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FIXED POINT RESULTS ON GRAPHICAL PARTIAL METRIC SPACES WITH AN APPLICATION

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Abstract. Aim is to present fixed point theorems for contractive mappings in the settings of partial metric spaces equipped with graph. To substantiate the claims and importance of newly obtained fixed point results, we present an application and non-trivial examples. In the light of an application, we ensure the existence of a solution of the linear integral equation via fixed point results. In this way, we generalize, extend and modify some important recent fixed point results of the existing literature, that is, in the settings of partial metric spaces equipped with graph.

1. Introduction

Fixed point theory is a flourishing branch of mathematical analysis that deals with an invariant point of a system that remains unchanged whichever transformation it undergoes. That particular point is considered to be the solution of the mathematical problem corresponding to that system. The versatility of fixed point theorems permits its generalizations and extensions in various branches of mathematical sciences.

The first pivotal result in fixed point theory that acclaimed immense attention was Banach Contraction Principle (Bcp) investigated by Stefan Banach [7] in 1922. In the Bcp symmetric contraction assumption of the self-mapping guaranteed the existence of the fixed point in the complete metric settings. The Bcp soon became the guiding light for many eminent researchers and generalized in the various metric settings and this process is still on. In this continuation, Matthews [14] extended the concept of metric space and introduced partial metric space by incorporating the idea that the of self distance of any point may not be zero and proved fixed point

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result in partial metric settings. Afterwards Oscar Valero [25] and B.E. Rhoades [22] presented some interesting generalizations of the Bcp on partial and usual metric spaces. Further this idea was followed, generalized and modified by many researchers in the various metric settings (see [2, 3, 4, 5, 6, 9, 10, 11, 15, 25]).

On the another point of note, Jachymski [12] established an interesting metrical fixed point results by incorporating the notion of graphical contraction mapping besides presenting an application to polynomial approximation concerning Bernstein operators via contraction principle on the spaces of continuous functions. Meanwhile the Bcp to ordered metric settings was highlighted by Ran and Reurings [23] and further extended by Nieto and López [16] besides presenting some interesting applications to ordinary differential equations. Afterwards this notion was extended and utilize by several researcher and there exist detailed generalization on this theme but keeping in view of requirement of this presentation, we merely refer [1, 13, 17, 18, 19, 20, 24].

Motivated by this work, the main goal of this research presentation is to investigate a graphical variant of fixed point theorem in the settings of partial metric spaces equipped with graph and some other weaker graphical metrical notions. Moreover, we present an application of the main results by solving the linear integral equations in the domain of spaces of continuous functions which ensures the utility of such investigations.

2. Preliminaries

In this section, we discuss relevant necessary background for this presentation. Throughout the presentation, $\mathbb{N}$ the set of natural numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\})$ the set of non-negative integers.

Matthews [14] introduce the concept of partial metric space as follows :

Definition 2.1. A partial metric on a set X is a function $p : X \times X \to R^+$ such that for all $u, v, w \in X$:
(p_1) $u = v \Leftrightarrow p(u,u) = p(u,v) = p(v,v)$,
(p_2) $p(u,u) \leq p(u,v)$,
(p_3) $p(u,v) = p(v,u)$,
(p_4) $p(u,v) \leq p(u,w) + p(w,v) - p(w,w)$.
Note that the self-distance of any point need not to be zero, hence the idea of

generalizing metrics so that a metric on a non-empty set X is precisely partial metric p on X such that for any $u \in X,\ p(u,u) = 0$.

Example 2.2 ([14])**.** Let a function $p : R^+ \times R^+ \to R^+$ be defined by $p(u,v) = \max\{u,v\}$ for any $u,v \in R^+$. Then, (R^+,p) is a partial metric space where the self-distance for any point $u \in R^+$ is its value itself.

