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ON EVALUATIONS OF THE CUBIC CONTINUED FRACTION BY MODULAR EQUATIONS OF DEGREE 3 REVISITED

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Abstract. We derive modular equations of degree 3 to find corresponding theta-function identities. We use them to find some new evaluations of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{25}{3\cdot 4^{m-1}}$ and $\frac{4^{1-m}}{3\cdot 25}$, where $m = 0,\ 1,\ 2$.

1. Introduction

Ramanujan's cubic continued fraction $G(q)$, for $|q| < 1$, is defined by

$$G(q) = \frac{q^{1/3}}{1} \, + \, \frac{q+q^2}{1} \, + \, \frac{q^2+q^4}{1} \, + \, \frac{q^3+q^6}{1} \, + \, \cdots \, .$$

In 1984, Ramanathan [12] found the value of $G(e^{-\pi\sqrt{10}})$ by using Kronecker's limit formula. Andrews and Berndt [3] also evaluated $G(e^{-\pi\sqrt{10}})$ by using Ramanujan's class invariants. In 1995, Berndt, Chan, and Zhang [6] evaluated $G(e^{-\pi\sqrt{n}})$ for $n = 2,\ 10,\ 22,\ 58$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1,\ 5,\ 13,\ 37$ by using Ramanujan's class invariants. Chan [7] evaluated $G(e^{-\pi\sqrt{n}})$ for $n = \frac{2}{9},\ 1,\ 2,\ 4$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 1,\ 5$ by using some reciprocity theorems for the cubic continued fraction,

In the 2000s, Adiga, Vasuki, and Mahadeva Naika [2] evaluated $G(e^{-2\pi})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3},\ \frac{25}{3},\ \frac{49}{3},\ \frac{1}{75},\ \frac{1}{147}$ by employing modular equations. Adiga, Kim, Mahadeva Naika, and Madhusudhan [1] also evaluated $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{27}$, 1, 3, 5. Meanwhile, Yi [13] found the values of the cubic continued fraction as stated in Table 1.1 by using modular equations and some eta function identities

In the 2010s, Yi et al. [14] evaluated $G(e^{-\pi\sqrt{n}})$ for $n = \frac{1}{3}$, 1, 4, 9 and $G(-e^{-\pi\sqrt{n}})$ for $n = 4,\ 9$ by employing modular equations of degrees 3 and 9. Paek and Yi [9]

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derived some algorithms based on modular equations of degrees 3 and 9 to evaluate $G(e^{-\pi\sqrt{n}})$ for $n = \frac{4}{3}, \frac{16}{3}, \frac{64}{3}, 36, 81, 144, 324$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{4}{3}, \frac{16}{3}, 36, 81$. Paek and Yi [10] also showed systematic evaluations of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = 4^m, \frac{1}{4^m}, 2\cdot 4^m$ and $\frac{1}{2\cdot 4^m}$, where m is a nonnegative integer. Furthermore, Paek, Shin, and Yi [11] evaluated $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{2\cdot 4^m}{3}, \frac{1}{3\cdot 4^m}$, and $\frac{2}{3\cdot 4^m}$, where $m = 1, 2, 3, 4$, by using modular equations of degrees 3 and 9.

More recently, Yi and Paek [16] and Paek [8] used some theta-function identities to find some new evaluations of the cubic continued fraction. (See Table 1.1 for details). Table 1.1 shows some known values of n for $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$ in chronological order.

