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NEW EXTENSIONS OF THE HERMITE-HADAMARD INEQUALITIES BASED ON ψ-HILFER FRACTIONAL INTEGRALS

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Abstract. This article presents the above and below bounds for Midpoint and Trapezoid types inequalities for ψ-Hilfer fractional integrals with the assistance of the functions whose second derivatives are bounded. We also possess some extensions and generalizations of Hermite–Hadamard inequalities via ψ-Hilfer fractional integrals with the aid of the functions that have the conditions that will said.

1. Introduction

Inequalities have guided many studies in mathematical topics. Especially the Hermite-Hadamard inequality (H-H$\mathbb{I}$) has been the subject of many studies in the literature. In fact, the inequality is defined for convex function $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ by

$$\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \leq \int_{\grave{\imath}_1}^{\grave{\imath}_2} \frac{\varrho(\eta)}{\grave{\imath}_2-\grave{\imath}_1}\, \mathrm{d}\eta \leq \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2}. \tag{1}$$

In particular, studies on the right and left sides of the H-H$\mathbb{I}$ contribute to the literature. For example, Dragomir *et al.* proved refinements and extensions for Midpoint type inequalities (M$\mathbb{I}$) and Trapezoid type inequalities (T$\mathbb{I}$) with the help of the bounds of the twice differentiable functions in [11] and [12], respectively. Fractional calculus is an successful apparatus to clarify physical wonders additionally real-world issues. Fractional derivative and integral operators not as it were varied from each other in terms of peculiarity, territory, and bits but moreover brought advancements to fractional analysis in terms of their utilization zones and spaces [22, 15, 1].

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Fractional integrals solved many integrals in mathematics. Fractional integral types, which are also used in the field of inequality, have provided new extensions, refinements, and, generalizations in this field [13, 3, 21, 16, 9, 18]. Insome studies, by using the convexity of the function, in some research, by making use of the bounds of the second derivative, many studies that will contribute to the literature have been made. There are many generalizations of this type about the well-known H-H$\mathbb{I}$ with the help of fractional integrals in the literature. For instance, the authors in [17] consider H-H$\mathbb{I}$ (1) as the form

$$\varrho\left(\tfrac{\imath_1+\imath_2}{2}\right) \leq \int_0^1 \varrho[(1-\eta)\imath_1+\eta\imath_2]\,\mathrm{d}\eta \leq \tfrac{\varrho(\imath_1)+\varrho(\imath_2)}{2}, \tag{2}$$

for convex function ϱ. In 2016, Chen obtained extensions of the H-H$\mathbb{I}$ for convex functions involving Riemann-Liouville fractional integrals [7]. Mihai *et al.* studied the H-H$\mathbb{I}$ (1) for the classes of convex, log-convex and log-concave functions [19]. Budak *et al.* gave bounds for the left and right hand sides of fractional Hermite–Hadamard [4]. In [8], the extensions of the Riemann-Liouville fractional H-H$\mathbb{I}$ are given for harmonic convex functions. Some T$\mathbb{I}$ and M$\mathbb{I}$ are presented for generalized fractional integrals [5]. Also You *et al.* [24] considered some new H-H$\mathbb{I}$ for harmonic convex functions via generalized fractional integrals. Mumcu *et al.* proved H-H$\mathbb{I}$ via generalized proportional fractional integrals [20]. In [2], the authors obtained new refinements of H-H$\mathbb{I}$ for strongly convex functions.

In this article, we will give refinements, extensions, and generalizations of H-H$\mathbb{I}$ (1) with the aid of the functions that have the conditions

$$\varrho'(\imath_1+\imath_2-\mathfrak{r})-\varrho'(\mathfrak{r}) \geq 0, \qquad \mathfrak{r}\in\left[\imath_1, \tfrac{\imath_1+\imath_2}{2}\right], \tag{3}$$

involving ψ-Hilfer fractional integrals. We will provide the necessary preliminary information about mentioned H-H$\mathbb{I}$ via ψ-Hilfer fractional integrals.

In Section 2, we present preliminary information and results such as definition of Riemann-Liouville fractional integrals, H-H$\mathbb{I}$ based on Riemann-Liouville fractional integrals, bounds of T$\mathbb{I}$ and M$\mathbb{I}$ for Riemann and Riemann-Liouville integrals, ψ-Hilfer integral definitions and H-H$\mathbb{I}$ involving ψ-Hilfer integral definitions. In the main results section 3, with help of bounds of second derivative functions, we will obtain improvements and generalizations of T$\mathbb{I}$ and M$\mathbb{I}$ using fractional integrals of a function with respect to another function, which is the main purpose of our article. In addition, it has been shown which inequalities are generalized among these inequalities obtained with the help of special choices in Section 3. We will also observe the generalized ψ-Hilfer fractional H-H$\mathbb{I}$ under different conditions with application. Finally, we will give some suggestions to the reader in the conclusion section 4.

