J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2024.31.3.299 ISSN(Online) 2287-6081
Volume 31, Number 3 (August 2024), Pages 299–310

SOME COMMON FIXED POINT THEOREMS WITH CONVERSE COMMUTING MAPPINGS IN BICOMPLEX-VALUED PROBABILISTIC METRIC SPACE

Sarmila Bhattacharyya [a,*], Tanmay Biswas [b] and Chinmay Biswas [c]

Abstract. The probabilistic metric space as one of the important generalizations of metric space, was introduced by Menger [16] in 1942. Later, Choi et al. [6] initiated the notion of bicomplex-valued metric spaces (bi-CVMS). Recently, Bhattacharyya et al. [3] linked the concept of bicomplex-valued metric spaces and menger spaces, and initiated menger space with bicomplex-valued metric. Here, in this paper, we have taken probabilistic metric space with bicomplex-valued metric, i.e., bicomplex-valued probabilistic metric space and proved some common fixed point theorems using converse commuting mappings in this space.

1. Introduction

Fixed point theorems are of fundamental importance in many areas of Mathematics. Several fixed point theorems are established on metric space theory. In 1986, Jungck [9] introduced the notion of compatible mappings in metric space. Later on, Jungck et al. [10] studied the notion of weakly compatible mappings and improved the commutativity conditions in common fixed point theorems. In 2002, Lü [14] presented the concept of the converse commuting mappings as a reverse process of weakly compatible mappings and proved few common fixed point theorems for single-valued mappings in metric spaces. Then some interesting common fixed point theorems were established for converse commuting mappings by several researchers. For examples, one may see [5, 17, 18, 19, 23].

However, in 1942, K. Menger [16] was the first who thought distance distribution function in metric space and introduced the concept of probabilistic metric space.

Received by the editors April 4, 2024. Revised May 22, 2024, Accepted May 26, 2024.
2020 *Mathematics Subject Classification.* 30G35, 47H10, 54E70.
Key words and phrases. probabilistic metric spaces, conversely commuting mappings, commuting point, fixed point.

*Corresponding author.

He [16] replaced the distance function $d(x, y)$, the distance between two points x and y by distribution function $Fx, y(t)$, where the value of $Fx, y(t)$ is interpreted as the probability that the distance between x, y, i.e., $d(x, y)$ is less than t, $t > 0$.

In this connection, the definition of probabilistic metric space is given as follows:

Definition 1.1 ([16, 22]). A probabilistic metric space (briefly PM space) is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a mapping of $X \times X$ into a collection Δ_+ of all distribution functions F (a distribution function F is a nondecreasing and left continuous mapping from the set of real numbers to $[0, 1]$ with $\inf F(t) = 0$ and $\sup F(t) = 1$). The value of F at $(x, y) \in X \times X$ will be denoted by $F_{x,y}$. The function $F_{x,y}$, $x, y \in X$, are assumed to satisfy the following conditions:

(a) $F_{x,y}(t) = 1$ for all $t > 0$, if and only if $x = y$,

(b) $F_{x,y}(0) = 0$,

(c) $F_{x,y}(t) = F_{y,x}(t)$, and

(d) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $s, t \geq 0$.

After that, many mathematicians proved several fixed point results in probabilistic metric spaces and menger spaces. In 2013, Chauhan et al. [4] proved some common fixed point theorems for conversely commuting mappings using implicit relations in menger space.

On the other hand, in 2011, Azam et al. [1] introduced the notion of complex valued metric space (CVMS) as a generalization and extension of cone metric space and classical metric space. Considering the idea of CVMS, as introduced by Azam et al. [1], Kumar et al. [12] proved some common fixed point theorems for conversely commuting mapping in complex valued metric space in 2014. Again in 2017, Choi et al. [6] linked the concepts of bicomplex numbers and complex valued metric spaces and introduced bicomplex valued metric spaces (bi-CVMS). For more details in the direction of CVMS and bi-CVMS, we refer the researchers [2, 7, 8, 11, 13, 15].