Table 1.1. Some known values of n for $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$

Refs	n for $G(e^{-\pi\sqrt{n}})$	n for $G(-e^{-\pi\sqrt{n}})$
[12]	10	
[6]	2, 10, 22, 58	1, 5, 13, 37
[7]	$\frac{2}{9}$, 1, 2, 4	1, 5
[13]	$\frac{1}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{1}{9}, \frac{4}{9}$, 3, 6, 7, 8, 10, 12, 16, 28	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}$, 2, 3, 4, 7
[2]	4	$\frac{1}{3}, \frac{25}{3}, \frac{49}{3}, \frac{1}{75}, \frac{1}{147}$
[1]		$\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{27}$, 1, 3, 5
[14]	$\frac{1}{3}$, 1, 4, 9	4, 9
[9]	$\frac{4}{3}, \frac{16}{3}, \frac{64}{3}$, 36, 81, 144, 324	$\frac{4}{3}, \frac{16}{3}$, 36, 81
[10]	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}$, 1, 8, 16, 32, 64, 128, 256	$\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{128}$ 8, 16, 32, 64
[11]	$\frac{8}{3}, \frac{32}{3}, \frac{128}{3}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192}, \frac{1}{384}$	$\frac{8}{3}, \frac{32}{3}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192}, \frac{1}{384}$
[16]	$\frac{1}{5}, \frac{4}{5}, \frac{9}{5}, \frac{16}{5}, \frac{36}{5}, \frac{144}{5}, \frac{5}{9}, \frac{20}{9}, \frac{80}{9}, \frac{1}{27}, \frac{4}{27}, \frac{16}{27}$, $\frac{1}{45}, \frac{4}{45}, \frac{16}{45}$, 5, 20, 27, 45, 48, 80, 108, 180, 432, 720	$\frac{4}{5}, \frac{9}{5}, \frac{36}{5}, \frac{5}{9}, \frac{20}{9}, \frac{1}{45}, \frac{4}{45}$, 20, 27, 45, 180
[8]	$\frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{20}{3}, \frac{15}{4}, \frac{3}{5}, \frac{12}{5}, \frac{3}{8}, \frac{1}{15}, \frac{4}{15}, \frac{3}{20}, \frac{2}{27}$, $\frac{5}{27}, \frac{8}{27}, \frac{20}{27}, \frac{1}{54}, \frac{1}{60}, \frac{5}{108}, \frac{1}{135}, \frac{4}{135}, \frac{1}{216}, \frac{1}{540}$, 15, 24, 60	$\frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{20}{3}, \frac{15}{4}, \frac{3}{5}, \frac{12}{5}, \frac{1}{6}, \frac{3}{8}, \frac{3}{20}$, $\frac{2}{27}, \frac{5}{27}, \frac{8}{27}, \frac{20}{27}, \frac{1}{54}, \frac{1}{60}, \frac{5}{108}, \frac{1}{135}$, $\frac{4}{135}, \frac{1}{216}, \frac{1}{540}$, 6, 15, 24, 60

In this paper, we first derive modular equations of degree 3 to evaluate to find some theta-function identities. We then use them to find some new evaluations of the cubic continued fraction.

Ramanujan's theta function $\psi(q)$, for $|q|<1$, is defined by

$$\psi(q)=\sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For any positive real numbers k and n, define $l_{k,n}$ and $l'_{k,n}$ by

$$l_{k,n}=\frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \quad \text{and} \quad l'_{k,n}=\frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)},$$

where $q=e^{-\pi\sqrt{n/k}}$. Note that the following property of $l_{k,n}$ in [15] will be useful for evaluating the cubic continued fraction later on

$$l_{k,\frac{1}{n}}=l_{k,n}^{-1}. \tag{1.1}$$

Note also the following formulas for $G^3(e^{-\pi\sqrt{n/3}})$ and $G^3(-e^{-\pi\sqrt{n/3}})$ in terms of $l'_{3,n}$ and $l_{3,n}$, respectively, in [15, Theorem 6.2(ii) and (v)] such as

$$G^3(e^{-\pi\sqrt{n/3}})=\frac{1}{3\,l'^4_{3,n}-1} \tag{1.2}$$

and

$$G^3(-e^{-\pi\sqrt{n/3}})=\frac{-1}{3\,l^4_{3,n}+1}. \tag{1.3}$$

For brevity, we write l_n and l'_n for $l_{3,n}$ and $l'_{3,n}$, respectively.