2. Preliminaries

First, mathematical preliminaries of fractional calculus theory and bounded functions will be presented as follows. Let $\varrho \in L_1[\grave{\imath}_1, \grave{\imath}_2]$. The Riemann-Liouville integrals $\mathbb{J}^{\ell}_{\grave{\imath}_1+}\varrho$ and $\mathbb{J}^{\ell}_{\grave{\imath}_2-}\varrho$ of order $\ell > 0$ with $\grave{\imath}_1 \geq 0$ are introduced by

$$\mathbb{J}^{\ell}_{\grave{\imath}_1+}\varrho(\mathfrak{r}) = \int_{\grave{\imath}_1}^{\mathfrak{r}} \frac{(\mathfrak{r}-\eta)^{\ell-1}}{\Gamma(\ell)}\,\varrho(\eta)\,\mathrm{d}\eta, \qquad \mathfrak{r} > \grave{\imath}_1,$$

$$\mathbb{J}^{\ell}_{\grave{\imath}_2-}\varrho(\mathfrak{r}) = \int_{\mathfrak{r}}^{\grave{\imath}_2} \frac{(\eta-\mathfrak{r})^{\ell-1}}{\Gamma(\ell)}\,\varrho(\eta)\,\mathrm{d}\eta, \qquad \mathfrak{r} < \grave{\imath}_2,$$

respectively. Here $\mathbb{J}^{0}_{\grave{\imath}_1+}\varrho(\mathfrak{r}) = \mathbb{J}^{0}_{\grave{\imath}_2-}\varrho(\mathfrak{r}) = \varrho(\mathfrak{r})$ [22, 15]. Sarikaya *et al.* first give the following interesting integral inequalities of Hermite–Hadamard type involving Riemann-Liouville fractional integralsIn [23].

Theorem 2.1 ([23])**.** *Let $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be a positive mapping with $0 \leq \grave{\imath}_1 < \grave{\imath}_2$ and $\varrho \in L_1[\grave{\imath}_1, \grave{\imath}_2]$. If ϱ is a convex function on $[\grave{\imath}_1, \grave{\imath}_2]$, then the following inequalities involving Rieamann-Liouville fractional integrals hold:*

$$\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \leq \frac{\Gamma(\ell+1)}{2(\grave{\imath}_2-\grave{\imath}_1)^{\ell}}\left[\mathbb{J}^{\ell}_{\grave{\imath}_1+}\varrho(\grave{\imath}_2) + \mathbb{J}^{\ell}_{\grave{\imath}_2-}\varrho(\grave{\imath}_1)\right] \leq \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2}, \quad \ell > 0. \tag{4}$$

The inequalities of the M$\mathbb{I}$ in [11] and the boundaries of T$\mathbb{I}$ in [12] are obtained in Theorems 2.2 and 2.3.

Theorem 2.2 ([11])**.** *Let $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants $m_\circ$ and m° so that $m_\circ \leq \varrho'' \leq m^\circ$. Then, the following inequalities hold*

$$m_\circ\frac{(\grave{\imath}_2-\grave{\imath}_1)^2}{24} \leq \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \int_{\grave{\imath}_1}^{\grave{\imath}_2} \frac{\varrho(\eta)}{\grave{\imath}_2-\grave{\imath}_1}\,\mathrm{d}\eta \leq m^\circ\frac{(\grave{\imath}_2-\grave{\imath}_1)^2}{24}. \tag{5}$$

Theorem 2.3 ([10])**.** *Let the conditions of Theorem 2.2 be satisfied. Then the following inequalities yield,*

$$m_\circ\frac{(\grave{\imath}_2-\grave{\imath}_1)^2}{24} \leq \int_{\grave{\imath}_1}^{\grave{\imath}_2} \frac{\varrho(\eta)}{\grave{\imath}_2-\grave{\imath}_1}\,\mathrm{d}\eta - \varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \leq m^\circ\frac{(\grave{\imath}_2-\grave{\imath}_1)^2}{24}. \tag{6}$$

Chen presented extension and refinement of the T$\mathbb{I}$ and M$\mathbb{I}$ for convex mappings utilizing Riemann-Liouville fractional integrals [7].

Theorem 2.4 ([7]). *Let $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be a positive, twice differentiable mapping with $\grave{\imath}_1 < \grave{\imath}_2$ and $\varrho \in L_1 [\grave{\imath}_1, \grave{\imath}_2]$ if ϱ'' is bounded $[\grave{\imath}_1, \grave{\imath}_2]$ then we get*

$$
\begin{aligned}
(7)\qquad & \tfrac{m_\circ \ell}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2} - \eta\right)^2 \left[(\grave{\imath}_2 - \eta)^{\ell-1} + (\grave{\imath}_1 - \eta)^{\ell-1}\right] \mathrm{d}\eta \\
& \le \tfrac{\Gamma(\ell+1)}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \left[\mathbb{J}^{\varphi}_{\grave{\imath}_1+} \varrho(\grave{\imath}_2) + \mathbb{J}^{\varphi}_{\grave{\imath}_1-} \varrho(\grave{\imath}_1)\right] - \varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \\
& \le \tfrac{m^\circ \ell}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2} - \eta\right)^2 \left[(\grave{\imath}_2 - \eta)^{\ell-1} + (\grave{\imath}_1 - \eta)^{\ell-1}\right] \mathrm{d}\eta,
\end{aligned}
$$

for $\ell > 0$, where

$$
(8)\qquad m_\circ = \inf_{\mathfrak{r} \in [\grave{\imath}_1, \grave{\imath}_2]} \varrho''(\mathfrak{r}), \qquad m^\circ = \sup_{\mathfrak{r} \in [\grave{\imath}_1, \grave{\imath}_2]} \varrho''(\mathfrak{r}).
$$

Theorem 2.5 ([7]). *Assume the conditions of Theorem 2.3. Then the following inequality is obtained.*

$$
\begin{aligned}
(9)\qquad & \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \tfrac{-m^\circ \ell (\eta-\grave{\imath}_1)(\grave{\imath}_2-\eta)}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \left[(\grave{\imath}_2 - \eta)^{\ell-1} + (\grave{\imath}_1 - \eta)^{\ell-1}\right] \mathrm{d}\eta \\
& \le \tfrac{\Gamma(\ell+1)}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \left[\mathbb{J}^{\varphi}_{\grave{\imath}_1+} \varrho(\grave{\imath}_2) + \mathbb{J}^{\varphi}_{\grave{\imath}_2-} \varrho(\grave{\imath}_1)\right] - \tfrac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} \\
& \le \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \tfrac{-m_\circ \ell (\eta-\grave{\imath}_1)(\grave{\imath}_2-\eta)}{2(\grave{\imath}_2-\grave{\imath}_1)^\ell} \left[(\grave{\imath}_2 - \eta)^{\ell-1} + (\grave{\imath}_1 - \eta)^{\ell-1}\right] \mathrm{d}\eta,
\end{aligned}
$$

for $\ell > 0$, where $m_\circ$ and m° are defined in Equation (8).

