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BEST PROXIMITY POINT THEOREMS FOR ψ-ϕ-CONTRACTIONS IN METRIC SPACES

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Abstract. In this paper, some best proximity points results for ψ-ϕ-contractions on complete metric spaces are proved. These results extend and generalize some best proximity and fixed point results on complete metric spaces. An example and some corollaries are provided that demonstrate the results proved herein.

1. Introduction and Preliminaries

In mathematics, most of the issues can be resolved by offering a solution to the equation $fx = x$, where f is a self-mapping of a non-empty set X, and such a point $x \in X$ is called the fixed point of the mapping f. Fixed point of a mapping plays an important role in the existence problems where a solution of the given problem in form of a fixed point of a suitable mapping is desired. Many authors work in this domain and generalized the existing results (see e.g., [8–10, 15, 19–22]).

An interesting case arises when the mapping f is defined from a nonempty subset P of set X into another subset Q of X, i.e., the mapping is a non-self-mapping. In such a case, the existence of a fixed point cannot be assured, and we seek an appropriate solution x that is optimal in the sense that the distance between x and fx is minimal in some sense.

In 1969, Fan [12] gave his famous theorem on the best approximation which is stated as follows: "If K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space E and $N\colon K \to E$ is a continuous mapping, then there exists an element μ in K such that $d(\mu, N\mu) = d(N\mu, K)$." There are several extensions and variants of the result of Fan [12], see e.g. [11, 18, 23, 24] etc. A unification of such approaches is given by Vetrivel et al. [25]. These results provide a point representing the best approximate solution, not the optimal solution.

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If (X, d) is a metric space, $P, Q \subseteq X$ and $f\colon P \to Q$ is a non-self-mapping. Then, because $d(x, fx) \geq d(P, Q)$, one can assume that an optimal solution is a point $\mu \in P$ such that $d(\mu, f\mu) = d(P, Q)$. Such a point μ is called the best proximity point.

The best proximity point theory is beneficial for discovering and using the concept as a strong tool to address many integral equations, differential equations, boundary value problems, simultaneous equations, and other associated areas. This theory is a stunning blend of, topology, analysis, and geometry. It is also a very powerful and important technique for solving a variety of applied problems in mathematical sciences. Eldred and Veeramani [2] proved the best proximity theorem which ensures the existence of the best proximity points of a cyclic mapping. Let P and Q be two non-empty closed subsets of a complete metric space (X, d). A mapping $N\colon P \cup Q \to P \cup Q$ is called cyclic if $N(P) \subseteq Q$ and $N(Q) \subseteq P$. For some $k \in (0, 1)$ the cyclic mapping N satisfying the condition

$$d(Nx, Ny) \leq kd(x, y) \text{ for all } x \in P, y \in Q \tag{1}$$

then $P \cap Q \neq \emptyset$ and so N has a fixed point in the intersection of P and Q (see [26]). This result is more general than the Banach contraction principle [16] because N is not necessarily continuous.

S. Basha [17] gave necessary and sufficient conditions for the existence of proximity points for proximal contraction of the first and second kind. Therefore, the best proximity point theory seeks the attention of several authors such as [3–7, 29]. Sanhan et al. [27] generalized the notion of proximal contractions of the first and the second kinds and established the best proximity point theorems for these classes. Their results improve and extend the recent results of S. Basha [17] and some authors.

If P and Q are two non-empty subsets of a metric space (X, d), then we define the following:

$$\begin{aligned} d(P, Q) &= \inf\{d(p, q)\colon p \in P, q \in Q\}; \\ P_0 &= \{p \in P\colon\ d(p, q) = d(P, Q) \text{ for some } q \in Q\}; \\ Q_0 &= \{q \in Q\colon\ d(p, q) = d(P, Q) \text{ for some } p \in P\}. \end{aligned}$$

Throughout the paper, P and Q will represent two non-empty disjoint subsets of a complete metric space (X, d), where P_0 and Q_0 contained in the boundaries of P

and Q respectively, provided P and Q are closed subsets of complete metric space such that $d(P,Q) > 0$.

Definition 1 (Basha [17])**.** Let P and Q be two non-empty subsets of metric space (X,d). A non-self mapping $N\colon P \to Q$ is said to be a *proximal contraction of the first kind* if there exists $\alpha \in [0,1)$ such that

$$d(u,Np) = d(v,Nq) \ = d(P,Q) \implies d(u,v) \leq \alpha d(Np,Nq)$$

for all $u,v,p,q \in P$.

Definition 2 (Basha [17])**.** Let P and Q be two non-empty subsets of metric space (X,d). A non-self mapping $N\colon P \to Q$ is said to be a *proximal contraction of the second kind* if there exists $\alpha \in [0,1)$ such that

$$d(u,Np) = d(v,Nq) \ = d(P,Q) \implies d(Nu,Nv) \leq \alpha d(Np,Nq)$$

for all $u,v,p,q \in P$.

