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GENERALIZED (α, β, γ) ORDER AND GENERALIZED (α, β, γ) TYPE ORIENTED SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we discuss on the growth properties of composite entire and meromorphic functions on the basis of generalized (α, β, γ) order and generalized (α, β, γ) type comparing to their corresponding left and right factors.

1. Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $g = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$. We hope that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [3, 9]. We also use the standard notations and definitions of the theory of entire functions which are available in [8, 9] and therefore we do not explain those in details. For meromorphic function f, the Nevanlinna's characteristic function $T_f(r)$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

where $m_f(r)$ and $N_f(r)$ are respectively called as the proximity function of f and the counting function of poles of f in $|z| \leq r$. For details about $T_f(r), m_f(r)$ and $N_f(r)$ one may see [3, p.4]. For an entire function g, the Nevanlinna's Characteristic function $T_g(r)$ of g is defined as

$$T_g(r) = m_g(r).$$

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Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\rightarrow+\infty$. Further we say that $\alpha\in L_1$, if $\alpha\in L$ with $\alpha(a+b)\leq\alpha(a)+\alpha(b)+c$ for all $a,b\geq R_0$ and fixed $c\in(0,+\infty)$. We say that $\alpha\in L_2$, if $\alpha\in L$ and $\alpha(x+O(1))=(1+o(1))\alpha(x)$ as $x\rightarrow+\infty$. Finally, $\alpha\in L_3$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)$ for all $a,b\geq R_0$, i.e., α is subadditive. Clearly $L_3\subset L_1$. Particularly, when $\alpha\in L_3$, then one can easily verify that $\alpha(mr)\leq m\alpha(r)$, $m\geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0)\geq 0$, then for $t\in[0,1]$,

$$\begin{aligned}\alpha(tx) &= \alpha(tx+(1-t)\cdot 0)\\ &\geq t\alpha(x)+(1-t)\alpha(0)\geq t\alpha(x),\end{aligned}$$

so that by choosing $t=\frac{a}{a+b}$ or $t=\frac{b}{a+b}$,

$$\begin{aligned}\alpha(a+b) &= \frac{a}{a+b}\alpha(a+b)+\frac{b}{a+b}\alpha(a+b)\\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right)+\alpha\left(\frac{b}{a+b}(a+b)\right)\\ &= \alpha(a)+\alpha(b),\ a,b\geq 0.\end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r)\leq\alpha(r+R_0)\leq\alpha(r)+\alpha(R_0)$$

for any $R_0\geq 0$. This yields that $\alpha(r)\sim\alpha(r+R_0)$ as $r\rightarrow+\infty$. Throughout the present paper we take $\alpha,\alpha_1,\alpha_2,\alpha_3\in L_1$, $\beta\in L_2$, $\gamma\in L_3$.

Recently Heittokangas et al. [4] have introduced a new concept of φ-order of entire and meromorphic function considering φ as subadditive function. For details one may see [4]. Later on Belaïdi et al. [1] have extended the concept and have introduced the definitions of generalized (α,β,γ) order and generalized (α,β,γ) lower order of a meromorphic function f, which are as follows:

Definition 1.1 ([1])**.** The generalized (α,β,γ) order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and generalized (α,β,γ) lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f are defined as:

$$\rho_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\rightarrow+\infty}\frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}\text{ and }\lambda_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\rightarrow+\infty}\frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}.$$

Remark 1.2. Let $\alpha(r)=\log^{[p]}r$, $(p\geq 0)$, $\beta(r)=\log^{[q]}r$, $(q\geq 0)$ and $\gamma(r)=r$, where $\log^{[k]}x=\log(\log^{[k-1]}x)$ $(k\geq 1)$, with convention that $\log^{[0]}x=x$. If $p=0$

and $q = 0$, i.e., $\alpha(r) = \beta(r) = r$, the Definition 1.1 coincides with the usual order and lower order, when $\alpha(r) = \log^{[p-1]} r$, $(p \geq 1)$, $\beta(r) = r$, we obtain the iterated p-order and iterated lower p-order (see [7]), moreover when $\alpha(r) = \log^{[p-1]} r$ and $\beta(r) = \log^{[q-1]} r$, $(p \geq q \geq 1)$, we get the (p,q)-order and lower (p,q)-order (see [5, 6]).