The Definitions of the following ψ-Hilfer fractional integrals are given in [22, 15].

Definition 2.6 ([15, 22]). Let $\psi : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\grave{\imath}_1, \grave{\imath}_2]$, having a continuous derivative $\psi'(\mathfrak{r})$ on $(\grave{\imath}_1, \grave{\imath}_2)$. The left-sided and right-sided fractional integrals of ϱ with resperct to the function ψ on $[\grave{\imath}_1, \grave{\imath}_2]$ of order $\ell > 0$ are defined by

$$
\begin{aligned}
\mathbb{I}^{\ell}_{\grave{\imath}_1+;\psi} \varrho(\mathfrak{r}) &= \int_{\grave{\imath}_1}^{\mathfrak{r}} \tfrac{\psi'(\eta)}{\Gamma(\ell)(\psi(\mathfrak{r})-\psi(\eta))^{1-\ell}} \varrho(\eta)\, \mathrm{d}\eta, \qquad \mathfrak{r} > \grave{\imath}_1, \\
\mathbb{I}^{\ell}_{\grave{\imath}_2-;\psi} \varrho(\mathfrak{r}) &= \int_{\mathfrak{r}}^{\grave{\imath}_2} \tfrac{\psi'(\eta)}{\Gamma(\ell)(\psi(\eta)-\psi(\mathfrak{r}))^{1-\ell}} \varrho(\eta)\, \mathrm{d}\eta, \qquad \mathfrak{r} < \grave{\imath}_2
\end{aligned}
$$

respectively, provided that the integral exists.

Jlelli *et al.* gave the following H-H$\mathbb{I}$ for ψ-Hilfer fractional integrals in [14].

Theorem 2.7. *Let ψ is defined in Definition 2.7 with the same features with having a continuos derivative $\psi'(\mathfrak{r})$ on $(\grave{\imath}_1, \grave{\imath}_2)$. Let ϱ is a convex function on $[\grave{\imath}_1, \grave{\imath}_2]$ and $\ell > 0$, then the following inequalities hold*

$$\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \leq \frac{\Gamma(\ell+1)}{4[\psi(\grave{\imath}_2)-\psi(\grave{\imath}_1)]^\ell}\left[\mathbb{I}^\ell_{\grave{\imath}_1+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\grave{\imath}_2-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \leq \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2}, \tag{10}$$

where

$$\widehat{\varrho}(\mathfrak{r}) = \varrho(\mathfrak{r}) + \varrho(\grave{\imath}_1 + 2 - \mathfrak{r}). \tag{11}$$

Budak gave the following new version of H-H$\mathbb{I}$ with the aid of the ψ-Hilfer fractional integrals [6].

Theorem 2.8 ([6])**.** *Let the assumptions of Theorem 2.7 hold. Then we have*

$$\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \leq \frac{\Gamma(\ell+1)}{2\mathcal{F}^\ell_\psi}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \leq \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2}, \tag{12}$$

where $\widehat{\varrho}$ is defined with Equation (11) *and*

$$\mathcal{F}^\ell_\psi = \left[\psi\left(\grave{\imath}_2\right) - \psi\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)\right]^\ell + \left[\psi\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) - \psi\left(\grave{\imath}_1\right)\right]^\ell. \tag{13}$$

3. Main results

Let's assume the properties of the ψ function that we will use in this section as follows. Let $\psi : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\grave{\imath}_1, \grave{\imath}_2]$ and a continuous derivative $\psi'(\mathfrak{r})$ on $(\grave{\imath}_1, \grave{\imath}_2)$.

Theorem 3.1. *Let $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants $m_\circ$ and m° so that $m_\circ \leq \varrho'' \leq m^\circ$. Then, the following inequalities hold*

$$\begin{aligned} &\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{m_\circ \ell \mathcal{F}_\psi(\eta)}{2\mathcal{F}^\ell_\psi}\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2} - \eta\right)^2 \mathrm{d}\eta \\ &\leq \frac{\Gamma(\ell+1)}{2\mathcal{F}^\ell_\psi}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] - \varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \\ &\leq \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{m^\circ \ell \mathcal{F}_\psi(\eta)}{2\mathcal{F}^\ell_\psi}\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2} - \eta\right)^2 \mathrm{d}\eta, \end{aligned} \tag{14}$$

where $\mathcal{F}^\ell_\psi$ is defined by Equation (13) *and*

$$\mathcal{F}_\psi(\mathfrak{r}) = (\psi(\grave{\imath}_2) - \psi(\grave{\imath}_1 + \grave{\imath}_2 - \mathfrak{r}))^{\ell-1}\psi'(\grave{\imath}_1 + \grave{\imath}_2 - \mathfrak{r}) + (\psi(\mathfrak{r}) - \psi(\grave{\imath}_1))^{\ell-1}\psi'(\mathfrak{r}). \tag{15}$$