The set of bicomplex numbers denoted by $\mathbb{C}_2$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C} = \{z = x + iy | x, y \in \mathbb{R} \text{ and } i^2 = -1\}$, where $\mathbb{R}$ be the sets of real numbers.For the idea and characteristics of bi-CVMS, one may see [20, 21]. However, we discuss briefly about the bicomplex numbers as follows:

$$\mathbb{C}_2 = \{w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3 | p_k \in \mathbb{R}, k = 0, 1, 2, 3\}.$$

Each element w in $\mathbb{C}_2$ be written as

$$\begin{aligned} w &= p_0 + i_1 p_1 + i_2(p_2 + i_1 p_3) \\ \text{or } w &= z_1 + i_2 z_2 (z_1, z_2 \in \mathbb{C}). \end{aligned}$$

So, we can also express $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{w = z_1 + i_2 z_2 | z_1, z_2 \in \mathbb{C}\}$$

where $z_1 = p_0 + i_1 p_1$, $z_2 = p_2 + i_1 p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The products of all units are commutative and satisfy

$$i_1 i_2 = j, i_1 j = -i_2, i_2 j = -i_1.$$

Let $u = u_1 + i_2 u_2 \in \mathbb{C}_2$ and $v = v_1 + i_2 v_2 \in \mathbb{C}_2$. A partial order relation $\precsim_{i_2}$ defined on $\mathbb{C}_2$, for details one may see [6]. A norm of a bicomplex number $w = z_1 + i_2 z_2$ denoted by $||w||$ is defined by

$$||w|| = ||z_1 + i_2 z_2|| = \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}}$$

which, upon choosing $w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ ($p_k \in \mathbb{R}$, $k = 0, 1, 2, 3$), gives

$$||w|| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}.$$

For details about bicomplex numbers, one may see [20]. For any two bicomplex numbers $u, v \in \mathbb{C}_2$, one can easily verify that $0 \precsim_{i_2} u \precsim_{i_2} v$ which implies $||u|| \leq ||v||$; $||u + v|| \leq ||u|| + ||v||$; $||\alpha u|| \leq \alpha ||u||$ where α is non-negative real number. Choi et al. [6] have defined a bicomplex-valued metric as follows:

Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}_2$ be a bicomplex-valued metric on X if it satisfies the following properties: For $x, y, z \in X$,

(M_1) $0 \precsim_{i_2} d(x, y)$ for all $x, y \in X$;

(M_2) $d(x, y) = 0$ if and only if $x = y$;

(M_3) $d(x, y) = d(y, x)$ for all $x, y \in X$; and

(M_4) $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then (X, d) is called a bicomplex-valued metric space.

Let (X, d) be a metric space and $S, T : X \to X$ be two mappings. A point $x \in X$ is said to be a common fixed point of S and T if and only if

$$Sx = Tx = x.$$

The self maps S and T are said to be commuting (see [14]) if $STx = TSx$ for all $x \in X$, and the point x is called commuting point. Two self maps S and T are said to be conversely commuting if $STx = TSx$ implies $Sx = Tx$ for all $x \in X$. The set of converse commuting points of S and T is denoted by $C(S, T)$.

Recently, Bhattacharyya et al. [3] linked the concept of bicomplex valued metric spaces and menger spaces, where they interpreted $Fx, y(t)$ as the probability that the norm of the distance between x and y is less than t, i.e., $||d(x, y)|| < t$, $t > 0$ and they initiated menger space with bicomplex valued metric. They [3] have also proved certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLR_g) or $(E.A)$ property in this space. Here, in this paper, we have taken probabilistic metric space with bicomplex-valued metric, i.e., probabilistic bicomplex-valued metric space and proved some common fixed point theorems using converse commuting mappings in this space.

2. Main Results

In this section, we have established the main results of this paper.