Each partial metric p on X generates a T_0 topology T_p on X, which has as a base the family of open p-balls $B_p(u,\epsilon), u \in X, \epsilon > 0$, where

$$B_p(u,\epsilon) = \{v \in X : p(u,v) < p(u,u) + \epsilon\}$$

for all $u \in X$ and $\epsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ defined by

$$p^s(u,v) = 2p(u,v) - p(u,u) - p(v,v)$$

is a metric on X.

Definition 2.3 ([14])**.** Let (X,p) be a partial metric space and $\{u_n\}$ be a sequence in X. Then
(a) $\{u_n\}$ converges to a point $u \in X$ if and only if $p(u,u) = \lim_{n\to\infty} p(u,u_n)$,
(b) $\{u_n\}$ is a Cauchy sequence if there exists (and is finite) $\lim_{n,\,m\to\infty} p(u_n,u_m)$.

Definition 2.4 ([14])**.** A partial metric space (X,p) is said to be *complete* if every Cauchy sequence $\{u_n\}$ in X converges with respect to T_p to a point $u \in X$ such that $p(u,u) = \lim_{n,m\to\infty} p(u_n,u_m)$.

Lemma 2.5 ([14])**.** *Let (X,p) be a partial metric space. Then*
(a) $\{u_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),
(b) (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(u_n,u) = 0$ *if and only if*

$$p(u,u) = \lim_{n\to\infty} p^s(u_n,u) = \lim_{n,\,m\to\infty} p(u_n,u_m).$$

Jachymski [12] introduced notion of graph in metric spaces to obtained the graphical analogous of Banach fixed point theorem. Consider X to be a non-empty set and Δ be the notation for the diagonal points of $X \times X$. Then G can be referred as directed graph with the set of vertices $V(G)$ concurring with X and $E(G)$ being the edge set which contains edges of the graph including all the loops that is

$\Delta \subset E(G)$. Also, assume that the graph omits the parallel edges, so that the ordered pair $(V(G); E(G))$ represents the graph. Also, by assigning distance of two vertices to the edge, graph can be referred as a weighted graph. The graph obtained by reversing the direction of edges is termed as conversion of graph and is denoted by G^{-1}. So, we have

$$E(G) = \{(u, v) \in X \times X\}.$$

Let $\breve{G}$ denote the graph obtained without mentioning direction of edges and treating the graph as directed one as the edge set is symmetric for directed graph.

Definition 2.6 ([12]). Let G be a graph defined on the non-empty set X. Then the sequence $\{u_n\} \subset X$ is :
(a) edge-preserving if $(u_n, u_{n+1}) \in E(G), n \in \mathbb{N}_0$
(b) (X, d) is G-complete if every edge-preserving Cauchy sequence converges in X.

Definition 2.7. Let (X, d) be a metric space and let G be a directed graph on a non-empty set X, then G is p- selfclosed if for any edge preserving sequence $\{u_n\} \subset X$, so that $u_n \to u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $(\{u_{n_k}\}, \{u_n\}) \in E(G),\ k \in \mathbb{N}_0$.

Definition 2.8 ([19]). Let T be a self mappings of a metric space (X, d) equipped with a graph G on X, then G is T_G-transitive if for any $u, v, w \in X$, we have

$$(Tu, Tv), (Tv, Tw) \in E(G) \implies (Tu, Tw) \in E(G).$$

Definition 2.9 ([12]). Let T be a self mappings of a metric space (X, d) equipped with a graph G on X, then T is G-continuous at u if for any edge preserving sequence $\{u_n\}$ so that $u_n \to u$, we have $Tu_n \to Tu$.

Definition 2.10 ([12]). A graph G is connected if there is a path between any two vertices, and is called *weakly connected* if $\breve{G}$ is connected, where $\breve{G}$ denotes the conversion of graph, that is the graph obtained from G by reversing the direction of edges.

Also define $\overset{*}{G} = G \ \cup \breve{G}$. Also it is observed that $\breve{G}\ \subset G$ is also a graph, where $(u, v)\ \in\ \breve{G}$ if $(u, v) \in G$ such that u, v are distinct.