Proof. With the aid of the Definition 2.6, we get

$$
\begin{aligned}
&\tfrac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell(1)}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2)+\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right]\\
&\quad=\tfrac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell(1)}\left[\tfrac{1}{\Gamma(\ell)}\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_2}(\psi(\grave{\imath}_2)-\psi(\eta))^{\ell-1}\psi'(\eta)\widehat{\varrho}(\eta)\,\mathrm{d}\eta\right.\\
&\qquad\left.+\tfrac{1}{\Gamma(\ell)}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}(\psi(\eta)-\psi(\grave{\imath}_1))^{\eta-1}\psi'(\eta)\widehat{\varrho}(\eta)\,\mathrm{d}\eta\right]\\
&\quad=\tfrac{\ell}{2\mathcal{F}_\psi^\ell(1)}\left[\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}(\psi(\grave{\imath}_2)-\psi(\grave{\imath}_1+\grave{\imath}_1-\eta))^{\ell-1}\psi'(\grave{\imath}_1+\grave{\imath}_2-\eta)\,\widehat{\varrho}(\grave{\imath}_1+\grave{\imath}_2-\eta)\,\mathrm{d}\eta\right. \quad (16)\\
&\qquad\left.+\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}(\psi(\eta)-\psi(\grave{\imath}_1))^{\ell-1}\psi'(\eta)\,\widehat{\varrho}(\eta)\,\mathrm{d}\eta\right]\\
&\quad=\tfrac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\Big[(\psi(\grave{\imath}_2)-\psi(\grave{\imath}_1+\grave{\imath}_2-\eta))^{\ell-1}\psi'(\grave{\imath}_1+\grave{\imath}_2-\eta)\\
&\qquad+(\psi(\eta)-\psi(\grave{\imath}_1))^{\ell-1}\psi'(t)\Big](\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta))\,\mathrm{d}\eta\\
&\quad=\tfrac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\mathcal{F}_\psi(\eta)\left[\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta)\right]\mathrm{d}\eta.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\tfrac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2)+\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right]-\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \quad (17)\\
&\quad=\tfrac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\mathcal{F}_\psi(\eta)\left[\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta)-2\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)\right]\mathrm{d}\eta.
\end{aligned}
$$

By using the facts that

$$
\begin{aligned}
\varrho(\mathfrak{r})-\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)&=\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\mathfrak{r}}\varrho'(\eta)\,\mathrm{d}\eta,\\
\varrho(\grave{\imath}_1+\grave{\imath}_2-t)-\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)&=\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-t}\varrho'(\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\varrho(t)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r})-2\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \quad (18)\\
&\qquad=\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{t}\varrho'(\eta)\,\mathrm{d}\eta+\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}\varrho'(\eta)\,\mathrm{d}\eta
\end{aligned}
$$

$$= \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} \varrho'(\eta)\,\mathrm{d}\eta - \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} \varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)\,\mathrm{d}\eta$$
$$= \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} \left[\varrho'(s) - \varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)\right]\,\mathrm{d}\eta.$$

We also get

$$\varrho'(\mathfrak{r}) - \varrho'(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}) = \int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\mathfrak{r}} \varrho''(\eta)\,\mathrm{d}\eta. \tag{19}$$

By utilizing $m_\circ \le \varrho''(\mathfrak{r}) \le m^\circ\,(\forall\,\mathfrak{r}\in[\grave{\imath}_1,\grave{\imath}_2])$, with the aid of the equality (19), we obtain

$$\int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\mathfrak{r}} m_\circ\,\mathrm{d}\eta \le \int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\mathfrak{r}} \varrho''(\eta)\,\mathrm{d}\eta \le \int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\mathfrak{r}} m^\circ\,\mathrm{d}\eta,$$

which gives

$$m_\circ(2\mathfrak{r}-\grave{\imath}_1-\grave{\imath}_2) \le \varrho'(\mathfrak{r}) - \varrho'(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}) \le m^\circ(2\mathfrak{r}-\grave{\imath}_1-\grave{\imath}_2).$$

From equality (18), we define

$$m_\circ \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} (2\eta-\grave{\imath}_1-\grave{\imath}_2)\,\mathrm{d}\eta \le \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} \left[\varrho'(\eta) - \varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)\right]\,\mathrm{d}\eta$$
$$\le m^\circ \int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} (2\eta-\grave{\imath}_1-\grave{\imath}_2)\,\mathrm{d}\eta.$$

That is,

$$m_\circ\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}-\mathfrak{r}\right)^2 \le \varrho(\mathfrak{r}) + \varrho(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}) - 2\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \le m^\circ\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}-\mathfrak{r}\right)^2. \tag{20}$$

Multiplying the inequality (20) by $\frac{\ell\mathcal{F}\psi(\mathfrak{r})}{2\mathcal{F}_\psi^\ell}$ and then integrating with respect to $\mathfrak{r}$ on the interval $\left[\grave{\imath}_1, \frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right]$, we posses

$$\frac{m_\circ\ell}{2\mathcal{F}_\psi^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \mathcal{F}_\psi(\eta)\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}-\eta\right)^2\,\mathrm{d}\eta$$
$$\le \frac{\ell}{2\mathcal{F}_\psi^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \mathcal{F}_\psi(\eta)\left[\varrho(\eta) + \varrho(\grave{\imath}_1+\grave{\imath}_2-\eta) - 2\varrho\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)\right]\,\mathrm{d}\eta$$
$$\le \frac{m^\circ\ell}{2\mathcal{F}_\psi^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \mathcal{F}_\psi(\eta)\left(\tfrac{\grave{\imath}_1+\grave{\imath}_2}{2}-\eta\right)^2\,\mathrm{d}\eta.$$

