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PADOVAN AND LUCAS-PADOVAN QUATERNIONS

Gwangyeon Lee [a] and Kisoeb Park [b,*]

Abstract. In this paper, we introduce the Lucas-Padovan quaternions sequence. Initiating the studies based on the Padovan quaternion coefficients in relation to their recurrence, their matrix representation is then defined. We investigate various aspects of these quaternions, including summation formulas and binomial sums.

1. Introduction

Quaternions are used in such fields as quantum physics, computer science, differential equations, and group theory [1, 6, 8, 7]. Moreover, quaternions are useful for the representation and generalization of large quantities.

Recently, Fibonacci quaternions cover a wide range of interest in modern math as they appear in the comprehensive works of [3, 4] and [13]. Also, several authors worked on Padovan quaternions, Pell quaternions and their generalizations in [2, 5, 9, 10, 11, 12].

A quaternion is described by:

$$q = a + bi + cj + dk,$$

where a, b, c, d are real numbers and i, j, k are the orthogonal parts at the base in $\mathbf{R}^3$. We note that the quaternion multiplication is defined using the rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

The conjugate and norm of the quaternions are defined by

$$q^* = a - bi - cj - dk$$

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*Corresponding author.
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and

$$N(q) = a^2 + b^2 + c^2 + d^2.$$

The Padovan sequence is the sequence of integers P_n defined by initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}$$

for all $n \geq 3$. The first a few values of P_n are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, \cdots .$$

The Padovan quaternions are defined by

$$Q(P_n) = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k,$$

where P_n is the nth Padovan number and i, j and k are orthonormal bases.

Now, we will define a new sequence to be called *Lucas-Padovan* (abbr. *Ludovan*) sequence $\{\ell_n\}$.

Definition 1.1. The Ludovan sequence is defined by the following rules; let $\ell_0 = 1$ and, for $n \geq 1$,

$$\ell_n = P_{n-1} + P_{n+1},$$

where P_n is the nth Padovan number.

The first a few values of Ludovan sequence ℓ_n, for $n \geq 0$, are

$$1, 2, 3, 3, 5, 6, 8, 11, 14, 19, 25, 33, 44, 58, 77, 102, 135, \cdots .$$

Definition 1.2. The Ludovan quaternions are defined by

$$Q(\ell_n) = \ell_n + \ell_{n+1}i + \ell_{n+2}j + \ell_{n+3}k,$$

where ℓ_n is the nth Ludovan number and i, j and k are orthonormal bases.

In this paper, we study the Padovan quaternions and Ludovan quaternions. Moreover, we also describe their properties using matrix representation.

2. Padovan Quaternions

In [11], the author gave the following theorem about the generating function for the Padovan quaternions.

Theorem 2.1 ([11])**.** *The generating function of the Padovan quaternions is*

$$g(x)=\frac{(1+i+j+2k)+(1+i+2j+2k)x+(i+j+k)x^2}{1-x^2-x^3}.$$

From Theorem 2.1, we have the following result. Note that $\binom{m}{n}+\binom{m}{\ell}$ if $n>m$ or $\ell<0$.

Theorem 2.2. *For the nth Padovan quaternion* $Q(P_n)$*,*

$$\begin{aligned}Q(P_n)&=\sum_{m=0}^{n}\binom{m+1}{n-2m}+i\sum_{m=0}^{n}\left\{\binom{m+1}{n-2m}+\binom{m}{n-2m-2}\right\}\\&\quad+j\sum_{m=0}^{n}\binom{m+2}{n-2m}+k\sum_{m=0}^{n}\left\{2\binom{m+1}{n-2m}+\binom{m}{n-2m-2}\right\}.\end{aligned}$$

Proof. Let us put $1+i+j+2k$ as A, $1+i+2j+2k$ as B, and $i+j+k$ as C. Then we have the generating function of the Padovan quaternions as follows:

$$\begin{aligned}g(x)&=(A+Bx+Cx^2)\frac{1}{1-x^2-x^3}\\&=(A+Bx+Cx^2)\sum_{s\ge 0}(x^2+x^3)^s\\&=A\sum_{s\ge 0}(1+x)^s x^{2s}+B\sum_{s\ge 0}(1+x)^s x^{2s+1}+C\sum_{s\ge 0}(1+x)^s x^{2s+2}\\&=A\sum_{s\ge 0}\sum_{t=0}^{s}\binom{s}{t}x^{2s+t}+B\sum_{s\ge 0}\sum_{t=0}^{s}\binom{s}{t}x^{2s+t+1}+C\sum_{s\ge 0}\sum_{t=0}^{s}\binom{s}{t}x^{2s+t+2}\\&=A\sum_{s\ge 0}\sum_{n=2s}^{3s}\binom{s}{n-2s}x^n+B\sum_{s\ge 0}\sum_{n=2s+1}^{3s+1}\binom{s}{n-2s-1}x^n\\&\quad+C\sum_{s\ge 0}\sum_{n=2s+2}^{3s+2}\binom{s}{n-2s-2}x^n\\&=\sum_{n\ge 0}\left\{A\sum_{m=0}^{n}\binom{m}{n-2m}+B\sum_{m=0}^{n}\binom{m}{n-2m-1}+C\sum_{m=0}^{n}\binom{m}{n-2m-2}\right\}x^n.\end{aligned}$$