2. Modular Equations

In this section, we derive modular equations of degree 3 to establish relations between l_n, l_{25n}, l'_n, and l'_{25n}.

Lemma 2.1 ([5], Entry 11, Chapter 20)**.** *Let α, β, γ, and δ be of the first, third, fifth, and fifteenth degrees, respectively. Let m be the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. Let t be such that $m'=mt^2$. Then,*

(i)
$$\alpha=\frac{(m-1)(3+m)^3}{16m^3}, \qquad \beta=\frac{(m-1)^3(3+m)}{16m},$$
$$\gamma=\frac{(m'-1)(3+m')^3}{16m'^3}, \qquad \delta=\frac{(m'-1)^3(3+m')}{16m'},$$
$$1-\alpha=\frac{(m+1)(3-m)^3}{16m^3}, \quad 1-\beta=\frac{(m+1)^3(3-m)}{16m},$$
$$1-\gamma=\frac{(m'+1)(3-m')^3}{16m'^3}, \quad 1-\delta=\frac{(m'+1)^3(3-m')}{16m'},$$

(ii) $\left(1+\dfrac{1}{t}\right)^5(1-t)=(m^2-1)(9m'^{-2}-1),$

$\left(1+\dfrac{1}{t}\right)(1-t)^5=(m'^2-1)(9m^{-2}-1),$

(iii) $m^2+\dfrac{9}{m^2t^4}=\dfrac{t^6+5t^5+5t^4-5t^2+5t-1}{t^5},$

(iv) $2t^5m^2=t^6+5t^5+5t^4-5t^2+5t-1-4t^2(t^2+2t-1)RS,$

and

(v) $\left(\dfrac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}=\dfrac{(R-S)^6}{(t^{-1}-t)^3},$

where

$$4t^2R^2=t^4+t^3+2t^2-t+1 \quad \text{and} \quad 4t^2S^2=t^4+5t^3+2t^2-5t+1.$$

Theorem 2.2. *If* $P=\dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $Q=\dfrac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}$, *then*

$$(PQ)^2+\left(\frac{3}{PQ}\right)^2=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}\right)^2+5\left(\frac{Q}{P}-\frac{P}{Q}\right)-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3. \tag{2.1}$$

Proof. Let α, β, γ, and δ be of degrees 1, 3, 5, 15, respectively. Let m and m' be the multipliers as in Lemma 2.1. Then, by [5, Entry 11(ii), Chapter 17],

$$P=\sqrt{m}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} \quad \text{and} \quad Q=\sqrt{m'}\left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{1/8}.$$

Thus

$$\frac{P}{Q}=\sqrt{\frac{m'}{m}}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}.$$

By Lemma 2.1(i) and (ii), it follows that

$$\frac{P}{Q}=\frac{1}{t}\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4}=\frac{1-t}{1+t},$$

or equivalently

$$t=\frac{Q-P}{P+Q}.$$

We are now in position to complete the proof of (2.1). By Lemma 2.1(v),

$$(PQ)^2=mm'\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}=m^2t^2\frac{(t^{-1}-t)^3}{(R-S)^6}.$$