Theorem 2.1. *Let X be a set of elements and (X, F) be a bicomplex-valued probabilistic metric space. Let P, Q, S and T be four self maps on X such that*

(i). (P, S) and (Q, T) are conversely commuting,

(ii). P and S have a commuting point,

(iii). Q and T have a commuting point, and

(iv). $F_{Px,Qy}(t) \geq \max\{F_{Px,Sx}(\frac{t}{\alpha}), F_{Sx,Ty}(\frac{t}{\alpha}), F_{Qy,Ty}(\frac{t}{\alpha})\}$ where $\alpha \in (0, 1)$, $t > 0$ and for all $x, y \in X$.

Then P, Q, S and T have a unique common fixed point.

Proof. From (ii), P and S have a commuting point, say a. So,

$$PSa = SPa. \tag{2.1}$$

Also, from (iii), Q and T have a commuting point, say b. So,

$$QTb = TQb. \tag{2.2}$$

Since P and S are conversely commuting, therefore

$$Pa = Sa. \tag{2.3}$$

Also, as Q and T are conversely commuting, so

$$Qb = Tb. \tag{2.4}$$

Then from (2.1), (2.2), (2.3) and (2.4), we have

$$PPa = PSa = SPa = SSa \tag{2.5}$$

and

$$QQb = QTb = TQb = TTb. \tag{2.6}$$

Now from (iv) using (2.3) and (2.4), we get by putting $x = a$ and $y = b$ that,

$$\begin{aligned} F_{Pa,Qb}(t) &\geq \max\{F_{Pa,Sa}(\frac{t}{\alpha}), F_{Sa,Tb}(\frac{t}{\alpha}), F_{Qb,Tb}(\frac{t}{\alpha})\} \\ &= \max\{1, F_{Sa,Tb}(\frac{t}{\alpha}), 1\} \\ &= 1. \end{aligned}$$

Which implies that

$$F_{Pa,Qb}(t) \geq 1.$$

But

$$F_{Pa,Qb}(t) \ngtr 1.$$

Therefore,

$$F_{Pa,Qb}(t) = 1 \text{ for all } t > 0.$$

Which implies that

$$Pa = Qb. \tag{2.7}$$

Now we show that Pa is a fixed point of the mapping P. Taking $x = Pa$ and $y = b$ in (iv), and using (2.4) and (2.5), we obtain that

$$\begin{aligned} F_{PPa,Qb}(t) &\geq \max\{F_{PPa,SPa}(\frac{t}{\alpha}), F_{SPa,Tb}(\frac{t}{\alpha}), F_{Qb,Tb}(\frac{t}{\alpha})\} \\ &= \max\{1, F_{SPa,Tb}(\frac{t}{\alpha}), 1\} \\ &= 1. \end{aligned}$$

Since $F_{PPa,Qb}(t) \ngtr 1$, we must have $F_{PPa,Qb}(t) = 1$ for all $t > 0$. Hence, by (2.7), we get that

$$\begin{aligned} F_{PPa,Pa}(t) &= 1 \text{ for all } t > 0. \\ \text{So, } PPa &= Pa. \end{aligned} \tag{2.8}$$

Therefore, Pa is a fixed point of P.

Again with the help of (2.3) and (2.6) and putting $x = a$ and $y = Qb$ in (iv), we obtain that

$$\begin{aligned} & F_{Pa,QQb}(t) \\ \geq \; & \max\{F_{Pa,Sa}(\frac{t}{\alpha}), F_{Sa,TQb}(\frac{t}{\alpha}), F_{QQb,TQb}(\frac{t}{\alpha})\} \\ = \; & \max\{1, F_{Sa,TQb}(\frac{t}{\alpha}), 1\} \\ = \; & 1. \end{aligned}$$

Therefore, $F_{Pa,QQb}(t) = 1$ for all $t > 0$, as $F_{Pa,QQb}(t) \ngtr 1$.

So, using (2.7), we get that $F_{Qb,QQb}(t) = 1$ for all $t > 0$. Then

$$Qb = QQb. \tag{2.9}$$

Therefore, from (2.7) and (2.9), it follows that

$$\begin{aligned} Pa &= Qb = QQb = QPa, \\ \text{i.e., } QPa &= Pa. \end{aligned} \tag{2.10}$$

Hence Pa is a fixed point of Q.