Definition 3 (Basha [17])**.** A pair of mappings (N,H), where $N\colon P \to Q$ and $H\colon Q \to P$, is said to be a *proximal cyclic contraction pair* if there exists a non-negative number $\alpha < 1$ such that for all $u,p \in P$ and $v,q \in Q$

$$d(u,Np) = d(v,Hq) \ = d(P,Q) \implies d(u,v) \leq \alpha d(p,q) + (1-\alpha)d(P,Q).$$

Definition 4 (Basha [17])**.** Let $N\colon P \to Q$ and $g\colon P \cup Q \to P \cup Q$ is an isometry. Then we say that the mapping N preserves the *isometric distance* with respect to g if: for all $p \in P$ and $q \in Q$

$$d(Ngp,Ngq) \quad = \quad d(Np,Nq).$$

Definition 5 (Sanhan et al. [27])**.** Let P and Q be two non-empty subsets of metric space (X,d). A non-self mapping $N\colon P \to Q$ is said to be a *generalized proximal ψ-contraction of the first kind* if for all $p,q,u,v \in P$ the following condition is satisfied:

$$d(u,Np) = d(v,Nq) = d(P,Q) \implies d(u,v) \leq \psi d(Np,Nq)$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is an upper semi continuous from the right such that $\psi(t) < t$ for all $t > 0$.

Definition 6 (Sanhan et al. [27])**.** Let P and Q be two non-empty subsets of metric space (X,d). A non-self mapping $N\colon P \to Q$ is said to be a *generalized proximal*

ψ-contraction of the second kind if for all $p, q, u, v \in P$ the following condition is satisfied:

$$d(u, Np) = d(v, Nq) = d(P, Q) \implies d(Nu, Nv) \leq \psi d(Np, Nq)$$

where $\psi\colon [0, \infty) \to [0, \infty)$ is an upper semi continuous from the right such that $\psi(t) < t$ for all $t > 0$.

Theorem 1 (Sanhan et al. [27]). *Let (X, d) be a complete metric space and let P and Q be two non-empty closed subsets of X such that P_0 and Q_0 are non-empty. Let $N\colon P \to Q$, $H\colon Q \to P$, and $g\colon P \cup Q \to P \cup Q$ be a generalized cyclic contraction such that the following conditions are satisfied:*

(a) N and H are generalized proximal ψ-contraction of first kind;
(b) g is an isometry;
(c) the pair (N, H) is proximal cyclic contraction;
(d) $N(P_0) \subseteq Q_0$ and $H(Q_0) \subseteq P_0$;
(e) $P_0 \subseteq g(P_0)$ and $Q_0 \subseteq g(Q_0)$.

Then there exist two unique points $p \in P$ and $q \in Q$ such that

$$d(gp, Np) = d(gq, Hq) = d(p, q) = d(P, Q).$$

Moreover, for any fixed $p_0 \in P_0$, the sequence $\{p_n\}$ defined by $d(gp_{n+1}, Np_n) = d(P, Q)$ converges to the element p. For any fixed $q_0 \in Q_0$, the sequence $\{q_n\}$ defined by $d(gq_{n+1}, Hq_n) = d(P, Q)$ converges to the element q. On the other hand, a sequence $\{u_n\}$ in P converges to p, if there is a sequence of positive number $\{\epsilon_n\}$ such that $\lim_{n\to\infty} \{\epsilon_n\} = 0$, $d(u_{n+1}, \xi_{n+1}) \leq \epsilon$, where $\xi_{n+1} \in P$, satisfies the condition that $d(g\xi_{n+1}, Nu_n) = d(P, Q)$.

Theorem 2 (Sanhan et al. [27]). *Let (X, d) be a metric space and let P and Q be two non-empty closed subsets of X such that P_0 and Q_0 are non-empty. Let $N\colon P \to Q$ and $g\colon P \to P$ satisfies the following conditions:*

(a) N is generalised proximal ψ-contraction of first and second kind;
(b) g is an isometry;
(c) N preserves an isometric distance with respect to g;
(d) $N(P_0) \subseteq Q_0$;
(e) $P_0 \subseteq g(P_0)$.

Then there exists a unique point $p \in P$ such that $d(gp, Np) = d(P, Q)$. Moreover, for any fixed $p_0 \in P_0$, the sequence $\{p_n\}$ defined by $d(gp_{n+1}, Np_n) = d(P, Q)$ converges to the element p. For any fixed $q_0 \in Q_0$, the sequence $\{q_n\}$ defined by $d(gq_{n+1}, Hq_n) = d(P, Q)$ converges to the element q. On the other hand, a sequence $\{u_n\}$ in P converges to p, if there is a sequence of positive number $\{\epsilon_n\}$ such that $\lim_{n\to\infty}\{\epsilon_n\} = 0$, $d(u_{n+1}, \xi_{n+1}) \le \epsilon$, where $z_{n+1} \in P$, satisfies the condition that $d(z_{n+1}, Nu_n) = d(P, Q)$.

This paper aims to introduce a new class of generalized proximal ψ-ϕ-contraction of first and second kind, which are more general than the class of proximal ψ-contraction of first and second kind. We present the necessary conditions to have the best proximity points of these classes of mappings and an illustrative example of our main results are also provided. The results of this paper are extensions and generalizations of the main results of Sanhan et al. [27] and some other results in the literature.

In the next section, we establish our main results.

2. Main Result

Let us consider the following two classes of functions introduced in [14]:

$$\Psi = \{\psi\colon [0,\infty) \to [0,\infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\},$$
$$\Phi = \{\phi\colon [0,\infty) \to [0,\infty) \text{ such that } \phi \text{ is lower semicontonuous}\},$$

where $\psi(t) = \phi(t) = 0$ if and only if $t = 0$.