Belaïdi et al. [2] have recently introduced the definition of another growth indicator, called generalized (α,β,γ) type of a meromorphic function f in the following way:

Definition 1.3 ([2])**.** The generalized (α,β,γ) type denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive generalized (α,β,γ) order $\left(0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ is defined as :

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

In this line, further one may introduce the definition of generalized (α,β,γ) lower type of a meromorphic function f which is as follows:

The generalized (α,β,γ) lower type denoted by $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive generalized (α,β,γ) order $\left(0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ are defined as :

$$\overline{\sigma}_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[f] \leq +\infty$.

Analogously, to determine the relative growth of two meromorphic functions having same non-zero finite generalized (α,β,γ) lower type, one may write the definitions of generalized (α,β,γ) weak type and generalized (α,β,γ) upper weak type of a meromorphic function f of finite positive generalized (α,β,γ) lower order as follows:

Definition 1.4. The generalized (α,β,γ) weak type denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$ and generalized (α,β,γ) upper weak type denoted by $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f having finite positive generalized (α,β,γ) lower order $\left(0 < \lambda_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ are defined as :

$$\begin{aligned}\overline{\tau}_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty}\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}}\\ \text{and } \tau_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty}\frac{\exp(\alpha(\log(T_f(r))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}}.\end{aligned}$$

It is obvious that $0 \le \tau_{(\alpha,\beta,\gamma)}[f] \le \overline{\tau}_{(\alpha,\beta,\gamma)}[f] \le +\infty$.

In this paper we study some growth properties of composite entire and meromorphic functions on the basis of generalized (α,β,γ) order, generalized (α,β,γ) type and generalized (α,β,γ) weak type as compared to the growth of their corresponding left and right factors.

2. Main Results

In this section, the main results of the paper are presented.

Theorem 2.1. *Let f be a meromorphic function and g be an entire function such that $0 < \lambda_{(\alpha_1,\beta,\gamma)}[f(g)] \le \rho_{(\alpha_1,\beta,\gamma)}[f(g)] < +\infty$ and $0 < \lambda_{(\alpha_2,\beta,\gamma)}[f] \le \rho_{(\alpha_2,\beta,\gamma)}[f] < +\infty$. Then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]} &\le \liminf_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)}\\ &\le \min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\le \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \le \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Proof. From the definitions of $\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]$, $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]$, $\lambda_{(\alpha_2,\beta,\gamma)}[f]$, $\rho_{(\alpha_2,\beta,\gamma)}[f]$ and we have for arbitrary positive ε and for all sufficiently large values of r such that

$$\alpha_1\left(\log(T_{f(g)}(r))\right) \geqslant \left(\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon\right)\beta(\log(\gamma(r))), \tag{2.1}$$

$$\alpha_1\left(\log(T_{f(g)}(r))\right) \le \left(\rho_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon\right)\beta(\log(\gamma(r))), \tag{2.2}$$

$$\alpha_2\left(\log(T_f(r))\right) \geqslant \left(\lambda_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right)\beta(\log(\gamma(r))) \tag{2.3}$$

$$\text{and } \alpha_2\left(\log(T_f(r))\right) \le \left(\rho_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right)\beta(\log(\gamma(r))). \tag{2.4}$$

Again for a sequence of values of r tending to infinity,

$$\alpha_1\left(\log(T_{f(g)}(r))\right) \leq \left(\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon\right)\beta(\log(\gamma(r))), \tag{2.5}$$

$$\alpha_1\left(\log(T_{f(g)}(r))\right) \geqslant \left(\rho_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon\right)\beta(\log(\gamma(r))), \tag{2.6}$$

$$\alpha_2\left(\log(T_f(r))\right) \leq \left(\lambda_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right)\beta(\log(\gamma(r))) \tag{2.7}$$

$$\text{and } \alpha_2\left(\log(T_f(r))\right) \geqslant \left(\rho_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right)\beta(\log(\gamma(r))). \tag{2.8}$$

Now from (2.1) and (2.4) it follows for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{2.9}$$

which is the first part of the theorem.

Combining (2.5) and (2.3), we have for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.10}$$

Again from (2.1) and (2.7), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.11}$$

Now, it follows from (2.3) and (2.2), for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.12}$$

Which is the last part of the theorem.

Now from (2.2) and (2.8), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.13}$$

So combining (2.4) and (2.6), we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log(T_{f(g)}(r))\right)}{\alpha_2\left(\log(T_f(r))\right)} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.14}$$

So, the second part of the theorem follows from (2.10) and (2.13) ,the third part is trivial and fourth part follows from (2.11) and (2.14) .