Definition 2.11. Let (X, p) be a partial metric space endowed with graph G. A subset $N \subset X$is said to be *G-precomplete* if each edge-preserving Cauchy sequence $\{u_n\} \subset N$ converges to some $u \in X$.

The following useful lemmas are necessary to prove the main results.

Lemma 2.12 ([14]). *Let (X, p) be a partial metric space*
(a) A sequence $\{u_n\}$ is Cauchy in (X, p) if and only if it is Cauchy in (X, d_p)
(b) (X, p) is complete if and only if (X, d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(u_n, u) = 0 \iff p(u, u) = \lim_{n\to\infty} d_p(u_n, u_m)$.

Where d_p is the usual metric and relation between p and d_p is given as

$$d_p((u_1, u_2) = 2p(u_1, u_2) - p(u_1, u_1) - p(u_2, u_2)$$

Lemma 2.13 ([14]). *Let (X, p) be a partial metric space and $\{u_n\} \subset X$ such that $u_n \to u$, for some $u \in X$ with $p(u, u) = 0$. Then for any $u^* \in X$, we have $\lim_{n\to\infty} p(u_n, u^*) = p(u, u^*)$.*

3. Main Results

In this section, at first, we establish fixed point results under non-linear contraction in partial metric spaces. Then, we prove uniqueness result and furnish a suitable example, corollaries and some important remarks.

Let Ψ be the set of all mappings $\psi : [0, \infty) \to [0, \infty)$ which satisfies the following:
$(\Psi_1) : \psi$ is non-decreasing;
$(\Psi_2) : \psi(\delta) = 0$ if and only if $0 = \delta$ and $\lim_{n\to\infty} \psi(\delta_n) > 0$ if $\lim_{n\to\infty} \delta_n > 0$.

Theorem 3.1. *Let (X, p) be a partial metric space, Also let G be directed graph and T be a self-mapping on X. Assume that the subsequent assumptions hold:*
(a) (X, p) is G-complete,
(b) $T(X, G)$ is non-empty that is there exists $u_0 \in X$ such that $(u_0, Tu_0) \in E(G)$,
(c) there exists $N \subseteq X \times X$ such that N is $\breve{G}$-precomplete and $E(G) \subseteq N$,
(d) G is p-G self closed and T_G-Transitive,
(e) T is either G-continuous or G is p-G self-closed,
(f) G is weakly connected,
(g) T is generalized contraction, that is,

$$p(Tu, Tv) \leq p(u, v) - \psi(p(Tu, Tv)) \tag{3.1}$$

with $(u, v) \in E(G)$ and $\psi \in \Psi$. Then T has a fixed point.

Proof. Consider $u_0 \in X$ in light of the assumption (b), we construct a sequence $\{u_n\} \subset X$ defined by $u_n = Tu_{n-1} = T^{\mathrm{n}}u_0$. If there exists some $m_0 \in \mathbb{N}_0$ such that

$u_{m_0}= u_{m_0+1}$, then u_{m_0} is the fixed point of T and hence the proof is completed. Otherwise assume that $u_n \neq u_{n+1}$, for every $n \in \mathbb{N}_0$, which along with the assumption (c) ensures that $(u_n, u_{n+1}) \in E(G)$, for all $n \in \mathbb{N}_0$. By using assumption (g), we have

$$p(Tu_{n-1}, Tu_n) \leq p(u_{n-1}, u_n) - \psi(p(Tu_{n-1}, Tu_n)) \tag{3.2}$$

which implies

$$p(u_n, u_{n+1}) = p(Tu_{n-1}, Tu_n) \leq p(Tu_{n-1}, Tu_n) \tag{3.3}$$

that is, $\{p(u_n, u_{n+1})\}$ is a non-decreasing sequence of positive real numbers bounded below by 0. So there exists $j \geq 0$ such that $\lim_{n\to\infty} p(u_n, u_{n+1}) = j$. Now we show that $j = 0$. Assume the contrary that $j > 0$. Applying limit on (1) we obtain,