Since $A=1+i+j+2k$, $B=1+i+2j+2k$, $C=i+j+k$, and $g(x)=\sum_{n\ge 0}Q(P_n)x^n$ we have

$$
\begin{aligned}
Q(P_n) = &\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m}{n-2m-1}\right\}\\
&+i\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m}{n-2m-1}+\binom{m}{n-2m-2}\right\}\\
&+j\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+2\binom{m}{n-2m-1}+\binom{m}{n-2m-2}\right\}\\
&+k\sum_{m=0}^{n}\left\{2\binom{m}{n-2m}+2\binom{m}{n-2m-1}+\binom{m}{n-2m-2}\right\}\\
=&\sum_{m=0}^{n}\binom{m+1}{n-2m}+i\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m+1}{n-2m-1}\right\}\\
&+j\sum_{m=0}^{n}\binom{m+2}{n-2m}+k\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m+2}{n-2m}\right\}.
\end{aligned}
$$

Therefore, the proof is completed. □

Corollary 2.3. *For the nth Padovan number P_n, we have the following identities.*

(i) $P_n = \sum_{m=0}^{n}\binom{m+1}{n-2m}$

(ii) $P_{n+1} = \sum_{m=0}^{n+1}\binom{m+1}{n-2m+1} = \sum_{m=0}^{n}\binom{m+1}{n-2m+1}$

(iii) $P_{n+2} = \sum_{m=0}^{n+2}\binom{m+1}{n-2m+2} = \sum_{m=0}^{n+1}\binom{m+1}{n-2m+2}$

(iv) $P_{n+3} = \sum_{m=0}^{n+3}\binom{m+1}{n-2m+3} = \sum_{m=0}^{n+1}\binom{m+1}{n-2m+3}$

(v) $\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m+1}{n-2m-1}\right\} = \sum_{m=0}^{n}\binom{m+1}{n-2m+1}$

(vi) $\sum_{m=0}^{n}\binom{m+2}{n-2m} = \sum_{m=0}^{n+1}\binom{m+1}{n-2m+2}$

(vii) $\sum_{m=0}^{n}\left\{\binom{m}{n-2m}+\binom{m+2}{n-2m}\right\} = \sum_{m=0}^{n+1}\binom{m+1}{n-2m+3}$

(viii) $\sum_{m=0}^{n+1}\binom{m+1}{n-2m+3} = \sum_{m=0}^{n}\binom{m+1}{n-2m+1}+\sum_{m=0}^{n}\binom{m+1}{n-2m} = \sum_{m=0}^{n}\binom{m+2}{n-2m+1}$

Proof. Since $Q(P_n) = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k$ and $P_{n+3} = P_{n+1} + P_n$, all identities can be readily verified using Theorem 2.2. □

3. LUCAS-PADOVAN QUATERNIONS

In this section, we consider the Ludovan quaternions.

In [11], the author gave the following theorem about the Binet-like Formula for the Padovan quaternions.

Theorem 3.1 ([11]). *For $n \geq 0$, the Binet-like Formula for the Padovan quaternions is*

$$Q(P_n) = a\alpha r_1^n + b\beta r_2^n + c\gamma r_3^n,$$

where r_1, r_2 and r_3 are the root of the equation $x^3 - x - 1 = 0$, and

$$a = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)},\ b = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)},\ c = \frac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)},$$
$$\alpha = 1 + r_1 i + r_1 j + r_1 k,\ \beta = 1 + r_2 i + r_2 j + r_2 k,\ \gamma = 1 + r_3 i + r_3 j + r_3 k.$$

From the Definitions 1.1, 1.2 and Theorem 3.1, we have the following theorem.