With help of the equation (17), we obtain the desired result. □

Example 3.2. Define function $\varrho : [0.1, 1] \to \mathbb{R}$ by $\varrho(\mathfrak{r}) = \frac{1}{\mathfrak{r}}$ with $[\imath_1, \imath_2] = [0.1, 1]$. Without a doubt ϱ is a twice differentiable map and

$$m_\circ = 1 \leq \varrho''(\mathfrak{r}) = \frac{2}{\mathfrak{r}^3} \leq 10 = m^\circ.$$

Now, by using (13), (15) and consider $\psi(\mathfrak{r}) = \mathfrak{r}$, we have

$$\begin{aligned} \mathcal{F}_\psi^\ell &= \left[\psi(\imath_2) - \psi\left(\tfrac{\imath_1+\imath_2}{2}\right)\right]^\ell + \left[\psi\left(\tfrac{\imath_1+\imath_2}{2}\right) - \psi(\imath_1)\right]^\ell \\ &= \left[\psi(1) - \psi\left(\tfrac{1.1}{2}\right)\right]^\ell + \left[\psi\left(\tfrac{1.1}{2}\right) - \psi(0.1)\right]^\ell \simeq \begin{cases} 1.7742, & \ell = 0.15, \\ 1.3416, & \ell = 0.50, \\ 0.9366, & \ell = 0.95, \end{cases} \end{aligned}$$

$$\begin{aligned} \mathcal{F}_\psi(\mathfrak{r}) &= (\psi(\imath_2) - \psi(\imath_1 + \imath_2 - \mathfrak{r}))^{\ell-1}\psi'(\imath_1 + \imath_2 - \mathfrak{r}) + (\psi(\mathfrak{r}) - \psi(\imath_1))^{\ell-1}\psi'(\mathfrak{r}) \\ &= \left[(1 - \psi(1.1 - \mathfrak{r}))^{\ell-1}\right] + (\psi(\mathfrak{r}) - \psi(0.1))^{\ell-1} \simeq \begin{cases} 4.3579, & \ell = 0.15, \\ 3.1622, & \ell = 0.50, \\ 2.0937, & \ell = 0.95, \end{cases} \end{aligned}$$

for $\mathfrak{r} = 0.5$, and

Table 1. Numerical results of Inequalities (14) for $\ell \in \{0.15, 0.5, 0.95\}$.

$\mathfrak{r}$	Left side	Ineqs. (14)	Right side	Left side	Ineqs. (14)	Right side	Left side	Ineqs. (14)	Right side
		$\ell = 0.15$			**$\ell = 0.5$**			**$\ell = 0.95$**	
0.1000	0.2910	2.9095	1.5924	0.2770	2.7700	0.9757	0.2676	2.6760	0.6427
0.2000	0.2910	2.9095	1.5924	0.2770	2.7700	0.9757	0.2676	2.6760	0.6427
0.3000	0.2910	2.9095	1.5924	0.2770	2.7700	0.9757	0.2676	2.6760	0.6427
0.4000	0.2910	2.9095	1.5924	0.2770	2.7700	0.9757	0.2676	2.6760	0.6427
0.5000	0.2910	2.9095	1.5924	0.2770	2.7700	0.9757	0.2676	2.6760	0.6427

$$\begin{aligned} &\frac{m_\circ \ell}{2\mathcal{F}_\psi^\ell} \int_{\imath_1}^{\frac{\imath_1+\imath_2}{2}} \mathcal{F}_\psi(\eta)\left(\tfrac{\imath_1+\imath_2}{2} - \eta\right)^2 \mathrm{d}\eta \simeq \begin{cases} 0.2909, & \ell = 0.15, \\ 0.2770, & \ell = 0.50, \\ 0.2676, & \ell = 0.95, \end{cases} \\ &\quad \leq \begin{cases} 1.5923, & \ell = 0.15, \\ 0.9756, & \ell = 0.50, \\ 0.6426, & \ell = 0.95, \end{cases} \\ &\quad \simeq \frac{\ell}{2\mathcal{F}_\psi^\ell} \int_{\imath_1}^{\frac{\imath_1+\imath_2}{2}} \mathcal{F}_\psi(\eta)\left[\varrho(\eta) + \varrho(\imath_1 + \imath_2 - \eta) - 2\varrho\left(\tfrac{\imath_1+\imath_2}{2}\right)\right] \mathrm{d}\eta \\ &\quad \leq \begin{cases} 2.9095, & \ell = 0.15, \\ 2.7700, & \ell = 0.50, \\ 2.6760, & \ell = 0.95, \end{cases} \simeq \frac{m^\circ \ell}{2\mathcal{F}_\psi^\ell} \int_{\imath_1}^{\frac{\imath_1+\imath_2}{2}} \mathcal{F}_\psi(\eta)\left(\tfrac{\imath_1+\imath_2}{2} - \eta\right)^2 \mathrm{d}\eta. \end{aligned}$$

Table 1 shows these results. Figures 1 and 2a, 2b, 2c show graphical representation of the variables. Thus, Inequalities (14) in Thorem 3.1 hold.