$$j \leq j - \liminf_{n\to\infty} \psi(p(u_n, u_{n+1})); \tag{3.4}$$

which is a contradiction due to (Ψ2). So, we have

$$\lim_{n\to\infty} p(u_n, u_{n+1}) = 0. \tag{3.5}$$

Also, the relation

$$\begin{aligned} d_p(u_n, u_{n+1}) &= 2p(u_n, u_{n+1}) - p(u_n, u_n) - p(u_{n+1}, u_{n+1}) \\ &\leq 2p(u_n, u_{n+1}), \end{aligned}$$

holds, which after applying limit and using (4) gives, $\lim_{n\to\infty} d_p(u_n, u_{n+1}) = 0$.
Next, assuming that sequence $\{u_n\}$ is a Cauchy sequence in (N, d_p). If possible, $\{u_n\}$ is not a Cauchy sequence then for some small positive number $\epsilon > 0$ and a smallest integer n_k there exist two subsequences, $\{u_{m_k}\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$n_k > m_k > k \textit{ and } d_p(u_{m_k}, u_{n_k}) \geq \epsilon. \tag{3.6}$$

Since, $d_p(u, v) \leq 2p(u, v)$, for all $u, v \in X$, so in view of equation (3.6), we have

$$n_k > m_k > kp(u_{m_k}, u_{n_k}) \geq \epsilon/2 \textit{ and } p(u_{m_k}, u_{n_k}) \geq \epsilon/2. \tag{3.7}$$

. By using triangular inequality, we have

$$\begin{aligned} &\epsilon/2 \leq p(u_{m_k}, u_{n_k}) \leq p(u_{m_k}, u_{n_k-1}) + p(u_{m_{k-1}}, u_{n_k}) - p(u_{m_k-1}, u_{n_k-1}) \\ &\epsilon/2 < \epsilon/2 + p(u_{n_k-1}, u_{n_k}). \end{aligned}$$

Letting $k \to \infty$ in the light of (3.4), we have

$$\lim_{k\to\infty} p(u_{m_k}, u_{n_k}) = \epsilon/2. \tag{3.8}$$

Now using (3.3), (3.6) and (p4) assumption of partial metric, we have

$$\begin{aligned}
\epsilon/2 &\leq p(u_{m_k}, u_{n_k}) \\
\epsilon/2 &\leq p(u_{m_k-1}, u_{n_k-1}) \\
\epsilon/2 &\leq p(u_{m_k-1}, u_{m_k}) + p(u_{m_k}, u_{n_k-1}) - p(u_{m_k}, u_{n_k}) \\
\epsilon/2 &< p(u_{m_k-1}, u_{m_k}) + \epsilon/2
\end{aligned}$$

Letting $k \to \infty$

$$\lim_{k\to\infty} p(u_{m_k-1}, u_{n_k-1}) = \epsilon/2.$$

By using T-transitivity assumption, we have $p(u_{m_k-1}, u_{n_k-1}) \in E(G)$, therefore (3.1) implies that

$$p(u_{m_k}, u_{n_k}) \leq p(u_{m_k-1}, u_{n_k-1}) - \psi(p(u_{m_k}, u_{n_k})).$$

Now applying limit $k \to \infty$ in the light of (3.7) and (3.8), we have

$$\epsilon/2 \leq \epsilon/2 - \lim_{k\to\infty} \inf \psi(p(u_{m_k}, u_{n_k})$$

which is a contradiction, hence sequence $\{u_n\}$ is an edge preserving Cauchy sequence in (N, d_p). The precompleteness of N in X assures existence of a point $u^* \in X$ such that

$$\lim_{n\to\infty} u_n = u^*, \text{ that is, } \lim_{n\to\infty} p(u_n, u^*) = p(u^*, u^*) \implies \lim_{n\to\infty} d_p(u_n, u^*) = 0$$

Again, from Lemma 2.1, we have

$$\lim_{n\to\infty} p(u^*, u^*) = \lim_{n\to\infty} p(u_n, u^*) = \lim_{m,n\to\infty} p(u_m, u_n).$$