Definition 7. Let P and Q be two non-empty subsets of metric space (X, d). A non-self mapping $N\colon P \to Q$ is said to be a *generalized proximal ψ-ϕ-contraction of the first kind* if for all $p, q, u, v \in P$ the following condition is satisfied:

$$d(u, Np) = d(v, Nq) = d(P, Q) \implies \psi(d(u, v)) \le \psi(d(p, q)) - \phi(d(p, q)).$$

Definition 8. Let P and Q be two non-empty subsets of metric space (X, d). A non-self mapping $N\colon P \to Q$ is said to be a *generalized proximal ψ-ϕ-contraction of the second kind* if for all $p, q, u, v \in P$ the following condition is satisfied:

$$d(u, Np) = d(v, Nq) = d(P, Q) \implies \psi(d(Nu, Nv)) \le \psi(d(Np, Nq)) - \phi(d(Np, Nq)).$$

Now we extend the result of Sanhan [27] and Banach's contraction principle for non-self mappings satisfying the generalized proximal ψ-ϕ-contractive condition of first and second kinds.

Theorem 3. *Let (X,d) be a metric space and let P and Q be two non-empty closed subsets of X such that P_0 and Q_0 are non-empty. Let $N\colon P \to Q$ and $H\colon Q \to P$ and $g\colon\ P\cup Q \to P\cup Q$ satisfy the following conditions:*

(a) N and H are to generalized proximal ψ-ϕ-contractions of first kind;

(b) g is an isometry;

(c) the pair (N,H) is proximal cyclic contraction;

(d) $N(P_0)\subseteq Q_0$ and $H(Q_0)\subseteq P_0$;

(e) $P_0\subseteq g(P_0)$ and $Q_0\subseteq g(Q_0)$.

Then there exists a unique point $p\in P$ and a unique point $q\in Q$ such that

$$d(gp,Np)=d(gq,Hq)\ =d(p,q)\ =d(P,Q)$$

and for any fixed $p_0\in P_0, q_0\in Q_0$, the sequences $\{p_n\},\{q_n\}$ defined by

$$d(gp_{n+1},Np_n)=d(gq_{n+1},Hq_n)=d(P,Q)$$

converge to the points p and q respectively. Furthermore, a sequence $\{u_n\}$ in P converges to p if there is a sequence $\{\xi_n\}$ in P with $g\xi_n\in P$ for all n such that the sequence of nonnegative numbers $\{d(u_n,p_n)\}$ is convergent and

$$\lim_{n\to\infty} d(\xi_n,u_n)=0 \text{ and } d(g\xi_{n+1},Nu_n)=d(P,Q).$$

Proof. Let p_0 be a fixed element in P_0. In view of the fact that $N(P_0)\subseteq Q_0$ we have $Np_0\in Q_0$, hence by definition of Q_0 there exists $p\in P$ such that $d(p,Np_0)=d(P,Q)$. Hence, $p\in P_0$. Again, since $P_0\subseteq g(P_0)$, it is ascertained that there exists an element $p_1\in P_0$ such that $p=gp_1\in P_0$, hence

$$d(gp_1,Np_0) \;=\; d(P,Q).$$

Again, since $N(P_0)\subseteq Q_0$ and $P_0\subseteq g(P_0)$, following a similar reasoning, there exists an element $p_2\in P_0$ such that

$$d(gp_2,Np_1) \;=\; d(P,Q).$$

Similarly, we can find $p_n\in P_0$ such that $gp_n\in P_0$. Having chosen p_n one can determine an element $p_{n+1}\in P_0$, such that

$$d(gp_{n+1},Np_n) \;=\; d(P,Q). \tag{2}$$

Because, g is an isometry, $N(P_0) \subseteq Q_0$ and $P_0 \subseteq g(P_0)$, and

$$d(gp_{n+1}, Np_n) = d(gp_n, Np_{n-1}) = d(P, Q)$$

hence by the definition of generalized proximal ψ-ϕ-contractions of first kind, for each $n \in \mathbb{N}$ we have

$$\psi(d(p_{n+1}, p_n)) = \psi(d(gp_{n+1}, gp_n)) \leq \psi(d(p_n, p_{n-1})) - \phi(d(p_n, p_{n-1})) \tag{3}$$

which implies that $\psi(d(p_{n+1}, p_n)) \leq \psi(d(p_n, p_{n-1})$, and ψ is non-decreasing, hence

$$d(p_{n+1}, p_n) \leq d(p_n, p_{n-1}).$$

So the sequence$\{d(p_{n+1}, p_n)\}$ is non-increasing sequence in $\mathbb{R}^+$, and thus it is convergent to $t \in \mathbb{R}^+$. We claim that $t = 0$. Suppose, on the contrary, that $t > 0$. Taking limit as $n \to \infty$ in equation (3) we get

$$\psi(t) \leq \psi(t) - \phi(t) \tag{4}$$

which implies that $\phi(t) = 0$, that is, $t = 0$. This is a contradiction. Hence $t = 0$, that is

$$\lim_{n\to\infty} d(p_{n+1}, p_n) = 0. \tag{5}$$

We claim that $\{p_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{p_n\}$ is not a cauchy sequence. Then, there exist $\epsilon > 0$ and two subsequence $\{p_{m_k}\}$, $\{p_{n_k}\}$ of $\{p_n\}$ such that $n_k > m_k \geq k$ with

$$d(p_{m_k}, p_{n_k}) \geq \epsilon \text{ and } d(p_{n_k}, p_{m_k-1}) < \epsilon.$$

for $k = 1, 2, 3....$ Then

$$\epsilon \leq d(p_{m_k}, p_{n_k}) \leq d(p_{n_k}, p_{m_k-1}) + d(p_{m_k-1}, p_{m_k}) < \epsilon + d(p_{m_k-1}, dp_{m_k}).$$