On the other hand, using (2.5) and (2.8), we get that

$$\begin{aligned} Pa &= PPa = PSa = SPa, \\ \text{i.e., } SPa &= Pa. \end{aligned} \tag{2.11}$$

Also, using (2.6), (2.7) and (2.9), we obtain that

$$\begin{aligned} Pa &= Qb = QQb = TQb = TPa, \\ \text{i.e., } TPa &= Pa. \end{aligned} \tag{2.12}$$

Therefore, from (2.8), (2.10), (2.11) and (2.12), we see that Pa is a common fixed point of P, Q, S and T.

Now, to show the uniqueness, let, if possible, ω be another fixed point of P, Q, S and T in X.

Taking $x = Pa$ and $y = \omega$ in (iv), we have

$$F_{PPa,Q\omega}(t) \geq \max\{F_{PPa,SPa}(\frac{t}{\alpha}), F_{SPa,T\omega}(\frac{t}{\alpha}), F_{Q\omega,T\omega}(\frac{t}{\alpha})\}.$$

Since Pa and ω are common fixed points of P, Q, S and T, so from above we get

$$\begin{aligned} F_{Pa,\omega}(t) &\geq \max\{F_{Pa,Pa}(\frac{t}{\alpha}), F_{Pa,\omega}(\frac{t}{\alpha}), F_{\omega,\omega}(\frac{t}{\alpha})\} \\ &= \max\{1, F_{Pa,\omega}(\frac{t}{\alpha}), 1\} \\ &= 1. \end{aligned}$$

Hence $F_{Pa,\omega}(t) = 1$ for all $t > 0$, as $F_{Pa,\omega}(t) \ngtr 1$.
Which implies $Pa = \omega$.
Therefore, Pa is the unique common fixed point of P, Q, S and T in X. □

Example 2.2. Let $X = [1, \infty)$ and a mapping $d : X \times X \to \mathbb{C}_2$ be defined by

$$d(x, y) = (1 + i_1 + i_2 + i_1 i_2)|x - y|, \; x, y \in X,$$

where the symbol $|\;|$ denotes the usual real modulus. So, d is a bicomplex-valued metric on X. Now we define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+||d(x,y)||}, & t > 0 \\ 0, & t = 0 \end{cases} \quad \text{for all } x, y \in X.$$

Then (X, F) is a probabilistic metric space with bicomplex-valued metric. Now the self-mappings P, Q, S, T are defined on X as

$$P(x) = \begin{cases} 2x - 1, & x < 2, \\ 1, & x \geq 2, \end{cases}$$
$$Q(x) = \begin{cases} 2x - 1, & x < 2, \\ 2, & x \geq 2, \end{cases}$$
$$S(x) = \begin{cases} x^2, & x < 2, \\ x + 3, & x \geq 2, \end{cases}$$
$$T(x) = \begin{cases} 3x^2 - 2, & x < 2, \\ x^2 + 1, & x \geq 2. \end{cases}$$

Here the pairs $(P, S)$and (Q, T) are conversely commuting. All conditions of the Theorem 2.1 are satisfied by the mappings and 1 is the unique common fixed point of the mappings P, Q, S, T.

Corollary 2.3. *Let X be a set of elements and (X, F) is a bicomplex-valued probabilistic metric space. Let P and S be self maps on X such that*

(i). the pair (P, S) is conversely commuting,

(ii). P and S have a commuting point, and

(iii). $F_{Px,Py}(t) \geq \max\{F_{Px,Sx}(\frac{t}{\alpha}), F_{Sx,Sy}(\frac{t}{\alpha}), F_{Py,Sy}(\frac{t}{\alpha})\}$ where $\alpha \in (0, 1)$, $t > 0$ and for all $x, y \in X$.

Then P and S have a unique common fixed point in X.

