$$\|Sz\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,z(s))\,ds \leq \frac{1}{8}\mu_3\|z\|_\infty.$$

It follows that for all $x, y, z \in Y$, ($x \neq y$, $x \neq z$ and $y \neq z$), we obtain

$$\begin{aligned} d(Sx,Sy,Sz) &= \|Sx\|_\infty + \|Sy\|_\infty \\ &\leq \frac{1}{8}\mu_1\|x\|_\infty + \frac{1}{8}\mu_2\|y\|_\infty \\ &\leq \frac{1}{8}\left(\|x\|_\infty + \|y\|_\infty\right) \\ &= \frac{1}{8}d(x,y,z). \end{aligned}$$

Therefore,

$$\begin{aligned} d(Sx,Sy,Sz) \leq \frac{1}{8}\max\{&d(x,y,z), d(x,Sx,S^2x), d(y,Sy,S^2y), d(z,Sz,S^2z), \\ &d(x,Sx,z), d(y,Sx,y), d(z,Sy,z)\}. \end{aligned} \tag{3.2}$$

On the other hand, obviously (3.2) holds. Therefore all condition of Corollary 2.20 are satisfied and so S has a unique fixed point. □

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References

1. R.P. Agarwal, E. Karapnar & A.F. Roldán-López-de-Hierro: Fixed point theorems in quasimetric spaces and applications to multidimensional fixed point theorems on G-metric spaces. J. Nonlinear Convex Anal. **16** (2015), no. 9, 1787-1816. https://doi.org/10.1007/978-3-319-24082-4-11
2. M.U. Ali, T. Kamran & E. Karapnar: On (α-ψ-η)-contractive multivalued mappings. Fixed Point Theory Appl. **2014**, Article ID 7, 8, pp. 2014. https://doi.org/10.1186/1687-1812-2014-7
3. A. Aliouche & V. Popa: General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications. Novi Sad. J. Math. **39** (2009), no. 1, 89-109.
4. M.A. Alghamdi & E. Karapnar: G-β-ψ-contractive-type mappings and related fixed point theorems. J. Inequal. Appl. **2013**, Article ID 70, 16, pp. 2013. https://doi.org/10.1186/1029-z42x-2013-70
5. M.A. Alghamdi, E. Karapnar: G-β-ψ-contractive type mappings in G-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 123, 17, pp. 2013. https://doi.org/10.1187-1812-2013-123
6. V. Berinde: Approximating fixed points of implicit almost contractions. Hacet. J. Math. Stat. (2012) **41**, no. 1, 93-102.
7. V. Berinde & F. Vetro: Common fixed points of mappings satisfying implicit contractive conditions. Fixed Point Theory Appl. **2012**, Article ID 105, 8, pp. 2012. https://doi.org/10.1186/1687-1812-2012-105
8. L.B. Ćirić: A generalization of Banach's contraction principle. Proc. Amer. Math. Soc. (1974) **45**, no. 2, 267-273. https://doi.org/10.1090/s0002-9939-1974-0356011-2
9. R.C. Dimri & G. Prasad: Coincidence theorems for comparable generalized nonlinear contractions in ordered partial metric spaces. Comm. Korean Math. Soc. **32** (2017), no. 2, 375-387. https://doi.org/10.4134/CKMS.c160127
10. M. Imdad, S. Kumar & M.S. Khan: Remarks on some fixed point theorems satisfying implicit relations. Rad. Math. **11** (2012), no. 1, 135-143. https://doi.org/MR1971330-135-143
11. M. Jleli, E. Karapnar & B. Samet: Best proximity points for generalized alpha-psi-proximal contractive type mappings. J. Appl. Math. **2013**, Article ID 534127, 10, pp. 2013. https://doi.org/10.1155/2013/534127
12. M. Jleli, E. Karapnar & B. Samet: Fixed point results for α-ψ_λ-contractions on gauge spaces and applications. Abstr. Appl. Anal. **2013**, Article ID 730825, 7, pp. 2013. https://doi.org/10.1155/2013/730825

13. M. Jleli & B. Samet: Remarks on G metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, Article ID 210, 7, pp. 2012. https://doi.org/10.1186/1687-1812-2012-210
14. E. Karapinar & B. Samet: Generalized α-ψ-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, Article ID 793486, 17, pp. 2012. https://doi.org/10.1155/2012-793486
15. E. Karapinar: Fixed point theory for cyclic weak ϕ-contraction. Appl. Math. Lett. **24** (2011), 822-825. https://doi.org/10.1016/j.aml.2010.12.016
16. V. La Rosa & P. Vetro: Common fixed points for α-ψ-φ-contractions in generalized metric spaces. Nonlinear Anal. Model. Control **19**, no. 1, 43-54. https://doi.org/10.1186/1029-242x-2014-439
17. B. Mohammadi, Sh. Rezapour & N. Shahzad: Some results on fixed points of α-ψ-Ciric generalized multifunctions. Fixed Point Theory Appl. **2013**, Article ID 24, 10, pp. 2013. https://doi.org/10.1186/1687-1812-2013-24
18. Z. Mustafa & B. Sims: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7** (2006), 289-297. https://ds.doi.org/10.22075/ijnaa.2022.27489.3619
19. M. Pacurar & I.A. Rus: Fixed point theory for cyclic φ-contractions. Nonlinear Anal., Theory Methods Appl., Ser. A **72** (2010), no. 34, 1181-1187. https://doi.org/10.1016/j.na.2009.08.002
20. V. Popa: Fixed point theorems for implicit contractive mappings. Stud. Cercet. Ştiint., Ser. Mat. Univ. Bacău **7** (1997), 129-133.
21. V. Popa: Some fixed point theorems for compatible mappings satisfying an implicit relation. Demonstr. Math. **32** (1999), 157-163. https://doi.org/10.1515/dema-199-0117
22. V. Popa: A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation. Filomat **19** (2005), 45-51. https://doi.org/10.2298/FIL0519045p
23. V. Popa & A.M. Patriciu: A general fixed point theorem for mappings satisfying an ϕ-implicit relation in complete G-metric spaces. Gazi Univ. J. Science **25** (2012), no. 2, 403-408. https://doi.org/10.35219/ann-ugal-mah-phys-mee.2018.2004
24. V. Popa & A.M. Patriciu: A General fixed point theorem for pairs of weakly compatible mappings in G-metric spaces. J. Nonlinear Sci. App. **5** (2012), no. 2, 151-160. https://dx.doi.org/10.22436/jnsa.005.02.08
25. G. Prasad & H́useyin Iik: On solution of boundary value problem via week contractions. J. Fun. **2022**, Article ID 6799205. pp. 2022. https://doi.org/10.1155/2022/6799205
26. G. Prasad: Coincidence points of relational Ψcontractions and an application. Afr. Mat. **32** (2021), 1475-1490. https://doi.org/10.1007/s13370-021-00913-6

27. G. Prasad & R.C. Dimri: Fixed point theorems via comparable mappings in ordered metric spaces. J. Anal. **27** (2019), no. 4, 1139-1150. https://doi.org/10.1007/s41478-019-00165-5
28. S. Reich & A.J. Zaslawski: Well-posedness of fixed point problems. Far East J. Math. Sci., Spec. Vol., Part **III** (2001), 393-401. https://doi.org/10.1007/s11784-018-0538-1
29. B. Samet, C. Vetro & P. Vetro: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl., Ser. A **75** (2012), no. 4, 2154-2165. https://doi.org/10.1016/j.na.2011.100014

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