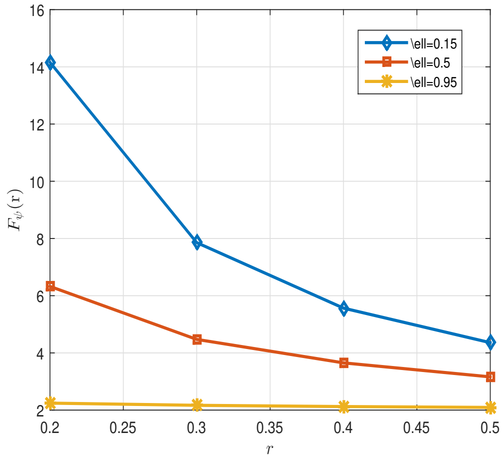
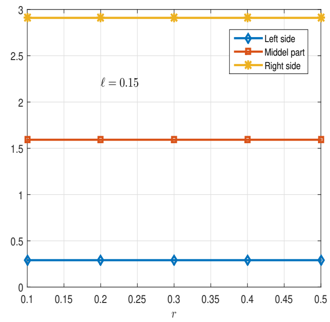
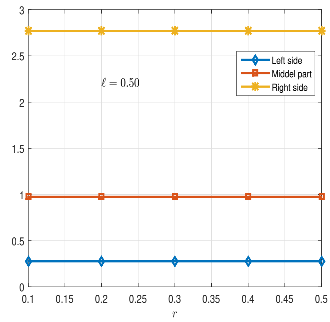


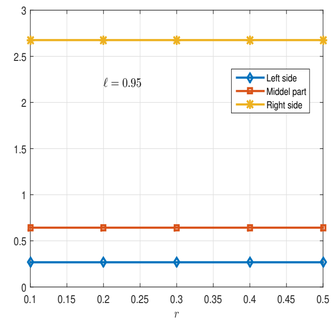
Figure 1. 2D-graph of $\mathcal{F}_\psi$ for $\mathfrak{r} \in [0.1, 1]$ in Example 3.2.



(A) $\ell = 0.15$



(B) $\ell = 0.50$



(C) $\ell = 0.95$

Figure 2. Graphical representation of inequalities (14) for $\ell \in \{0.15, 0.50, 0.95\}$.

Corollary 3.3. *If we choose $\psi(\mathfrak{r}) = \mathfrak{r}$ in Theorem 3.1, then inequalities* (14) *is reduced to inequality* (7).

Theorem 3.4. *Let $\varrho : [\grave{\imath}_1, \grave{\imath}_2] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants $m_\circ$ and m° so that $m_\circ \le \varrho'' \le m^\circ$. Then, we establish*

$$\begin{aligned}(21)\qquad & \frac{m_\circ \ell}{2\mathcal{F}_\psi^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \mathcal{F}_\psi(\eta)(\grave{\imath}_2-\eta)(\eta-\grave{\imath}_1)\,\mathrm{d}\eta \\ & \le \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \frac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \\ & \le \frac{m^\circ \ell}{2\mathcal{F}_\psi^\ell} \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \mathcal{F}_\psi(\eta)(\grave{\imath}_2-\eta)(\eta-\grave{\imath}_1)\,\mathrm{d}\eta\end{aligned}$$

where $\mathcal{F}_\psi$ is stated by as in (15).

Proof. By the equation (16), we have

$$\begin{aligned}(22)\qquad & \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \frac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+i\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \\ & = \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{\ell\mathcal{F}_\psi(\eta)}{2\mathcal{F}_\psi^\ell}\left[\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta)\right]\mathrm{d}\eta \\ & = \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{\ell\mathcal{F}_\psi(\eta)}{2\mathcal{F}_\psi^\ell}\left[\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)-(\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta))\right]\mathrm{d}\eta.\end{aligned}$$

With help of the equalities

$$\varrho(\grave{\imath}_1)-\varrho(t) = -\int_{\grave{\imath}_1}^{\mathfrak{r}} \varrho'(\eta)\,\mathrm{d}\eta, \qquad \varrho(\grave{\imath}_2)-\varrho(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}) = \int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\grave{\imath}_2} \varrho'(\eta)\,\mathrm{d}\eta,$$

we have

(23)

$$\begin{aligned}\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)-(\varrho(\mathfrak{r})+\varrho(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r})) &= \int_{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}}^{\grave{\imath}_2} \varrho'(\eta)\,\mathrm{d}\eta - \int_{\grave{\imath}_1}^{\mathfrak{r}} \varrho'(\eta)\,\mathrm{d}\eta \\ = \int_{\grave{\imath}_1}^{\mathfrak{r}} \varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)\,\mathrm{d}\eta - \int_{\grave{\imath}_1}^{\mathfrak{r}} \varrho'(\eta)\,\mathrm{d}\eta &= \int_{\grave{\imath}_1}^{\mathfrak{r}} \left[\varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)-\varrho'(\eta)\right]\mathrm{d}\eta.\end{aligned}$$

We also obtain

$$(24)\qquad \varrho'(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r})-\varrho'(\mathfrak{r}) = \int_{\mathfrak{r}}^{\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}} \varrho''(\eta)\,\mathrm{d}\eta.$$

By (24) and the condition that $m_\circ \le \varrho'' \le m^\circ$, we establish

$$(25)\qquad m_\circ(\grave{\imath}_1+\grave{\imath}_2-2\mathfrak{r}) \le \varrho'(\mathfrak{r})-\varrho'(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r}) \le m^\circ(\grave{\imath}_1+\grave{\imath}_2-2\mathfrak{r}).$$

From equality (23) and the inequality (25), we derive

$$\int_{\grave{\imath}_1}^{\mathfrak{r}} m_\circ(\grave{\imath}_1+\grave{\imath}_2-2\eta)\,\mathrm{d}\eta \le \int_{\grave{\imath}_1}^{\mathfrak{r}} \left[\varrho'(\eta)-\varrho'(\grave{\imath}_1+\grave{\imath}_2-\eta)\right]\,\mathrm{d}\eta \le \int_{\grave{\imath}_1}^{\mathfrak{r}} m^\circ(\grave{\imath}_1+\grave{\imath}_2-2\eta)\,\mathrm{d}\eta,$$

i.e.