It follows from equation (5)

$$\lim_{k\to\infty} d(p_{m_k}, p_{n_k}) = \epsilon. \tag{6}$$

Also, by the following two inequalities

$$\begin{aligned} d(p_{m_k}, p_{n_k}) &\leq d(p_{m_k}, p_{m_k+1}) + d(p_{m_k+1}, p_{n_k+1}) + d(p_{n_k+1}, p_{n_k}); \\ d(p_{m_k+1}, p_{n_k+1}) &\leq d(p_{m_k+1}, p_{m_k}) + d(p_{m_k}, p_{n_k}) + d(p_{n_k}, p_{n_k+1}) \end{aligned}$$

we must have

$$\lim_{k\to\infty} d(p_{m_k+1}, p_{n_k+1}) = \epsilon. \tag{7}$$

On the other hand, by the construction of sequence $\{p_n\}$ we have

$$d(gp_{m_k+1}, Np_{m_k}) = d(gp_{n_k+1}, Np_{n_k}) = d(P, Q).$$

Since N is a generalized proximal ψ-ϕ-contraction of first kind and g as an isometry, we have

$$\psi(d(p_{m_k+1}, p_{n_k+1})) = \psi(d(gp_{m_k+1}, gp_{n_k+1})) \leq \psi(d(p_{m_k}, p_{n_k})) - \phi(d(p_{m_k}, p_{n_k})).$$

If $k \to \infty$, then from equation (6) and (7) and the properties of ψ and ϕ we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

This implies that $\phi(\epsilon) = 0$ that is $\epsilon = 0$. This is a contradiction, therefore $\{p_n\}$ is a Cauchy sequence and converges to an element $p \in P$. Similarly, $H(Q_0) \subseteq P_0$ and $P_0 \subseteq g(P_0)$, hence we can construct a sequence $\{q_n\}$ such that

$$d(gq_{n+1}, Hq_n) = d(P, Q)$$

and $\{q_n\}$ converges to an element $q \in Q$. Since the pair (N, H) is proximal cyclic contraction and g is an isometry, we have

$$d(p_{n+1}, q_{n+1}) = d(gp_{n+1}, gq_{n+1}) \leq \alpha d(p_n, q_n) + (1 - \alpha) d(P, Q). \tag{8}$$

Letting $n \to \infty$ in inequality (8) we get

$$d(p, q) \leq \alpha d(p, q) + (1 - \alpha) d(P, Q). \tag{9}$$

This shows that $d(p, q) \leq d(P, Q) = \inf\{d(p, q) \colon p \in P, q \in Q\}$. Hence

$$d(p, q) = d(P, Q). \tag{10}$$

Thus, we conclude that $p \in P_0$ and $q \in Q_0$. Since

$$N(P_0) \subseteq Q_0, \ H(Q_0) \subseteq P_0$$

there are $u \in P$ and $v \in Q$ such that

$$d(u, Np) = d(v, Hq) = d(P, Q). \tag{11}$$

Since $d(u, Np) = d(P, Q) = d(gp_{n+1}, Np_n)$ and N is a generalised proximal ψ-ϕ-contraction of first kind, we get

$$\psi(d(u, gp_{n+1})) \leq \psi(d(p, p_n)) - \phi(d(p, p_n)).$$

Letting $n \to \infty$ in the above inequality we get

$$\psi(d(u, gp)) \leq \psi(0) - \phi(0).$$

This shows that $u = gp$. Therefore

$$d(gp, Np) = d(P, Q). \tag{12}$$

Similarly, we can show that $v = gq$ and

$$d(gq, Hq) = d(P, Q). \tag{13}$$

From equation (10), (12) and (13), we get

$$d(p, q) = d(gp, Np) = d(gq, Hq) = d(P, Q). \tag{14}$$

As, $gp_n \in P_0 \subseteq P$ for all $n \in \mathbb{N}$, g is isometry and $p_n \to p$, by closedness of P we must have $gp \in P$. Similarly, $gq \in Q$. Next, we prove uniqueness. Then, its sufficient to show that for any $\bar{p}_0 \in P_0$ and $\bar{q}_0 \in Q_0$ the sequences $\{\bar{p}_n\}$ and $\{\bar{q}_n\}$ defined by

$$d(g\bar{p}_{n+1}, N\bar{p}_n) = d(g\bar{q}_{n+1}, H\bar{q}_n) = d(P, Q)$$

converge to the points p and q respectively. Then, starting with $\bar{p}_0 \in P_0$ and $\bar{q}_0 \in Q_0$, and repeating the same process as above, one can show that the sequences $\{\bar{p}_n\}$ and $\{\bar{q}_n\}$ are convergent to the limits $\bar{p}, \bar{q}$ respectively (say) such that

$$d(g\bar{p}, N\bar{p}) = d(g\bar{q}, N\bar{q}) = d(P, Q) \text{ for all } n \geq 0$$

and $g\bar{p} \in P, g\bar{q} \in Q$. We show that $\bar{p} = p, \bar{q} = q$. On the contrary, suppose that $\bar{p} \neq p$. Since g is isometry, N is a generalized proximal ψ-ϕ-contraction of first kind, we must have

$$\psi(d(\bar{p}, p)) = \psi(d(g\bar{p}, gp)) \leq \psi(d(\bar{p}, p)) - \phi(d(\bar{p}, p)).$$

Since $\bar{p} \neq p$, the above inequality yields a contradiction. Therefore, we must have $\bar{p} = p$. Similarly, $\bar{q} \neq q$. This proves the uniqueness. Furthermore, let a sequence $\{u_n\}$ in P be such that there is a sequence $\{\xi_n\}$ in P with $g\xi_n \in P$ for all n such that the sequence of nonnegative numbers $\{d(u_n, p_n)\}$ is convergent,

$$\lim_{n\to\infty} d(\xi_n, u_n) = 0 \text{ and } d(g\xi_{n+1}, Nu_n) = d(P, Q).$$