Proof. The proof can be established easily by taking $P = Q$ and $S = T$ in Theorem 2.1. □

Theorem 2.4. *Let (X, F) be a bicomplex-valued probabilistic metric space. Let P, Q, S and T are self mappings on X such that*

(i). the pairs (P, T) and (Q, S) are conversely commuting,

(ii). P and T have a commuting point,

(iii). Q and S have a commuting point, and

(iv). $F_{Px,Qy}(t) \geq \max\{\frac{F_{Px,Tx}(\frac{t}{\alpha})+F_{Tx,Sy}(\frac{t}{\alpha})}{2}, \frac{F_{Px,Tx}(\frac{t}{\alpha})+F_{Sy,Qy}(\frac{t}{\alpha})}{2}, \frac{F_{Px,Sy}(\frac{t}{\alpha})+F_{Tx,Qy}(\frac{t}{\alpha})}{2}\}$
for all $x, y \in X$ and $\alpha \in (0, 1)$, $t > 0$. Then P, Q, S and T have a unique common fixed point in X.

Proof. Let a and b be commuting points of the pairs (P, T) and (Q, S) respectively.
Therefore,

$$PTa = TPa \tag{2.13}$$

$$\text{and } QSb = SQb. \tag{2.14}$$

Again, the pairs (P, T) and (Q, S) are conversely commuting, so

$$Pa = Ta \tag{2.15}$$

$$\text{and } Qb = Sb. \tag{2.16}$$

From (2.13), (2.14), (2.15) and (2.16), we have

$$PPa = PTa = TPa = TTa \tag{2.17}$$

$$\text{and } QQb = QSb = SQb = SSb. \tag{2.18}$$

Now we try to establish

$$Pa = Qb.$$

In (iv), putting $x = a$, $y = b$, we have

$$\begin{aligned} & F_{Pa,Qb}(t) \\ \geq\ & \max\left\{\frac{F_{Pa,Ta}(\frac{t}{\alpha}) + F_{Ta,Sb}(\frac{t}{\alpha})}{2},\right. \\ & \left.\frac{F_{Pa,Ta}(\frac{t}{\alpha}) + F_{Sb,Qb}(\frac{t}{\alpha})}{2}, \frac{F_{Pa,Sb}(\frac{t}{\alpha}) + F_{Ta,Qb}(\frac{t}{\alpha})}{2}\right\}. \end{aligned}$$

Hence in view of (2.15) and (2.16), we get that

$$\begin{aligned} & F_{Pa,Qb}(t) \\ \geq\ & \max\left\{\frac{1+F_{Ta,Sb}(\frac{t}{\alpha})}{2}, 1, \frac{F_{Pa,Sb}(\frac{t}{\alpha})+F_{Ta,Qb}(\frac{t}{\alpha})}{2}\right\} \\ =\ & 1. \end{aligned}$$

Since $F_{Pa,Qb}(t) \not> 1$, we must have $F_{Pa,Qb}(t) = 1$ for all $t > 0$.

Which implies that

$$Pa = Qb. \tag{2.19}$$

Next we show that

$$P^2a = PPa = Pa.$$

Taking $x = Pa$, $y = b$ in (iv), we have

$$\begin{aligned} & F_{P^2a,Qb}(t) \\ \geq\ & \max\left\{\frac{F_{PPa,TPa}(\frac{t}{\alpha})+F_{TPa,Sb}(\frac{t}{\alpha})}{2},\right. \\ & \left.\frac{F_{PPa,TPa}(\frac{t}{\alpha})+F_{Sb,Qb}(\frac{t}{\alpha})}{2}, \frac{F_{PPa,Sb}(\frac{t}{\alpha})+F_{TPa,Qb}(\frac{t}{\alpha})}{2}\right\}. \end{aligned}$$

Now using (2.16) and (2.17) , we obtain that

$$\begin{aligned} & F_{P^2a,Qb}(t) \\ \geq\ & \max\left\{\frac{1+F_{TPa,Sb}(\frac{t}{\alpha})}{2}, 1, \frac{F_{PPa,Sb}(\frac{t}{\alpha})+F_{TPa,Qb}(\frac{t}{\alpha})}{2}\right\} \\ =\ & 1. \end{aligned}$$

Since $F_{P^2a,Qb}(t) \not> 1$, we must have $F_{P^2a,Qb}(t) = 1$ for all $t > 0$.