Thus the theorem follows from (2.9) , (2.10) , (2.11), (2.12) , (2.13) and (2.14) . □

Remark 2.2. If we take " $0<\lambda_{(\alpha_3,\beta,\gamma)}[g]\leq\rho_{(\alpha_3,\beta,\gamma)}[g]<+\infty$" instead of " $0<\lambda_{(\alpha_2,\beta,\gamma)}[f]\leq\rho_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.1 remains true with "$\lambda_{(\alpha_3,\beta,\gamma)}[g]$", "$\rho_{(\alpha_3,\beta,\gamma)}[g]$" and "$\alpha_3\left(\log(T_g(r))\right)$" in place of "$\lambda_{(\alpha_2,\beta,\gamma)}[f]$", "$\rho_{(\alpha_2,\beta,\gamma)}[f]$" and "$\alpha_2\left(\log(T_f(r))\right)$" respectively in the denominator.

Theorem 2.3. *Let f be a meromorphic function and g be a non-constant entire function such that $0<\lambda_{(\alpha,\beta,\gamma)}[f]\leq\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f(g)]=+\infty$. Then*

$$\lim_{r\to+\infty}\frac{\alpha(\log(T_{f(g)}(r)))}{\alpha(\log(T_f(r)))}=+\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta>0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log(T_{f(g)}(r)))\leq\Delta\cdot\alpha(\log(T_f(r))). \tag{2.15}$$

Again from the definition of $\rho_{(\alpha,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\alpha(\log(T_f(r)))\leq(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r))). \tag{2.16}$$

From (2.15) and (2.16), for a sequence of values of r tending to $+\infty$, we have

$$\alpha(\log(T_{f(g)}(r))) \le \Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r)))$$
$$i.e.,\ \frac{\alpha(\log(T_{f(g)}(r)))}{\beta(\log(\gamma(r)))} \le \Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)$$
$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha(\log(T_{f(g)}(r)))}{\beta(\log(\gamma(r)))} < +\infty.$$
$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f(g)] < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 2.4. If we take " $0<\lambda_{(\alpha,\beta,\gamma)}[g]\le\rho_{(\alpha,\beta,\gamma)}[g]<+\infty$" instead of " $0<\lambda_{(\alpha,\beta,\gamma)}[f]\le\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.3 remains true with "$\alpha(\log(T_g(r)))$" in replace of "$\alpha\left(\log(T_f(r))\right)$" in the denominator.

Remark 2.5. Theorem 2.3 and Remark 2.4 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f(g)]=+\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f(g)]=+\infty$" and the other conditions remain the same.

Theorem 2.6. *Let f be a meromorphic function and g be an entire function such that* $0<\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]\le\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]<+\infty$, $0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\le\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$ *and* $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_2,\beta,\gamma)}[f]$. *Then*

$$\begin{aligned}\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]} &\le \liminf_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\\ &\le \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]},\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\le \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]},\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\le \limsup_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))} \le \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Proof. From the definitions of $\sigma_{(\alpha_2,\beta,\gamma)}[f]$, $\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$, $\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]$ and $\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]$, we have for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large values of r that (2.17)

$$\exp(\alpha_1(\log\left(T_{f(g)}(r)\right))) \le \left(\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]},$$

(2.18)
$$\exp(\alpha_1(\log\left(T_{f(g)}(r)\right))) \geq \left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]},$$

$$\exp(\alpha_2(\log(T_f(r)))) \leq \left(\sigma_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{2.19}$$

$$\exp(\alpha_2(\log(T_f(r)))) \geq \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.20}$$

Again for a sequence of values of r tending to infinity, we get that

$$\exp(\alpha_1(\log\left(T_{f(g)}(r)\right))) \geqslant (\sigma_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}, \tag{2.21}$$

(2.22)
$$\exp(\alpha_1(\log\left(T_{f(g)}(r)\right))) \leq \left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]},$$

$$\exp(\alpha_2(\log(T_f(r)))) \leq \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{2.23}$$

$$\exp(\alpha_2(\log(T_f(r)))) \geq \left(\sigma_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.24}$$

Now from (2.18), (2.19) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.25}$$

Combining (2.22) and (2.20) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \leq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \leq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.26}$$

Now from (2.18), (2.23) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \geq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\geq\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.27}$$

In view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows from (2.20) and (2.17) for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\leq\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\leq\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.28}$$

Now from (2.17), (2.24) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\leq\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\leq\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.29}$$