Then, since

$$\begin{aligned} d(\xi_n, p_n) &\leq d(\xi_n, u_n) + d(u_n, p_n); \text{ and} \\ d(u_n, p_n) &\leq d(u_n, \xi_n) + d(\xi_n, p_n) \end{aligned}$$

we must have $\lim_{n\to\infty} d(\xi_n, p_n) = \lim_{n\to\infty} d(u_n, p_n)$. Again, since $d(g\xi_{n+1}, Nu_n) = d(P, Q)$ by the properties of N and g we have

$$\psi(d(\xi_{n+1}, p_{n+1})) = \psi(d(g\xi_{n+1}, gp_{n+1})) \leq \psi(d(u_n, p_n)) - \phi(d(u_n, p_n)).$$

As $\lim_{n\to\infty} d(\xi_n, p_n) = \lim_{n\to\infty} d(u_n, p_n)$ it follows from the above inequality that

$$\lim_{n\to\infty} d(\xi_n, p_n) = \lim_{n\to\infty} d(u_n, p_n) = 0.$$

Finally, by the use of triangular inequality, the sequence $\{u_n\}$ converges to p. □

If g is assumed to be an identity mapping then according to Theorem 3, we obtain the following result.

Corollary 4. *Let (X, d) be a metric space and let P and Q be two non-empty closed subsets of X such that P_0 and Q_0 are non-empty. Let $N\colon P \to Q$ and $H\colon Q \to P$ satisfy the following conditions:*

(a) N and H are to generalized proximal ψ-ϕ-contractions of first kind;
(b) the pair (N, H) is proximal cyclic contraction;
(c) $N(P_0) \subseteq Q_0$ and $H(Q_0) \subseteq P_0$.

Then there exists a unique point $p \in P$ and a unique point $q \in Q$ such that

$$d(p, Np) = d(q, Hq) = d(p, q) = d(P, Q)$$

and for any fixed $p_0 \in P_0, q_0 \in Q_0$, the sequences $\{p_n\}, \{q_n\}$ defined by

$$d(p_{n+1}, Np_n) = d(q_{n+1}, Hq_n) = d(P, Q)$$

converge to the points p and q respectively. Furthermore, a sequence $\{u_n\}$ in P converges to p if there is a sequence $\{\xi_n\}$ in P such that the sequence of nonnegative numbers $\{d(u_n, p_n)\}$ is convergent

$$\lim_{n\to\infty} d(\xi_n, u_n) = 0 \text{ and } d(\xi_{n+1}, Nu_n) = d(P, Q).$$

Example 1. *Consider the complete metric space $\mathbb{R}^2$ with metric d defined by*

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Let

$$P = \{(0, y); 0 \le y \le 1\}, Q = \{(1, y); 0 \le y \le 1\}$$

and define three mappings $N\colon P \to Q$, $H\colon Q \to P$ and $g\colon P \cup Q \to P \cup Q$ by:

$$\begin{aligned} N(0, y) &= (1, \frac{y - y^2}{2}) \text{ and } H(1, y) = (0, \frac{y - y^2}{2}) \\ g(x, y) &= (x, y) \end{aligned} \tag{15}$$

Then, $d(P,Q)=1$ *and* $P_0=P$ *and* $Q_0=Q$. *Now we claim that* N *and* H *are generalized proximal* ψ-ϕ-*contraction of the first kind. Suppose that the mappings* $\psi,\phi\colon [0,\infty)\to[0,\infty)$ *are defined by:*

$$\psi(t)=t,\ \text{and}\ \phi(t)=\frac{t^2}{2}\ \text{for all}\ t\geq 0.$$

Let $u=(0,x_1), v=(0,x_2), p=(0,p_1), q=(0,p_2)$ *be four elements in* P *satisfying*

$$d((0,x_1),N(0,p_1))=d(P,Q)=1$$

and

$$d((0,x_2),N(0,p_2))=d(P,Q)=1.$$

Then, it follows that

$$x_i \;=\; \frac{p_i-p_i^2}{2}\ \text{for}\ i=1,2.$$

Without loss of generality, we may assume that $p_1-p_2>0$. *Then*

$$\begin{aligned}
\psi(d(u,v)) &= \psi(d((0,x_1),(0,x_2)))\\
&= \psi\left(d\left(\left(0,\frac{p_1-p_1^2}{2}\right),\left(0,\frac{p_2-p_2^2}{2}\right)\right)\right)\\
&= \psi\left(\left|\left(\frac{p_1-p_1^2}{2}\right)-\left(\frac{p_2-p_2^2}{2}\right)\right|\right)\\
&= \frac{1}{2}\left|(p_1-p_2)-\left(p_1^2-p_2^2\right)\right|\\
&\leq (p_1-p_2)-\frac{(p_1-p_2)^2}{2}\\
&= \psi(p_1-p_2)-\phi(p_1-p_2)\\
&= \psi(d(p,q))-\phi(d(p,q)).
\end{aligned}$$

Hence, N *is a generalized proximal* ψ-ϕ-*contraction of the first kind. Similarly, we can see that* H *is a generalised proximal* ψ-ϕ-*contraction of the first kind.*
Let $(1,x_1),(1,x_2),(1,p_1),(1,p_2)$ *be four elements in* Q *such that*

$$d((1,x_1),H(1,p_1))=d(P,Q)=1,$$
$$d((1,x_2),H(1,p_2))=d(P,Q)=1$$

Then, it follows that $x_i=\dfrac{p_i-p_i^2}{2}$ *for* $i=1,2$. *Therefore, the pair* (N,H) *forms a proximal cyclic contraction with* $\alpha=\dfrac{1}{2}$, *and other suppositions are also satisfied. It* *is easy to see that the elements* $(0,0) \in P$ *and* $(1,0) \in Q$ *are such that*

$$d(g(0,0), N(0,0)) = d(g((1,0)), H((1,0))) \ = d((0,0),(1,0)) = d(P,Q).$$

Next, we prove a best proximity point theorem for mappings which are generalized proximal ψ-ϕ-contractions of first as well as of second kind.