Therefore, from (2.19) and above, we get that

$$\begin{aligned} P^2a &= Qb = Pa, \\ \text{i.e., } P^2a &= Pa. \end{aligned} \tag{2.20}$$

Similarly, we obtain that

$$Q^2b = Qb. \tag{2.21}$$

Hence it follows that

$$Pa = PPa = PTa = TPa,$$

which implies that

$$TPa = Pa. \tag{2.22}$$

Again by (2.18) and (2.21) , we have

$$Qb = QQb = QSb = SQb.$$

As $Pa = Qb$, so we get from above,

$$SPa = Pa. \tag{2.23}$$

Also, $QQb = Qb$ implies that

$$QPa = Pa. \tag{2.24}$$

From (2.20) , (2.22) , (2.23) and (2.24) , we can show that Pa is a common fixed point of P, Q, S and T.

To show the uniqueness, let, if possible, ω be another fixed point of P, Q, S and T in X.

Taking $x = Pa$ and $y = \omega$ in (iv), we have

$$F_{PPa,Q\omega}(t) \geq \max\left\{\frac{F_{PPa,TPa}(\frac{t}{\alpha}) + F_{TPa,S\omega}(\frac{t}{\alpha})}{2}, \frac{F_{PPa,TPa}(\frac{t}{\alpha}) + F_{S\omega,Q\omega}(\frac{t}{\alpha})}{2}, \frac{F_{PPa,S\omega}(\frac{t}{\alpha}) + F_{TPa,Q\omega}(\frac{t}{\alpha})}{2}\right\}.$$

Which implies that

$$\begin{aligned} F_{Pa,\omega}(t) &\geq \max\left\{\frac{F_{Pa,Pa}(\frac{t}{\alpha}) + F_{Pa,\omega}(\frac{t}{\alpha})}{2}, \frac{F_{Pa,Pa}(\frac{t}{\alpha}) + F_{\omega,\omega}(\frac{t}{\alpha})}{2}, \frac{F_{Pa,\omega}(\frac{t}{\alpha}) + F_{Pa,\omega}(\frac{t}{\alpha})}{2}\right\}. \\ &= \max\left\{\frac{1 + F_{Pa,\omega}(\frac{t}{\alpha})}{2}, 1, F_{Pa,\omega}(\frac{t}{\alpha})\right\} \\ &= 1. \end{aligned}$$

Hence $F_{Pa,\omega}(t) = 1$ for all $t > 0$, as $F_{Pa,\omega}(t) \ngtr 1$.

Which implies that $Pa = \omega$.

Therefore, Pa is the unique common fixed point of P, Q, S and T in X.

This completes the proof. □

ACKNOWLEDGMENT

The authors are very much thankful to the reviewers for their valuable suggestions to bring the paper in its present form.