$$m_\circ(\grave{\imath}_2-\mathfrak{r})(\mathfrak{r}-\grave{\imath}_1) \le \varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)-(\varrho(\mathfrak{r})+\varrho(\grave{\imath}_1+\grave{\imath}_2-\mathfrak{r})) \le m^\circ(\grave{\imath}_2-\mathfrak{r})(\mathfrak{r}-\grave{\imath}_1).$$

Multiplying the inequality (26) by $\frac{\ell\mathcal{F}_\psi(\mathfrak{r}}{2\mathcal{F}_\psi^\ell}$ and then integrating with respect to $\mathfrak{r}$ on the interval $\left[\grave{\imath}_1, \frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right]$, we derive

(26)

$$\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{m_\circ\ell(\grave{\imath}_2-\eta)(\eta-\grave{\imath}_1)}{2\mathcal{F}_\psi^\ell}\mathcal{F}_\psi(\eta)\,\mathrm{d}\eta \le \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \frac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\grave{\imath}_1+;\psi}\widehat{\varrho}(\grave{\imath}_2)+\mathbb{I}^\ell_{\grave{\imath}_2-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right]$$
$$\le \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{m^\circ\ell(\grave{\imath}_2-\eta)(\eta-\grave{\imath}_1)}{2\mathcal{F}_\psi^\ell}\mathcal{F}_\psi(\eta)\,\mathrm{d}\eta.$$

This completes the proof. □

Corollary 3.5. *If we take $\psi(\mathfrak{r})=\mathfrak{r}$ in Theorem 3.4, then the inequality* (21) *reduce to the inequality* (9).

In the next theorem, we will examine the proof of Theorem 2.7 under different conditions.

Theorem 3.6. *Let $\varrho:[\grave{\imath}_1,\grave{\imath}_2]\to\mathbb{R}$ be a positive and differentiable function and $\varrho\in L_1[\grave{\imath}_1,\grave{\imath}_2]$. If condition* (3) *holds, then the following inequalities via fractional integrals hold*

(27)
$$\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right) \le \frac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2)+\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \le \frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2}.$$

Proof. Thank to of the equalities (17) and (18), we posses

$$\frac{\Gamma(\ell+1)}{4[\psi(\grave{\imath}_2)-\psi(\grave{\imath}_1)]^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2)+\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right]-\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)$$
$$=\frac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\mathcal{F}_\psi(\eta)\left[\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta)-2\varrho\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)\right]\mathrm{d}\eta$$
$$=\frac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\mathcal{F}_\psi(\eta)\left[\int_{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}^{\grave{\imath}_1+\grave{\imath}_2-\eta}\left[\varrho'(\nu)-\varrho'(\grave{\imath}_1+\grave{\imath}_2-\nu)\right]\mathrm{d}\nu\right]\mathrm{d}\eta$$
$$=\frac{\ell}{2\mathcal{F}_\psi^\ell}\int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\mathcal{F}_\psi(\eta)\left[\int_{\eta}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}}\left[\varrho'(\grave{\imath}_1+\grave{\imath}_2-\nu)-\varrho'(\nu)\right]\mathrm{d}\nu\right]\mathrm{d}\eta\ge 0,$$

which proves the first inequality in (27). Similarly, by the equalities (22) and (23), we establish

$$\begin{aligned}
&\frac{\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)}{2} - \frac{\Gamma(\ell+1)}{2\mathcal{F}_\psi^\ell}\left[\mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)+;\psi}\widehat{\varrho}(\grave{\imath}_2) + \mathbb{I}^\ell_{\left(\frac{\grave{\imath}_1+\grave{\imath}_2}{2}\right)-;\psi}\widehat{\varrho}(\grave{\imath}_1)\right] \\
&\qquad = \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{\ell\mathcal{F}_\psi(\eta)}{2\mathcal{F}_\psi^\ell}\left[\varrho(\grave{\imath}_1)+\varrho(\grave{\imath}_2)-(\varrho(\eta)+\varrho(\grave{\imath}_1+\grave{\imath}_2-\eta))\right] \mathrm{d}\eta \\
&\qquad = \int_{\grave{\imath}_1}^{\frac{\grave{\imath}_1+\grave{\imath}_2}{2}} \frac{\ell\mathcal{F}_\psi(\eta)}{2\mathcal{F}_\psi^\ell}\left[\int_{\grave{\imath}_1}^{\eta}\left[\varrho'(\grave{\imath}_1+\grave{\imath}_2-\nu)-\varrho'(\nu)\right]\mathrm{d}\nu\right]\mathrm{d}\eta \geq 0
\end{aligned}$$

The proof of the Theorem 3.6 completed. □

4. Conclusion

In the present paper, we presented new H-H$\mathbb{I}$ based on ψ-Hilfer fractional integral operators with the aid of the mappings whose twice differentiable are bounded. Moreover, the ψ-Hilfer fractional H-H$\mathbb{I}$ was obtained under different conditions. As can be seen from our article, curious readers can use the conditional $m_\circ \leq \varrho''(\mathfrak{r}) \leq m^\circ$ for all $\mathfrak{r} \in [\grave{\imath}_1, \grave{\imath}_2]$ instead of convexity, and they can try to find better bounds using the condition (3). What's more, mathematicians will be able to study different types of fractional mappings. In addition, researchers will be able to derive new inequalities using different types of convexities.