Theorem 5. *Let* (X,d) *be a complete metric space and let* P *and* Q *be two non-empty closed subsets of* X *such that* P_0 *and* Q_0 *are non-empty. Let* $N\colon P \to Q$ *and* $g\colon P \to P$ *be two mappings satisfying the following conditions:*

(a) N *is generalised proximal* ψ-ϕ*-contraction of first and second kind;*

(b) g *is an isometry;*

(c) N *preserve isometric distance with respect to* g*;*

(d) $N(P_0) \subseteq Q_0$*;*

(e) $P_0 \subseteq g(P_0)$*.*

Then there exists a unique point $p \in P$ *such that*

$$d(gp, Np) = d(P,Q)$$

and for any fixed $p_0 \in P_0$*, the sequence* $\{p_n\}$ *defined by*

$$d(gp_{n+1}, Np_n) = d(P,Q)$$

converges to the point p*. Furthermore, a sequence* $\{p_n\}$ *in* P *converges to* p *if there is a sequence* $\{\xi_n\}$ *in* P *with* $g\xi_n \in P$ *for all* n *such that the sequence of nonnegative numbers* $\{d(u_n, p_n)\}$ *is convergent and*

$$\lim_{n\to\infty} d(\xi_n, u_n) = 0 \text{ and } d(g\xi_{n+1}, Nu_n) = d(P,Q).$$

Proof. Consider $p_0 \in P_0$. Since $N(P_0) \subseteq Q_0$ and $P_0 \subseteq g(P_0)$, it is ascertain that there exists an element $p_1 \in P_0$ such that

$$d(gp_1, Np_0) = d(P,Q).$$

Again, since $N(P_0) \subseteq Q_0$ and $P_0 \subseteq g(P_0)$, then there exists an element $p_2 \in P_0$ such that

$$d(gp_2, Np_1) = d(P,Q).$$

Similarly we can find p_n in P_0. Having chosen p_n one can determine an element $p_{n+1} \in P_0$, such that

$$d(gp_{n+1}, Np_n) = d(P,Q) \tag{16}$$

for all $n \geq 0$. As g is an isometry and N is generalized proximal ψ-ϕ-contraction of first kind, we find that

$$\psi(d(gp_{n+1}, gp_n)) \leq \psi(d(p_n, p_{n-1})) - \phi(d(p_n, p_{n-1})).$$

For all $n \in \mathbb{N}$. Similar to the proof of Theorem 3 one can see that the sequence $\{p_n\}$ is a Cauchy sequence and convergence to some $p \in P$. Since N is generalised proximal ψ-ϕ-contraction of the second kind and preserves isometric distance with respect to g we have

$$\begin{aligned} \psi(d(Np_{n+1}, Np_n)) &= \psi(d(Ngp_{n+1}, Ngp_n)) \\ &\leq \psi(d(Np_n, Np_{n-1}) - \phi(d(Np_n, Np_{n-1})). \end{aligned} \tag{17}$$

Hence, we must have $\psi(d(Np_{n+1}, Np_n)) \leq \psi(d(Np_n, Np_{n-1}))$, and so the sequence $\{d(Np_{n+1}, Np_n)\}$ is a nonincreasing sequence and is bounded below. Hence, there exists $t \geq 0$ such that

$$\lim_{n\to\infty} d(Np_{n+1}, Np_n) = t.$$

We claim that $t = 0$. Suppose on the contrary that $t > 0$. Letting $n \to \infty$ in (17), we get

$$\psi(t) \leq \psi(t) - \phi(t).$$

This implies that $\phi(t) = 0$. That is $t = 0$. Which is a contradiction, hence $t = 0$, that is

$$\lim_{n\to\infty} d(Np_{n+1}, Np_n) = 0. \tag{18}$$

We claim that $\{Np_n\}$ is a Cauchy sequence. Suppose that $\{Np_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and two subsequences subsequence $\{Np_{m_k}\}$ and $\{Np_{n_k}\}$ of $\{Np_n\}$ such that $n_k > m_k \geq k$ with

$$d(Np_{m_k}, Np_{n_k}) \geq \epsilon,\ d(Np_{n_k}, Np_{m_k-1}) < \epsilon.$$

for $k \in \mathbb{N}$, and so

$$\begin{aligned} \epsilon &\leq d(Np_{m_k}, Np_{n_k}) \\ &\leq d(Np_{n_k}, Np_{m_k-1}) + d(Np_{m_k-1}, Np_{m_k}) \\ &< \epsilon + d(Np_{m_k-1}, Np_{m_k}). \end{aligned}$$