Thus, by Lemma 2.1(iii), (iv), and (v),

$$
\begin{aligned}
&(PQ)^2+\left(\frac{3}{PQ}\right)^2\\
&=\frac{m^2(1-t^2)^3}{t(R-S)^6}+\frac{9t(R-S)^6}{m^2(1-t^2)^3}\\
&=\frac{m^2t^5(R+S)^6}{(1-t^2)^3}+\frac{9t(R-S)^6}{m^2(1-t^2)^3}\\
&=\frac{t}{(1-t^2)^3}\left(m^2t^4+\frac{9}{m^2}\right)(R^6+15R^4S^2+15R^2S^4+S^6)\\
&\quad+\frac{t}{(1-t^2)^3}\left(m^2t^4-\frac{9}{m^2}\right)(6R^5S+20R^3S^3+6RS^5)\\
&=\frac{t}{(1-t^2)^3}\left(m^2t^4+\frac{9}{m^2}\right)(R^6+15R^4S^2+15R^2S^4+S^6)\\
&\quad+\frac{t}{(1-t^2)^3}\left(2m^2t^4-\left(m^2t^4+\frac{9}{m^2}\right)\right)(6R^5S+20R^3S^3+6RS^5)\\
&=\frac{t^6+5t^5+5t^4-5t^2+5t-1}{(1-t^2)^3}(R^6+15R^4S^2+15R^2S^4+S^6)\\
&\quad-\frac{8t^2(t^2+2t-1)}{(1-t^2)^3}R^2S^2(3R^4+10R^2S^2+3S^4)\\
&=\frac{2(5t^6+16t^5+25t^4+16t-5)}{(t^2-1)^3}\\
&=\frac{-P^6-5P^5Q-5P^2Q^4+5P^2Q^4-5PQ^5+Q^6}{P^3Q^3}\\
&=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}\right)^2+5\left(\frac{Q}{P}-\frac{P}{Q}\right)-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3.
\end{aligned}
$$

□

The following result is an immediate consequence of the modular equation (2.1) and the definition of l_n.

Corollary 2.3. *For every positive real number n, we have*

$$
\begin{aligned}
(2.2)\qquad &3l_n^2\,l_{25n}^2+\frac{3}{l_n^2\,l_{25n}^2}\\
&=\left(\frac{l_{25n}}{l_n}\right)^3-5\left(\frac{l_{25n}}{l_n}\right)^2+5\left(\frac{l_{25n}}{l_n}-\frac{l_n}{l_{25n}}\right)-5\left(\frac{l_n}{l_{25n}}\right)^2-\left(\frac{l_n}{l_{25n}}\right)^3.
\end{aligned}
$$

Theorem 2.4. *If* $P = \dfrac{\psi(q)}{q^{1/4}\psi(q^3)}$ *and* $Q = \dfrac{\psi(q^5)}{q^{5/4}\psi(q^{15})}$, *then*

$$(2.3)\quad (PQ)^2 + \left(\frac{3}{PQ}\right)^2 = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{Q}{P} - \frac{P}{Q}\right) + 5\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3.$$

Proof. Let $T = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $U = \dfrac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}$. Then, by Theorem 2.2,

$$(TU)^2 + \left(\frac{3}{TU}\right)^2 = \left(\frac{U}{T}\right)^3 - 5\left(\frac{U}{T}\right)^2 + 5\left(\frac{U}{T} - \frac{T}{U}\right) - 5\left(\frac{T}{U}\right)^2 - \left(\frac{T}{U}\right)^3.$$

Replace q by $-q$, then $(TU)^2$, $\dfrac{U}{T}$, and $\dfrac{T}{U}$ are converted into $-(PQ)^2$, $-\dfrac{Q}{P}$, and $-\dfrac{P}{Q}$, respectively. Hence

$$-(PQ)^2 - \left(\frac{3}{PQ}\right)^2 = -\left(\frac{Q}{P}\right)^3 - 5\left(\frac{Q}{P}\right)^2 - 5\left(\frac{Q}{P} - \frac{P}{Q}\right) - 5\left(\frac{P}{Q}\right)^2 + \left(\frac{P}{Q}\right)^3,$$

which is equivalent to the modular equation (2.2). □

See [4, Theorem 2.1] for a different proof of Theorem 2.4.

The following result comes from the modular equation (2.2) and the definition of l'_n.

Corollary 2.5. *For every positive real number* n, *we have*

$$(2.4)\quad 3l'^2_n\, l'^2_{25n} + \frac{3}{l'^2_n\, l'^2_{25n}}$$
$$= \left(\frac{l'_{25n}}{l'_n}\right)^3 + 5\left(\frac{l'_{25n}}{l'_n}\right)^2 + 5\left(\frac{l'_{25n}}{l'_n} - \frac{l'_n}{l'_{25n}}\right) + 5\left(\frac{l'_n}{l'_{25n}}\right)^2 - \left(\frac{l'_n}{l'_{25n}}\right)^3.$$

For brevity, we write l_n and l'_n for $l_{3,n}$ and $l'_{3,n}$, respectively.