As T is continuous, implies that as $\{u_n\} \to u_*$ and $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}_0$

$$\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} u_{n+1} = Tu^*,$$

Thus, uniqueness of the limit implies that

$$Tu^* = u^*.$$

Alternatively, if $E(G)$ is p-selfclosed then for any edge preserving sequence $\{u_n\}$ in N with $u_n \to u^*$, there exists a subsequence $\{ u_{n_k} \}$ of $\{ u_n \}$ such that $[u_{n_k}, u^*] \in E(G)$, for all $k \in \mathbb{N}_0$. Now in view of assumption (e) with $u = u_{n_k}$ and $w = u^*$, we obtain

$$\begin{aligned}
p(Tu_{n_k}, Tu^*) &\leq p(u_{n_k}, u^*) - \psi(p(Tu_{n_k}, Tu^*) \\
&\leq p(u_{n_k}, u^*).
\end{aligned}$$

Passing the limit $n \to \infty$ and using Lemma 2.2, we get $p(u^*, Tu^*)) \leq 0$, thereby yielding $u^* = Tu^*$. This completes the proof. □

Next we present a theorem which ensures the uniqueness of the fixed point.

Theorem 3.2. *On adding the following assumption, to the assumptions of Theorem 3.1*
(h) if $T(X)$ is p-G-connected.
Then T has a unique fixed point.

Proof. Owing to Theorem 3.1, it is assured that at least one fixed point of T exists. Suppose, that T has two fixed points, say $u, u^* \in X$, then we have $u = Tu$ and $u^* = Tu^*$. Our claim is that $u = u^*$ as $u,\ u^* \in T(X) \subset N$, so condition (h) ensures that there exists a path, say $\{v_0, v_1, v_2, ..., v_k\} \subset X$ of finite length k in $E(G)$ from $u \to u^*$, where $v_0 = u$ and $v_k = u^*$.
Henceforth

$$[v_i, v_{i+1}] \in E(G) \text{ for each } 0 \leq i \leq k-1 \tag{3.9}$$

Define two constant sequences $\{v_n^0 = u\}$ and $\{v_n^* = u^*\}$,then we have

$$Tv_n^0 = Tu = u \textit{ and } Tv_n^k = Tu^* = u^*, \text{ for all } n \in \mathbb{N}_0.$$

Also put

$$v_0^i = v_i \text{ for each } 0 \ \leq \ i \ \leq \ k \tag{3.10}$$

and define sequences $\{v_n^1\}, \{v_n^2\}, ..., \{v_n^{k-1}\}$ by
$v_{n+1}^i = Tv_n^i,\ \textit{for all } n \in \mathbb{N}_0$ and for each $(1 \ \leq \ i \ \leq \ k-1)$
Hence

$$v_{n+1}^i = Tv_n^i \text{ for all } n \in N_0 \text{ and for each } 0 \ \leq \ i \ \leq \ k.$$

Next we prove that

$$[v_n^i, v_n^{i+1}] \in E(G) \text{ for all } n \in \mathbb{N}_0 \text{ for each } 0 \ \leq \ i \ \leq \ k-1.$$

In the light of (3.9) and (3.10), we obtain $[v_0^i, v_0^{i+1}] \in\ E(G)$ and furthermore *p*-*G*-closedness of G implies that

$$[v_0^i,\ v_0^{i+1}] \in E(G),\ \text{for each } 0 \ \leq \ i \ \leq \ k-1.$$

Now, for all values of $n \in \mathbb{N}_0$ and for each $0 \leq i \leq k-1$ define $\beta_n^i = p(v_n^1, v_n^{i+1})$. Our claim is that

$$\lim_{n\to\infty} \beta_n^i = 0,$$

Let us assume the contrary that $\lim_{n\to\infty} \beta_n^i = \beta > 0$. Since $[v_0^i, v_0^{i+1}] \in E(G)$, either $[v_0^i, v_0^{i+1}] \in E(G)$ or $[v_0^{i+1}, v_0^i] \in E(G)$ (and are distinct) for all $n \in \mathbb{N}_0$ and for each $0 \leq i \leq j-1$, so (3.1) gives