It follows from (18) that

$$\lim_{k\to\infty} d(Np_{m_k}, Np_{n_k}) = \epsilon. \tag{19}$$

Also, by the following two inequalities

$$\begin{aligned} d(Np_{m_k}, Np_{n_k}) &\leq d(Np_{m_k}, Np_{m_k+1}) + d(Np_{m_k+1}, Np_{n_k+1}) + d(Np_{n_k+1}, Np_{n_k}); \\ d(Np_{m_k+1}, Np_{n_k+1}) &\leq d(Np_{m_k+1}, Np_{m_k}) + d(Np_{m_k}, Np_{m_k}) + d(Np_{m_k}, Np_{m_k+1}) \end{aligned}$$

we must have

$$\lim_{k\to\infty} d(Np_{m_k+1}, Np_{n_k+1}) = \epsilon. \tag{20}$$

Since N is generalised proximal ψ-ϕ-contraction of the second kind and preserves isometric distance with respect to g we have

$$\begin{aligned} \psi(d(Np_{m_k+1}, Np_{n_k+1})) &= \psi(d(Ngp_{m_k+1}, Ngp_{n_k+1})) \\ &\leq \psi(d(Np_{m_k}, Np_{n_k})) - \phi(d(Np_{m_k}, Np_{n_k})). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (18), (19) and (20) we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

This implies that $\phi(\epsilon) = 0$, that is, $\epsilon = 0$. This is a contradiction hence $\{Np_n\}$ is a Cauchy sequence and so converges to some element $q \in Q$. Therefore, we conclude that

$$d(gp, q) = \lim_{n\to\infty} d(gp_{n+1}, Np) = d(P, Q).$$

This shows that $gp \in P_0$. Since $P_0 \subseteq g(P_0)$, we have $gp = g\xi$ for a $\xi \in P_0$ and then $d(gp, g\xi) = 0$. By the fact that g is an isometry we have $d(p, \xi) = d(gp, g\xi) = 0$. Hence, $p = \xi$, and so p becomes a point of P_0, and as $N(P_0) \subseteq Q_0$ we have

$$d(u, Np) = d(P, Q). \tag{21}$$

for some $u \in P$. Using (16), (21) and the fact that N is generalised proximal ψ-ϕ-contraction of first kind we obtain

$$\psi(d(u, gp_{n+1})) \leq \psi(d(p, p_n)) - \phi(d(p, p_n)).$$

For all $n \in N$. Letting $n \to \infty$, we get the sequence $\{gp_n\}$ converges to the point u. As g is continuous we have

$$gp_n \to gp, \text{ as } n \to \infty.$$

By the uniqueness of the limit of a sequence, we conclude that $u = gp$. Therefore it results that

$$d(gp, Np) = d(u, Np) = d(P, Q).$$

The uniqueness and the residual part of the proof are similar as in the Theorem 3. This completes the proof of the theorem. □

If g is assumed to be the identity mapping, then by Theorem 3 we obtain the following corollary.

Corollary 6. *Let (X, d) be a complete metric space and let P and Q be two non-empty closed subsets of X such that P_0 and Q_0 are non-empty. Let $N\colon P \to Q$ be a mapping such that the following conditions are satisfied:*

(a) N is generalised proximal ψ-ϕ-contraction of first and second kind;

(b) $N(P_0) \subseteq Q_0$.

Then there exists a unique point $p \in P$ such that

$$d(p, Np) = d(P, Q)$$

and for any fixed $p_0 \in P_0$, the sequence $\{p_n\}$ defined by

$$d(p_{n+1}, Np_n) = d(P, Q)$$

converges to the point p.

If $P = Q$ then the above corollary yields the following fixed point result:

Corollary 7. *let (X, d) be a complete metric space and $P \subseteq X$ be closed. Let $N\colon P \to P$ be a mapping satisfying: for all $p, q \in P$*

$$\psi(d(Np, Nq)) \leq \psi(d(p, q)) - \phi(d(p, q))$$

where $\psi \in \Psi$ and $\phi \in \Phi$. Then N has a unique fixed point in X.

3. Conclusion

In the presented work we have established some best proximity point results for ψ-ϕ-contractions on complete metric spaces and extended some known results. An example is given to illustrate the results proved herein. The results of this paper can be further extended to generalized spaces, e.g., vector-valued fuzzy metric spaces [19], graphical symmetric spaces [15] etc. Further, the results can be helpful to investigate the solutions of initial and boundary value problems, fractional calculus [15] and as well. It may be investigated that under what conditions the fuzzy metric version of our results are true.

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References

1. J. Anuradha Eldered & P. Veeramani: Proximal pointwise contraction. Topol. Appl. **156** (2009), no. 18, 2942-2948. https://doi.org/10.1016/j.topol.2009.01.017
2. A. Anthony Eldred & P. Veeramani: Existence and convergence of best proximity points. J. Math. Anal. Appl. **323** (2006), 1001-1006. https://doi.org/10.1016/j.jmaa.2005.10.081
3. A.D. Arvanitakis: A proof of the generalized Banach contraction conjecture. Proc. Amer. Math. Soc. **131** (2003), no. 12, 3647-3656. https://doi.org/10.2307/1194510
4. B.S. Choudhury & K.P. Das: A new contraction principle in Menger spaces. Acta Mathematica Sinica **24** (2008), no. 8, 1379-1386. https://doi.org/ 10.1007/s10114-007-6509-x
5. C. Mongkolkeha & P. Kumam: Fixed point and common fixed-point theorems for generalized weak contraction mappings of integral type in modular spaces. Internat. J. Math. & Math. Sci. **2011** (2011), Article ID 705943, 12 pages.
6. C. Mongkolkeha & P. Kumam: Fixed point theorems for generalized asymptotic pointwise ρ-contraction mappings involving orbits in modular function spaces. Applied Mathematics Letters **25** (2012), no. 10, 1285-1290. https://doi.org/ 10.1016/j.aml.2011.11.027
7. D.W. Boyd & J.S.W. Wong: On nonlinear contractions. Proc. Amer. Math. Soc. **20** (1969), 458-464. https://doi.org/ 10.1090/S0002-9939-1969-0239559-9
8. G. Prasad & D. Khantwal: Fixed points of JS-contractive mappings with applications. J. Anal. **31** (2023), 2687-2701. https://doi.org/10.1007/s41478-023-00598-z
9. G. Prasad & R.C. Dimri: Fixed point theorems for weakly contractive mappings in relational metric spaces with an application. J. Anal. **26** (2018), 151-162. https://doi.org/10.1007/s41478-018-0076-7
10. G. Prasad & R.C. Dimri: Coincidence theorems in new generalized metric spaces under locally g-transitive binary relation. J. Indian Math. Soc. **85** (2018), no. 3-4, 396-410.
11. J.B. Prolla: Fixed point theorems for set valued mappings and existence of best approximations. Numer. Funct. Anal. Optim. **5** (1982-1983), 449-455.
12. K. Fan: Extensions of two fixed point theorems of F. E. Browder. Mathematische Zeitschrift **112** (1969), no. 3, 234-240. https://api.semanticscholar.org/CorpusID:119627163