3. EVALUATIONS OF l_n AND l'_n

Theorem 3.1. *We have*

(i) $l_{25} = \dfrac{1}{3}\left(4 + \sqrt[3]{10} + \sqrt[3]{10^2} + 3\sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)$,

(ii) $l_{\frac{1}{25}} = -\dfrac{1}{3}\left(4 + \sqrt[3]{10} + \sqrt[3]{10^2} - 3\sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)$.

Proof. For (i), let $n = 1$ in (2.2) and set $l_{25} = x$. Using $l_1 = 1$ as in [15, Theorem 2.1(i)], we find that

$$x^6 - 8x^5 + 5x^4 - 5x^2 - 8x - 1 = 0.$$

Now putting $A = x - \frac{1}{x}$, we have

$$A^3 - 8A^2 + 8A - 16 = 0.$$

Solving this equation for A and using the fact that $A > 0$, we deduce that

$$A = \frac{2}{3}(4 + \sqrt[3]{10} + \sqrt[3]{10^2}).$$

Thus rewriting the last equation in terms of x, we have

$$x^2 - \frac{2}{3}(4 + \sqrt[3]{10} + \sqrt[3]{10^2})\, x - 1 = 0.$$

Solving the last equation for x and using the fact that $x > 0$, we complete the proof with the help of *Mathematica*.

For (ii), use the identity $l_{\frac{1}{25}} = l_{25}^{-1}$ as in [14, Theorem 2.1(ii)] to complete the proof. □

See [15, Theorem 4.9(viii)] for a different proof of Theorem 3.1(i).

We recall the following theta-function identities to find some more values of l_n and l'_n.

Lemma 3.2 ([11], Corollaries 3.2, 3.4)**.** *For any positive real number n, we have*

(i) $l'^4_n(\sqrt{3}\, l'^2_{4n} - 1) = l'^2_{4n}(l'^2_{4n} + \sqrt{3})$,

(ii) $l^4_n(\sqrt{3}\, l'^2_{4n} + 1) = l'^2_{4n}(l'^2_{4n} - \sqrt{3})$.

Note that Lemma 3.2(i) and (ii) follow from the modular equations $P^4(Q^2-1) = Q^2(Q^2+3)$ with $P = \frac{\psi(q)}{q^{1/4}\psi(q^3)}$, $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$ and $P^4(Q^2+1) = Q^2(Q^2+3)$ with $P = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}$, $Q = \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$, respectively.

We are in position to evaluate $l'_{\frac{25}{4^{m-1}}}$ for $m = 0, 1, 2$.

Theorem 3.3. *We have*

(i) $l'^4_{100} = \dfrac{1}{3}\left(a - 1 + \sqrt{a^2-4}\right)^2$,

(ii) $l'^4_{25} = \dfrac{a^2 - 4 + (a-1)\sqrt{a^2-4}}{3(a-2)}$,

(iii) $l'^4_{\frac{25}{4}} = 1 - \dfrac{(a+5)\sqrt{a-2} + (a-1)\sqrt{a+2}}{3\sqrt{a-2} - 3\sqrt{a^2 - 4 + (a-1)\sqrt{a^2-4}}}$,

where

$$a = \frac{5}{2} + \frac{1}{54}\left(4 + \sqrt[3]{10} + \sqrt[3]{10^2} + 3\sqrt{5 + 2\sqrt[3]{10} + \sqrt[3]{10^2}}\right)^4.$$

Proof. For (i), first note that $a = \frac{5}{2} + \frac{3}{2}\, l_{25}^4$. Letting $n = 25$ in Lemma 3.2(ii) and setting $x = l'_{100}$, we deduce that

$$3x^4 - 2\sqrt{3}\,(a-2)\,x^2 - 2a + 5 = 0.$$

Solving the last equation for x and using $x > 0$, we complete the proof.