$$p\ (v_n^i,\ v_n^{i+1}) \ \leq\ p\ (v_n^1,\ v_n^{i+1}) \ -\ \psi(p\ (v_n^i\ , v_n^{i+1})),$$

or

$$p\ (v_{n+1}^i\ , v_{n+1}^{i+1}) \ \leq\ p\ (v_n^i,\ v_n^{i+1}) \ -\ \psi(p\ (v_{n+1}^i,\ v_{n+1}^{i+1})),$$

by applying limit, it gives

$$\beta \ \leq\ \beta - \lim_{n\to\infty} \inf\ \psi\ (p(v_{n+1}^i,\ v_{n+1}^{i+1}));$$

which is a contradiction . Hence $\lim_{n\to\infty} \beta_n^i = 0$.
Next,we have

$$\begin{aligned} p(u, u^*) &= p\ (v_n^0, v_n^k)), \\ &\leq \sum_{i=0}^{j-1} p(v_n^i, v_n^{i+1}) \ -\ \sum_{i=1}^{j-1} p\ (v_n^i, v_n^{i+1}), \\ &\leq \sum_{i=0}^{j-1} p\ (v_n^i, v_n^{i+1}), \\ &= \sum_{i=0}^{j-1}\ \beta_n^i \to 0,\ (as\ n \to \infty). \end{aligned}$$

Hence (by $\Psi 1$ and $\Psi 2$) $u =\ u^*$ and this completes the proof. □

Corollary 3.3. *The conclusion of Theorem 3.2 remains true if the condition (h) is replaced by either of the following conditions:*
(h) G is complete.*
*(h**) $E(G)$ is $\breve{G}$- directed.*

Proof. If (h*) is valid , then any $u_1,\ u_2 \in T(X)$, we have $(u_1,\ u_2) \in \overset{*}{G} \subseteq G$ (by condition (c)), that is., $\{u_1, u_2\}$ is a path, length of which is 1 in $E(G)$ from u_1 *to* u_2. Hence, condition (h) of Theorem 3.2 is satisfied and the result is inferred by Theorem 3.2. Further, if condition (h * *) holds, then, for each $u_1,\ u_2 \in E(G)$, there exists $u_3 \in N$ such that $(u_1,\ u_3)$ and $(u_2,\ u_3) \in E(G)$. This concludes that there exists a path of length 2 (say $u_1, u_3, u_2) \in E(G)$ from u_1 to u_2. Hence, again by Theorem 3.2, the result follows. □

Example 3.4. Let $X = [0, 5)$ with partial metric $p : X \times X \to [0, \infty)$ defined by $p(u, v) = max\{u, v\}$, for all $u, v \in X$. Define a graph G such that

$$E(G) = \{(u, v) : p(u, u) = p(u, v) \text{ if } u = max\{u, v\}\}$$

Clearly, (X, d_p) is a complete metric space and therefore (X, p) too. Define $T : [0, 5) \to [0, 5)$ such that

$$Tu = \frac{u}{4}$$

Then, continuity of T imply $\breve{G}$-continuity. Also, it is trivial to prove conditions (a) and (b). Furthermore, let us define $\psi : [0, \infty) \to [0, \infty)$ as $\psi(\alpha) = \frac{5\alpha}{8}$, for all $\alpha \in [0, \infty)$.
For any $u, v \in X$ such that $(u, v) \in E(G)$, we have

$$\begin{aligned} p(Tu, Tv) = \ & \frac{u}{4} \leq p(u, u) - \psi(p(Tu, Tv)) \\ & = u - \left(\frac{5}{8}\right)\left(\frac{u}{4}\right), \\ & = \frac{29u}{32} \end{aligned}$$

Thus, conditions of Theorem 3.1 are fulfilled. Hence, T has $u = 0$ as fixed point. Further as X is G-complete then in the light of Corollary 3.3, we have that 0 is unique fixed point of T.