13. K. Ungchittrakool: A Best Proximity Point Theorem for Generalized Non-Self-Kannan-Type and Chatterjea-Type Mappings and Lipschitzian Mappings in Complete Metric Spaces. Journal of Function Spaces **2016** (2016), no. 1-2, 1-11. https://doi.org/10.1155/2016/9321082
14. N. Hussain, A.Latif & P. Salimi: Best Proximity Point Result in G-Metric Space. Abstract and Applied Analysis **2014** (2014) Artical ID 837943, 8. http://dx.doi.org/10.1155/2014/837943
15. N. Dubey, S. Shukla & R. Shukla: On Graphical Symmetric Spaces, Fixed-Point Theorems and the Existence of Positive Solution of Fractional Periodic Boundary Value Problems. Symmetry **16** (2024), no. 2, 182. https://doi.org/10.3390/sym16020182
16. S. Banach: Sur les opérations dans les ensembles abstraits et leur application aux équations integrals. Fundam. Math. **3** (1922), 133-181. https://api.semanticscholar.org/CorpusID:118543265
17. S. Sadiq Basha: Best proximity point theorems. Journal of Approximation theory **163** (2011), no. 11, 1772-1781. https://doi.org/10.1016/j.jat.2011.06.012
18. S. Reich: Approximate selections, best approximations, fixed points and invariant sets. J. Math. Anal. Appl. **62** (1978), 104-113. https://api.semanticscholar.org/CorpusID:21665412
19. S. Shukla, N. Dubey, J-J. Miñana: Vector-Valued Fuzzy Metric Spaces and Fixed Point Theorems. Axioms **13** (2024), no. 4, 252. https://doi.org/10.3390/axioms13040252
20. S. Shukla, S. Rai & R. Shukla: Some Fixed Point Theorems for α-Admissible Mappings in Complex-Valued Fuzzy Metric Spaces. Symmetry **15**(9) (2023), 1797. https://doi.org/10.3390/sym15091797
21. S. Shukla & S. Rai: Caristi type fixed point theorems in 1-M-complete fuzzy metric-like spaces. The Journal of Analysis **31** (2023), no. 3, 2247-2263. https://doi.org/10.1007/s41478-023-00562-x
22. S. Shukla, N. Dubey, R. Shukla & I. Mezník: Coincidence point of Edelstein type mappings in fuzzy metric spaces and application to the stability of dynamic markets. Axioms **12** (2023), no. 9, 854. https://doi.org/10.3390/axioms12090854
23. V.M. Sehgal & S.P. Singh: A generalization to multifunctions of fans best approximation theorem. Proc. Amer. Math. Soc. **102** (1988), 534-537.
24. V.M. Sehgal & S.P. Singh: A theorem on best approximations. Numer. Funct. Anal. Optim. **10** (1988), 181-184. https://doi.org/10.1080/01630568908816298
25. V. Vetrivel, P. Veeramani & P. Bhattacharyya: Some extensions of fans best approximation theorem. Numer. Funct. Anal. Optim. **13** (1992), 397-402. https://doi.org/10.1080/01630569208816486
26. W.A. Kirk, P.S. Srinivasan & P. Veeramani: Fixed points for mappings satisfying cyclic contractive conditions. Fixed Point Theory **4** (2003), 79-89.

27. W Sanhan, C. Mongkolkeha & P. Kumam: Generalized Proximalψ -Contraction Mappings and Best Proximity Points. Abstract and Applied Analysis **2012** (2012), Article ID 896912, 19. https://doi.org/10.1155/2012/896912
28. W. Sintunavarat & P. Kumam: Gregus-type common fixed-point theorems for tangential multivalued mappings of integral type in metric spaces. Internat. J. Math. & Math. Sci. **2011** (2011), Article ID 923458, 12 pages. https://doi.org/10.1155/2011/923458
29. W. Sintunavarat & P. Kumam: Weak condition for generalized multi-valued (f, α, β) weak contraction mappings. Applied Mathematics Letters **24** (2011), no. 4, 460-465. https://api.semanticscholar.org/CorpusID:31271775
30. X. Zhang: Common fixed-point theorems for some new generalized contractive type mappings. J. Math. Anal. Appl. **333** (2007), no. 2, 780-786. https://doi.org/10.1016/j.jmaa.2006.11.028

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