For (ii), let $n = 25$ in Lemma 3.2(i) and let putting the value of l'_{100} obtained in part (i), and using $l'_{25} > 0$, we complete the proof.

The proof of (iii) is similar to that of (ii). □

Note that $l'^4_{\frac{25}{4}}$ in Theorem 3.3(ii) can be evaluated by using the value of $l'^4_1 = 2+\sqrt{3}$ in [14, theorem 4.3(i)] and (2.4), but the evaluation is more complicated.

Theorem 3.4. *Let a be as in Theorem 3.3. Then we have*

(i) $l^4_{\frac{25}{4}} = -1 + \dfrac{(a+5)\sqrt{a-2} + (a-1)\sqrt{a+2}}{3\sqrt{a-2} + 3\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}$,

(ii) $l^4_{\frac{4}{25}} = \dfrac{3\sqrt{a-2} + 3\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{(a+2)\sqrt{a-2} + (a-1)\sqrt{a+2} - 3\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}$.

Proof. For (i), let $n = \frac{25}{4}$ in Lemma 3.2(ii) and put the value of l'_{25} in Theorem 3.3(ii), then we complete the proof. The proof of (ii) follows from (1.1) and (i). □

We now evaluate $l'_{\frac{4^{1-m}}{25}}$ for $m = 0,\, 1,\, 2$.

Theorem 3.5. *Let a be as in Theorem 3.3. Then we have*

(i) $l'^4_{\frac{4}{25}} = 3\left(\dfrac{a-1+\sqrt{a^2-4}}{2a-5}\right)^2$,

(ii) $l'^4_{\frac{1}{25}} = \dfrac{3(a^2-4+(a-1)\sqrt{a^2-4}\,)}{(a+2)(2a-5)}$,

(iii) $l'^4_{\frac{1}{100}} = 1 - \dfrac{3(a-1)\sqrt{a-2} + (5a-11)\sqrt{a+2}}{(2a-5)\sqrt{a+2} - 3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}$.

Proof. For (i), first note that a satisfies $l'^4_{\frac{4}{25}} = \frac{3}{2a-5}$. Letting $n = \frac{1}{25}$ in Lemma 3.2(ii), putting the value of $l'^4_{\frac{4}{25}}$ in terms of a, and setting $x = l'_{\frac{4}{25}}$, we deduce that

$$(2a-5)x^4 - 2\sqrt{3}\,(a-1)\,x^2 - 3 = 0.$$

Solving the last equation for x and using $x > 0$, we complete the proof.

For (ii), letting $n = \frac{1}{25}$ in Lemma 3.2(i), putting the value of $l'_{\frac{4}{25}}$ obtained in part (i), and using $l'_{\frac{1}{25}} > 0$, we complete the proof. The proof of (iii) is similar to that of (ii). □

Theorem 3.6. *Let a be as in Theorem 3.3. Then we have*

(i) $$l^4_{\frac{1}{100}} = -1 + \frac{3(a-1)\sqrt{a-2} + (5a-11)\sqrt{a+2}}{(2a-5)\sqrt{a+2} + 3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}},$$

(ii) $$l^4_{100} = \frac{(2a-5)\sqrt{a+2} + 3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{3(a-1)\sqrt{a-2} + 3(a-2)\sqrt{a+2} - 3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}.$$

Proof. For (i), let $n = \frac{1}{100}$ in Lemma 3.2(ii) and put the value of $l'_{\frac{1}{25}}$ in Theorem 3.5(ii), then we complete the proof. The proof of (ii) follows from (1.1) and (i). □

4. Evaluations of $G(q)$

In this section, we first evaluate $G(e^{-\pi\sqrt{n}})$ for $n = \frac{25}{3\cdot 4^{m-1}}$ and $\frac{4^{1-m}}{3\cdot 25}$, where $m = 0$, 1, 2.