In light of the fact that every metric is also a partial metric, we can conclude the following result:

Corollary 3.5. *Let (X, d) be a metric space endowed with graph G and T be a self mapping on X. Considering the following results hold true:*
(a) for $(u_0, Tu_0) \in G$ there exists $u_0 \in X$,
(b) G is T-closed and locally T-transitive,
(c) there exists $N \subseteq X$ such that N is $\breve{G}$-precomplete and $T(X) \subseteq N$,
(d) T satisfies the following generalized contraction,

$$d(T(u), T(v)) \leq d(u, v) - \psi\ (d\ (T(u), T(v)),$$

for all $u, v \in X$ with $(u, v) \in \breve{G}$ and $\psi \in \Psi$;
(e) T is $\breve{G}$-continuous or $\breve{G}$ is d-self closed.
Then T has a fixed point and it is unique if $T(X)$ is $\breve{G}$-connected.

Remark 3.6. The conclusion of Theorem 3.1 remains true if the condition (d) is replaced by the following condition:
(d*) T is weak contraction, that is,

$$d(Tu, Tv) \leq d(u, v) - \psi(d(u, v))$$

for all $u, v \in X$ with $(u, v) \in \widetilde{G}$ and $\psi \in \Psi$.

Corollary 3.7. *Let (X, d) be metric space endowed with a graph G and self mapping T. Assume that the following conditions are fulfilled:*
(a) there exists $u_0 \in X$ such that $(u_0, Tu_0) \in E(G)$;
(b) G is T-closed and locally T-transitive;
(c) there exists $N \subset X$ such that N is $\breve{G}$-precomplete and $T(X) \subset N$;
(d) there exists $w \in [0, 1)$ such that

$$d(Tu, Tv) \leq w\ d(u, v),$$

for all $u, v \in X$ with $(u, v) \in \breve{G}$;
(e) T is $\breve{G}$ -continuous or $\breve{G}$ is increasing and d -self closed.
Then T has a fixed point and it is unique if $T(X)$ is $\breve{G}$-connected.

4. An Application

In this section, we discuss the existence of a solution for the linear integral equations.

The theory of integral equations are nowadays a burning topic of research due to the its large applications in many branches of mathematical sciences. The main motivation behind this is due to the sharp rise of the subject integral calculus itself and of having extensive utilizations in various domains of mathematical sciences. In qualitative sciences, mostly problems are modeled as different kind of differential/integral equations.

Further, fixed point theory is a potent analytical tool to establish the solution of nearly all problems modeled by nonlinear relations by proving existence and uniqueness. Due to its applications, fixed point theory is highly appreciated and explored. In fact, the theory can be applied in many spaces, such as metric, Hilbert, Banach, and Sobolev etc. This feature of allowing operations in different spaces makes fixed point theory a valuable tool in studying numerous problems of practical sciences structured as differential and integral equations.

Now we study the condition that suffice for the existence of solution of the following integral equation in the partial metric setup endowed with graph:

$$u(s) = \int_0^s K(s, \eta, u(\eta))d\eta,\ s \in \omega = [0, S],\ S > 0, \tag{4.1}$$

where, $K : \omega \times \omega \times \mathbb{R} \to \mathbb{R}$ is continuous.
Here, we have taken partial metric space $X = C(\omega, \mathbb{R})$ into consideration. The space of all continuous functions from $\omega \to \mathbb{R}$, with partial metric p on X defined as

$$p(f, g) = \max\{\sup_{s\in\omega} |f(s)|, \sup_{s\in\omega} |g(s)|\}.$$

Also consider $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function with $\alpha(s, t) = 0$ iff $s = t$ and $\alpha\ (s, t)\ \leq\ 0$ and $\alpha\ (s, w)\ \leq\ 0\ \implies \alpha\ (t, w) \leq 0$.