Theorem 3.2. *For $n \geq 0$, the Binet-like Formula for the Ludovan quaternions is*

$$\begin{aligned} Q(\ell_n) &= a\alpha r_1^n \left(\frac{1}{r_1} + r_1\right) + b\beta r_2^n \left(\frac{1}{r_2} + r_2\right) + c\gamma r_3^n \left(\frac{1}{r_3} + r_3\right) \\ &= a\alpha r_1^n \left(r_1 - \frac{\sqrt{5}-1}{2}\right)\left(r_1 + \frac{\sqrt{5}+1}{2}\right) \\ &\quad + b\beta r_2^n \left(r_2 - \frac{\sqrt{5}-1}{2}\right)\left(r_2 + \frac{\sqrt{5}+1}{2}\right) \\ &\quad + c\gamma r_3^n \left(r_3 - \frac{\sqrt{5}-1}{2}\right)\left(r_3 + \frac{\sqrt{5}+1}{2}\right), \end{aligned}$$

where r_1, r_2 and r_3 are the root of the equation $x^3 - x - 1 = 0$, and

$$a = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)},\ b = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)},\ c = \frac{(r_1-1)(r_2-1)}{(r_3-r_1)(r_3-r_2)},$$
$$\alpha = 1 + r_1 i + r_1 j + r_1 k,\ \beta = 1 + r_2 i + r_2 j + r_2 k,\ \gamma = 1 + r_3 i + r_3 j + r_3 k.$$

Proof. From the definition of the Ludovan quaternions, we have

$$\begin{aligned} Q(\ell_n) &= \ell_n + \ell_{n+1} i + \ell_{n+2} j + \ell_{n+3} k \\ &= (P_{n-1} + P_{n+1}) + (P_n + P_{n+2})i + (P_{n+1} + P_{n+3})j + (P_{n+2} + P_{n+4})k \\ &= QP_{n-1} + QP_{n+1} \\ &= a\alpha r_1^{n-1}(1 + r_1^2) + b\beta r_2^{n-1}(1 + r_2^2) + c\gamma r_3^{n-1}(1 + r_3^2) \\ &= a\alpha r_1^n \left(\frac{1}{r_1} + r_1\right) + b\beta r_2^n \left(\frac{1}{r_2} + r_2\right) + c\gamma r_3^n \left(\frac{1}{r_3} + r_3\right). \end{aligned}$$

Since $x^3 - x - 1 = 0$, we have

$$\frac{1}{x} + x = x^2 + x - 1 = \left(x - \frac{\sqrt{5}-1}{2}\right)\left(x + \frac{\sqrt{5}+1}{2}\right).$$

Therefore, the proof is completed. □

Now, we consider the generating function of the Ludovan quaternions.

Lemma 3.3. *Let $\{\ell_n\}$ be the Ludovan sequence. Then, for $n \geq 3$, we have*

$$\ell_n = \ell_{n-2} + \ell_{n-3}.$$

Proof. Since $\ell_0 = 1$, $\ell_1 = 2$, $\ell_2 = 3$, and $\ell_3 = 3$, we have $\ell_3 = \ell_1 + \ell_0$. By induction on n, assuming $\ell_n = P_{n-1} + P_{n+1}$ and $P_n = P_{n-2} + P_{n-3}$, we have

$$\begin{aligned} \ell_{n-2} + \ell_{n-3} &= P_{n-3} + P_{n-1} + P_{n-4} + P_{n-2} \\ &= (P_{n-1} + P_{n-2}) + (P_{n-3} + P_{n-4}) \\ &= P_{n+1} + P_{n-1} = \ell_n. \end{aligned}$$

Therefore, for $n \geq 3$, we have

$$\ell_n = \ell_{n-2} + \ell_{n-3}.$$

□

From Lemma 3.3, we have

$$Q(\ell_n) = Q(\ell_{n-2}) + Q(\ell_{n-3}).$$

The following theorem is related with the generating function of the Ludovan quaternions.

Theorem 3.4. *The generating function of the Ludovan quaternions is*

$$f(x) = \frac{(1 + 2i + 3j + 3k) + (2 + 3i + 3j + 5k)x + (2 + i + 2j + 3k)x^2}{1 - x^2 - x^3}.$$

Proof. Let

$$f(x) = \sum_{n\geq 0} Q(\ell_n)x^n = Q(\ell_0) + Q(\ell_1)x + Q(\ell_2)x^2 + Q(\ell_3)x^3 + \cdots + Q(\ell_n)x^n + \cdots$$

be generating function of the Ludovan quaternions. Since

$$\begin{aligned} x^2 f(x) &= Q(\ell_0)x^2 + Q(\ell_1)x^3 + Q(\ell_2)x^4 + Ql_3x^5 + \cdots + Q(\ell_{n-2})x^n + \cdots, \\ x^3 f(x) &= Q(\ell_0)x^3 + Q(\ell_1)x^4 + Q(\ell_2)x^5 + Q(\ell_3)x^6 + \cdots + Q(\ell_{n-3})x^n + \cdots, \end{aligned}$$

we write

$$\begin{aligned} (1 - x^2 - x^3)f(x) = Q(\ell_0) &+ Q(\ell_1)x + (Q(\ell_2) - Q(\ell_0))x^2 + ((Q(\ell_3) - Q(\ell_1) - Q(\ell_0))x^3 \\ &+ (Q(\ell_4) - Q(\ell_2) - Q(\ell_1))x^4 + \cdots \\ &+ (Q(\ell_n) - Q(\ell_{n-2}) - Q(\ell_{n-3}))x^n + \cdots. \end{aligned}$$