References

1. A. Azam, F. Brain & M. Khan: Common fixed point theorems in complex valued metric spaces. Numer. Funct. Anal. Optim. **32** (2011), no. 3, 243-253. https://doi.org/10.1080/01630563.2011.533046
2. R. Agarwal, M.P. Goswami & R.P. Agarwal: Bicomplex version of Stieltjes transform and applications. Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **21** (2014), 229-246.
3. S. Bhattacharyya, C. Biswas & T. Biswas: Common fixed point theorems in connection with two weakly compatible mappings in menger space with bicomplex-valued metric. Electron. J. Math. Anal. Appl. **12**, (2024), no. (1, 9), 1-10. https://doi.org/10.21608/EJMAA.2024.243997.1087
4. S. Chauhan & H. Sahper: Common fixed point theorems for conversely commuting mappings using implicit relations. J. Oper. Hindawi **2013**, Article ID 391474, 5 pages. https://doi.org/10.1155/2013/391474
5. S. Chauhan & B.D. Pant: Fixed points of weakly compatible mappings using common (E.A) like property. Matematiche **68** (2013) no. 1, 99-116.
6. J. Choi, S.K. Datta, T. Biswas & Md N. Islam: Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Honam Math. J. **39** (2017), no. 1, 115-126. https://doi.org/10.5831/HMJ.2017.39.1.115
7. A.J. Gnanaprakasam, S.M. Boulaaras, G. Mani, B. Cherif & S.A. Idris: Solving system of linear equations via bicomplex valued metric space. Demonstr. Math. **54** (2021), 474-487. https://doi.org/10.1515/dema-2021-0046
8. Md N. Islam: Fixed point theorems based on well-posedness and periodic point property in ordered bicomplex valued metric spaces. Malaya J. Mat. **9** (2020), no. 1, 9-19. https://doi.org/10.26637/MJM0901/0002
9. G. Jungck: Compatible mappings and common fixed points. Int. J. Math. Math. Sci. **9** (1986), no. 4, 771-779. https://doi.org/10.1155/S0161171286000935
10. G. Jungck & B.E. Rhoades: Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. **29** (1998), no. 3, 227-238.
11. M.V.R. Kameswari & A. Bharathi: Common fixed points for a pair of maps in bicomplex valued generalized metric spaces. Adv. Appl. Math. Sci. **21** (2022), no. 5, 2345-2370.
12. J. Kumar, R. Sharma & A. Singh: Common fixed point theorem for converse commuting maps in complex valued metric spaces. Int. J. Math. Archive **5** (2014), no. 6, 215-220.
13. R.G. Lavoie, L. Marchildon & D. Rochon: In finite-dimensional bicomplex Hilbert spaces. Ann. Funct. Anal. **1** (2010), no. 2, 75-91. https://doi.org/10.15352/afa/1399900590

14. Z.X. Lü: Common fixed points for converse commuting selfmaps on a metric space. Acta Anal. Funct. Appl. **4** (2002), no. 3, 226-228 (Chinese).
15. G. Mani, A.J. Gnanaprakasam, R. George & G.D. Mitrovic: The existence of a solution of a nonlinear Fredholm integral equations over bicomplex b-metric spaces. Gulf J. Math. **14** (2023), no. 1, 69-83. https://doi.org/10.56947/gjom.v14i1.984
16. K. Menger: Statistical metrics. Proc. Nat. Acad. Sci., U.S.A. **28** (1942), 535-537. https://doi.org/10.1073/pnas.28.12.535
17. H.K. Pathak & R.K. Verma: Integral type contractive condition for converse commuting mappings. Int. J. Math. Anal. **3** (2009), no. 24, 1183-1190.
18. H.K. Pathak & R.K. Verma: An Integral type implicit relation for converse commuting mappings. Int. J. Math. Anal. **3** (2009), no. 24, 1191-1198.
19. V. Popa: A general fixed point theorem for converse commuting multi-valued mappings in symmetric spaces. Filomat **21** (2007), no. 2, 267-271.
20. D. Rochon & M. Shapiro: On algebraic properties of bicomplex and hyperbolic numbers. An. Univ. Oradea Fasc. Mat. **11** (2004), 71-110.
21. C. Segre: Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. Math. Ann. **40** (1892), 413-467. https://doi.org/10.1007/BF01443559
22. B. Schweizer & A. Sklar: Probabilistic metric spaces. North-Holland, Amsterdam, The Netherlands, 1983.
23. R. Vasuki: Common fixed points for R-weakly commuting maps in fuzzy metric spaces. Indian J. Pure Appl. Math. **30** (1999), 419-423.

[a]Assistant Professor: Department of Mathematics, Netaji Mahavidyalaya, P.O.- Arambagh, Dist.-Hooghly, PIN-712601, West Bengal, India.
Email address: sarmilabhattacharyya1982@gmail.com

[b]Independent Research Scientist: Rajbari, Rabindrapally, R. N. Tagore Road, P.O.-Krishnagar, P.S.-Katwali, Dist.-Nadia, PIN-741101, West Bengal, India.
Email address: tanmaybiswas_math@rediffmail.com

[c]Assistant Professor: Department of Mathematics, Nabadwip Vidyasagar College, Nabadwip, Dist.-Nadia, PIN-741302, West Bengal, India.
Email address: chinmay.shib@gmail.com