Theorem 4.1. *Consider that the conditions mentioned below hold true:*
(A_1) *there exists* $u_0\ \in X$ *such that*

$$\alpha\left(u_0(s), \int_0^s K(s, \eta, u_0(\eta))d\eta\right) \leq 0;$$

(A_2) *for all* $u_1, u_2 \in X$ *and* $s\ \in\ \omega$

$$\alpha(u_1, u_2) \leq 0 \implies \alpha\left(\int_0^s K(s, \eta, u_1(\eta))d\eta, \int_0^s K(s, \eta, u_2(\eta))d\eta\right) \leq 0;$$

(A_3) *for each* $u \in X$ *and* $s, \eta \in \omega$*, there exists a number* $z \in [0, 1)$ *such that*

$$\int_0^s K(s, \eta, u(\eta))d\eta\ \leq\ z\ u(s)$$

Then (4.1) has a solution, say $u^* \in X$*.*

Proof. Define a graph G on X such that

$$(u, v)\ \in\ G \iff \alpha(u_1(s), u_2(s)) \leq 0\ for\ all\ s \in \psi.$$

Also, define $T : X \to X$ by

$$Tu(s) = \int_0^s K(s, \eta, u(\eta))d\eta.$$

Then by condition (A_1), there exists u_0, such that $(u_0, Tu_0) \in G$. Now suppose $(u_1, u_2) \in G$ for some $u_1, u_2 \in X$, that is $\alpha(u_1(s), u_2(s)) \leq 0,$ for all $s \in \psi$. Then owing to condition (A_2), we get

$$\begin{aligned}\alpha(u_1(s), u_2(s)) \leq 0 &\implies \alpha\left(\int_0^s K(s, \eta, u_1(\eta))d\eta, \int_0^s K(s, \eta, u_2(\eta))d\eta\right) \leq 0;\\ &\implies \alpha((u_1, u_2)) \leq 0\\ &\implies (Tu_1, Tu_2) \in G,\end{aligned}$$

that is G is T-closed. Also for $(u_1, u_2) \in \breve{G}$, that is $\alpha(u_1(s), u_2(s)) \leq 0,$ for all $s \in \psi$, we have $p(Tu_1, Tu_2)$

$$\begin{aligned}
&= \max\{\underset{s\in\psi}{Sup}(Tu_1)s,\ \underset{s\in\psi}{Sup}(Tu_2)s\} \\
&= \max\{\underset{s\in\psi}{Sup}\int_0^s K(s,\eta,u_1(\eta))d\eta,\ \underset{s\in\psi}{Sup}\int_0^s\ K(s,\eta,u_2(\eta))d\eta\} \\
&\leq \max\{\underset{s\in\psi}{Sup}\ z\ u_1(s),\ \underset{s\in\psi}{Sup}\ z\ u_2(s)\} \\
&= z\max\{\underset{s\in\psi}{Sup}\ u_1(s),\ \underset{s\in\psi}{Sup}\ u_2(s)\} \\
&= z\ p(u_1, u_2)
\end{aligned}$$

Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(s) = (1 - z)\ s,\ z \in [0, 1)$. It is obvious that $\psi \in \Psi$. Now using it in above inequality, we obtain,

$$\begin{aligned}
p(Tu_1, Tu_2) &\leq p(u_1, u_2) - \psi(p\,(p(u_1, u_2)) \\
&= p\ (u_1, u_2) - \psi p\,(Tu_1, Tu_2)
\end{aligned}$$

Thus all the hypothesis of Theorem 3.1 are satisfied, so we conclude that (4.1) has a solution $u^* \in X$. □

Conclusion

In this work, fixed point results for non-linear contraction are proved for a contractive mappings on partial metric space in the light of some weaker graph theoretic metrical variants. Further, we present non-trivial example vindicating that the claims are new and original. Indeed, we present variant of prominent recent results on the graphical metric settings. In addition, to annotate the utility of such newly obtained results, we solve the linear integral equation. Thus, these findings supply yet another view on fixed point results with some new applications.

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