References

1. R.P. Agarwal: Certain fractional q-integrals and q-derivatives. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 66, No. 2, pp. 365-370). Cambridge University Press, 1969. `doi:10.1017/S0305004100045060`
2. M.U. Awan, M.A. Noor, T.S. Du & K.I. Noor: New refinements of fractional Hermite-Hadamard inequality. RACSAM, **113** (2019), no. 1, 21-29. `doi:10.1007/s13398-017-0448-x`
3. A. Berhail, N. Tabouche, J. Alzabut & M.E. Samei: Using Hilfer-Katugampola fractional derivative in initial value Mathieu fractional differential equations with application on particle in the plane. Advances in Continuous and Discrete Models: Theory and Applications 2022, 44 (2022). `doi:10.1186/s13662-022-03716-6`
4. H. Budak, H. Kara, M.Z. Sarikaya & M.E. Kiriş: New extensions of the Hermite–Hadamard inequalities involving Riemann-Liouville fractional integrals. Miskolc Mathematical Notes **21** (2020), no. 2, 665-678. `doi:10.1007/s13398-017-0448-x`

5. H. Budak, E. Pehlivan & P. Kosem: On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals. Sahand Communications in Mathematical Analysis **18** (2021), no. 1, 73-88. doi:10.22130/scma.2020.121963.759
6. H. Budak: New Hermite-Hadamard type inequalities for convex mappings utilizing generalized fractional integrals. Filomat **33** (2019), no. 8, 2329-2344. doi:10.2298/FIL1908329
7. F.X. Chen: Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals. J. Math. Inequal. **10** (2016), no. 1, 75-81. doi:10.7153/jmi-10-07
8. F.X. Chen: On the generalization of some Hermite-Hadamard Inequalities for functions with convex absolute values of the second derivatives via fractional integrals. Ukrainian Mathematical Journal **12** (2019), no. 70, 1953-1965. doi:10.1007/s11253-019-01618-7
9. A. Devi, A. Kumar, T.A. & A. Khan: Existence and stability analysis of solutions for fractional Langevin equation with nonlocal integral and anti-periodic type boundary conditions. Fractals **28** (2020), no. 8, 2040006. doi:10.1142/S0218348X2040006X
10. S.S. Dragomir & C.E.M. Pearce: Selected topics on Hermite-Hadamard inequalities and applications. RGMIA Monographs, Victoria University, 2000. Online: http://www.sta.vu.edu.au/RGMIA/monographs/hermite_hadamard.html.
11. S.S. Dragomir, P. Cerone & A. Sofo: Some remarks on the midpoint rule in numerical integration. Studia Univ. Babes-Bolyai, Math. **XLV** (2000), no. 1, 63-74.
12. S.S. Dragomir, P. Cerone & A. Sofo: Some remarks on the trapezoid rule in numerical integration. Indian J. Pure Appl. Math. **31** (2000), no. 5, 475-494.
13. S. Etemad, S. Rezapour & M.E. Samei: On a fractional caputo-hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property. Mathematical Methods in the Applied Sciences **37** (2020), 9719-9734. doi:10.1002/mma.6644
14. M. Jleli & B. Samet: On Hermite–Hadamard type inequalities via fractional integrals of a function with respect to another function. Journal of Nonlinear Sciences and Applications. **9** (2016), no. 3, 1252-1260. doi:10.22436/jnsa.009.03.50
15. A.A. Kilbas, H.M. Srivastava & J.J. Trujillo: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
16. A. Lachouri, M.S. Abdo, A. Ardjouni, B. Abdalla & T. Abdeljawad: On a class of differential inclusions in the frame of generalized hilfer fractional derivative. AIMS Mathematics **7** (2022), no. 3, 3477-3493. doi:10.3934/math.2022193
17. B. Li: Refinements of Hermite-Hadamard type inequalities for operator convex functions. Int. J. Contemp. Math. Sci. **8** (2013), no. 9-12, 463-467.

18. R. Mert, L. Erbe & T. Abdeljawad: A variational approach of the Strum-Liouville problem in fractional difference calculus. Dynamic Systems and Applications **27** (2018), no. 1, 137-148. doi:10.12732/dsa.v27i1.7
19. M.V. Mihai, M.U. Awan, M.A. Noor, J.K. Kim & K.I. Noor: Hermite-Hadamard inequalities and their applications. Journal of inequalities and applications **2018** (2018), no. 1, 1-9. doi:10.1186/s13660-018-1895-4
20. İ. Mumcu, E. Set, A.O. Akdemir & F. Jarad: New extensions of Hermite-Hadamard inequalities via generalized proportional fractional integral. Numerical Methods for Partial Differential Equations **2021** (2021), 1-12. doi:10.1002/num.22767
21. M.E. Samei, V. Hedayati & S. Rezapour: Existence results for a fraction hybrid differential inclusion with Caputo- Hadamard type fractional derivative. Advances in Difference Equations **2019** (2019), 163. doi:10.1186/s13662-019-2090- 8
22. S.G. Samko, A.A. Kilbas & O.I. Marichev: Fractional integrals and derivatives. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, 1993, (Vol. 1), Yverdon.
23. M.Z. Sarikaya, E. Set, H. Yaldiz & N. Basak: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Mathematical and Computer Modelling. **57** (2013), 2403-2407. DOI:10.1016/j.mcm.2011.12.048
24. X.X. You, M.A. Ali, H. Budak, P. Agarwal & Y.M. Chu: Extensions of Hermite–Hadamard inequalities for harmonically convex functions via generalized fractional integrals. Journal of Inequalities and Applications **2021** (2021), no. 1, 1-22. doi:10.1186/s13660-021-02638-3

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