Theorem 4.1. *Let a be as in Theorem 3.3. Then we have*

(i) $$G^3(e^{-\frac{10\pi}{\sqrt{3}}}) = \frac{-(a+1)(a-2) + (a-1)\sqrt{a^2-4}}{8(a-2)},$$

(ii) $$G^3(e^{-\frac{5\pi}{\sqrt{3}}}) = -\frac{1}{4}\left((a+1)(a-2) - (a-1)\sqrt{a^2-4}\right),$$

(iii) $$G^3(e^{-\frac{5\pi}{2\sqrt{3}}}) = \frac{-\sqrt{a-2} + \sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{(a+3)\sqrt{a-2} + (a-1)\sqrt{a+2} + 2\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}},$$

(iv) $$G^3(e^{-\frac{2\pi}{5\sqrt{3}}}) = \frac{-(a+2)(7a-13) + 9(a-1)\sqrt{a^2-4}}{8(a+2)(2a-5)},$$

(v) $$G^3(e^{-\frac{\pi}{5\sqrt{3}}}) = \frac{-(a+2)(7a-13) + 9(a-1)\sqrt{a^2-4}}{4(2a-5)^2},$$

(vi) $$G^3(e^{-\frac{\pi}{10\sqrt{3}}}) = \frac{-(2a-5)\sqrt{a+2} + 3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{9(a-1)\sqrt{a-2} + (11a-23)\sqrt{a+2} + 6\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}.$$

Proof. The proofs follow from Theorems 3.3, 3.5 and (1.2). □

We end this section by evaluating $G(-e^{-\pi\sqrt{n}})$ for $n = \frac{25}{3\cdot 4^{m-1}}$ and $\frac{4^{1-m}}{3\cdot 25}$, where $m = 0, 1, 2$.

Theorem 4.2. *Let a be as in Theorem 3.3. Then we have*

(i) $G^3(-e^{-\frac{10\pi}{\sqrt{3}}})$
$$= \frac{-(a-1)\sqrt{a-2}-(a-2)\sqrt{a+2}+\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{(a-1)\sqrt{a-2}+(3a-7)\sqrt{a+2}+2\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}},$$

(ii) $G^3(-e^{-\frac{5\pi}{\sqrt{3}}}) = -\frac{1}{4}\left(1-6\sqrt[3]{10}+3\sqrt[3]{10^2}-3\sqrt{-40+8\sqrt[3]{10}+5\sqrt[3]{10^2}}\right)$,

(iii) $G^3(-e^{-\frac{5\pi}{2\sqrt{3}}})$
$$= \frac{-\sqrt{a-2}-\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{(a+3)\sqrt{a-2}+(a-1)\sqrt{a+2}-2\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}},$$

(iv) $G^3(-e^{-\frac{2\pi}{5\sqrt{3}}})$
$$= \frac{-(a+2)\sqrt{a-2}-(a-1)\sqrt{a+2}+3\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{(a+11)\sqrt{a-2}+(a-1)\sqrt{a+2}+6\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}},$$

(v) $G^3(-e^{-\frac{\pi}{5\sqrt{3}}}) = -\frac{1}{4}\left(1-6\sqrt[3]{10}+3\sqrt[3]{10^2}+3\sqrt{-40+8\sqrt[3]{10}+5\sqrt[3]{10^2}}\right)$,

(vi) $G^3(-e^{-\frac{\pi}{10\sqrt{3}}})$
$$= \frac{-(2a-5)\sqrt{a+2}-3\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}{9(a-1)\sqrt{a-2}+(11a-23)\sqrt{a+2}-6\sqrt{2a-5}\sqrt{a^2-4+(a-1)\sqrt{a^2-4}}}.$$

Proof. The results are immediate consequences of (1.3) and Theorems 3.1, 3.4, and 3.6. □

See [2, Theorem 2.1] for a different proof of Theorem 4.2(ii).

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