So combining (2.19) and (2.21) and in view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log\left(T_f(r)\right)))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{2.30}$$

Thus the theorem follows from (2.25), (2.26), (2.27), (2.28), (2.29) and (2.30). □

Remark 2.7. If we take “ $0<\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]\leq\sigma_{(\alpha_3,\beta,\gamma)}[g]<+\infty$”and “$\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_3,\beta,\gamma)}[g]$” instead of “ $0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\leq\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$” and “$\rho_{(\alpha_1,\beta,\gamma)}[f(g)]=\rho_{(\alpha_2,\beta,\gamma)}[f]$” and other conditions remain same, the results of Theorem 2.6 remain true with “$\sigma_{(\alpha_3,\beta,\gamma)}[g]$”, “$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$” and “$\exp(\alpha_3(\log\left(T_g(r)\right)))$” instead of “$\sigma_{(\alpha_2,\beta,\gamma)}[f]$”, “$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$” and “$\exp(\alpha_2(\log\left(T_f(r)\right)))$” respectively in the denominator.

Remark 2.8. If we take " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" instead of " $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.6 remain true with "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\tau_{(\alpha_2,\beta,\gamma)}[f]$" in place of "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" and "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$" respectively in the denominator.

Remark 2.9. If we take " $0 < \tau_{(\alpha_3,\beta,\gamma)}[g] \leq \overline{\tau}_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_3,\beta,\gamma)}[g]$" instead of " $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.6 remain true with "$\tau_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log(T_g(r))))$" in place of "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$", "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log(T_f(r))))$" respectively in the denominator.

Now in the line of Theorem 2.6 , one can easily prove the following theorem using the notions of generalized (α, β, γ) weak type and generalized (α, β, γ) upper weak type and so the proof is omitted.

Theorem 2.10. *Let f be a meromorphic function and g be an entire function such that $0 < \tau_{(\alpha_1,\beta,\gamma)}[f(g)] \leq \overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)] < +\infty$, $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$. Then*

$$\begin{aligned}
\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \\
&\leq \min\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\leq \max\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\log\left(T_{f(g)}(r)\right)))}{\exp(\alpha_2(\log(T_f(r))))} \leq \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}.
\end{aligned}$$

Remark 2.11. If we take " $0 < \tau_{(\alpha_3,\beta,\gamma)}[g] \leq \overline{\tau}_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_3,\beta,\gamma)}[g]$" instead of " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.10 remain true with "$\tau_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log(T_g(r))))$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log(T_f(r))))$" respectively in the denominator.

Remark 2.12. If we take " $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" instead of " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]$

$= \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.10 remain true with "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$" and "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" respectively in the denominator.

Remark 2.13. If we take " $0 < \overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g] \leq \sigma_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_3,\beta,\gamma)}[g]$" instead of " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" and "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 2.10 remain true with "$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$", "$\sigma_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log(T_g(r))))$" in place of "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log(T_f(r))))$" respectively in the denominator.

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References

1. B. Belaïdi & T. Biswas: Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α, β, γ)-order. WSEAS Trans. Math. **21** (2022), 361-370. `doi:10.37394/23206.2022.21.43`
2. B. Belaïdi & T. Biswas: Growth of (α, β, γ)-order solutions of linear differential equations with entire coefficients. arXiv:2308.04208v1 [math.CV] **8** (2023), 1-25.
3. W.K. Hayman: Meromorphic Functions. The Clarendon Press, Oxford, 1964.
4. J. Heittokangas, J. Wang, Z.T. Wen & H. Yu: Meromorphic functions of finite φ-order and linear q-difference equations. J. Difference Equ. Appl. **27** (2021), no. 9, 1280-1309. `doi:10.1080/10236198.2021.1982919`
5. O.P. Juneja, G.P. Kapoor & S.K. Bajpai: On the (p, q)-order and lower (p, q)-order of an entire function. J. Reine Angew. Math. **282** (1976), 53-67.
6. O.P. Juneja, G.P. Kapoor & S.K. Bajpai: On the (p, q)-type and lower (p, q)-type of an entire function. J. Reine Angew. Math. **290** (1977), 180-190.
7. D. Sato: On the rate of growth of entire functions of fast growth. Bull. Amer. Math. Soc. **69** (1963), 411-414.
8. G. Valiron: Lectures on the general theory of integral functions. Chelsea Publishing Company, New York (NY) USA, 1949.
9. L. Yang: Value distribution theory. Springer-Verlag, Berlin, 1993.

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