Now using $Q(\ell_0) = 1+2i+3j+3k$, $Q(\ell_1) = 2+3i+3j+5k$, $Q(\ell_2)=3+3i+5j+6k$ and $Q(\ell_n) - Q(\ell_{n-2}) - Q(\ell_{n-3}) = 0$, we can get the conclusion we want.

Theorefore, the proof is completed. □

Theorem 3.5. *For the Ludovan quaternion $Q(\ell_n)$,*

$$\begin{aligned} Q(\ell_n) &= \sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m}{n-2m-2} \right\} \\ &\quad + i \sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m+1}{n-2m} \right\} \\ &\quad + j \sum_{m=0}^{n} \left\{ 3\binom{m+1}{n-2m} + 2\binom{m}{n-2m-2} \right\} \\ &\quad + k \sum_{m=0}^{n} \left\{ 2\binom{m+2}{n-2m} + \binom{m+1}{n-2m} + \binom{m}{n-2m-2} \right\}. \end{aligned}$$

Proof. Let us put $1+2i+3j+3k$ as A, $2+3i+3j+5k$ as B, and $2+i+2j+2k$ as C. Then, in the same way as the proof of Theorem 2.2, we have the generating function of the Ludovan quaternions as follows:

$$\begin{aligned} f(x) = \sum_{nge0} Q(\ell_n)x^n &= (A+Bx+Cx^2)\frac{1}{1-x^2-x^3} \\ &= \sum_{n\geq 0} \left\{ A\sum_{m=0}^{n} \binom{m}{n-2m} + B\sum_{m=0}^{n} \binom{m}{n-2m-1} \right. \\ &\qquad \left. + C\sum_{m=0}^{n} \binom{m}{n-2m-2} \right\} x^n. \end{aligned}$$

Now using $A = 1+2i+3j+3k$, $B = 2+3i+3j+5k$ and $C = 2+i+2j+3k$, we can get

$$\begin{aligned} Q(\ell_n) &= \sum_{m=0}^{n} \left\{ \binom{m}{n-2m} + 2\binom{m}{n-2m-1} + 2\binom{m}{n-2m-2} \right\} \\ &\quad + i \sum_{m=0}^{n} \left\{ 2\binom{m}{n-2m} + 3\binom{m}{n-2m-1} + \binom{m}{n-2m-2} \right\} \\ &\quad + j \sum_{m=0}^{n} \left\{ 3\binom{m}{n-2m} + 3\binom{m}{n-2m-1} + 2\binom{m}{n-2m-2} \right\} \\ &\quad + k \sum_{m=0}^{n} \left\{ 3\binom{m}{n-2m} + 5\binom{m}{n-2m-1} + 3\binom{m}{n-2m-2} \right\}. \end{aligned}$$

Using the properties of binomial coefficients, we can get the conclution we want.

Therefore, the proof is completed. □

Corollary 3.6. *For the nth Ludovan number ℓ_n, we have the following identities.*

(i) $\ell_n = \sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m}{n-2m-2} \right\}$

(ii) $\ell_{n+1} = \sum_{m=0}^{n+1} \left\{ \binom{m+2}{n-2m+1} + \binom{m}{n-2m-1} \right\}$

(iii) $\ell_{n+2} = \sum_{m=0}^{n+2} \left\{ \binom{m+2}{n-2m+2} + \binom{m}{n-2m} \right\}$

(iv) $\ell_{n+3} = \sum_{m=0}^{n+3} \left\{ \binom{m+2}{n-2m+3} + \binom{m}{n-2m+1} \right\}$

(v) $\sum_{m=0}^{n+1} \left\{ \binom{m+2}{n-2m+1} + \binom{m}{n-2m-1} \right\} = \sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m+1}{n-2m} \right\}$

(vi) $\sum_{m=0}^{n+2} \left\{ \binom{m+2}{n-2m+2} + \binom{m}{n-2m} \right\} = \sum_{m=0}^{n} \left\{ 3\binom{m+1}{n-2m} + 2\binom{m}{n-2m-2} \right\}$

(vii) $\sum_{m=0}^{n+3} \left\{ \binom{m+2}{n-2m+3} + \binom{m}{n-2m+1} \right\}$
$= \sum_{m=0}^{n} \left\{ 2\binom{m+2}{n-2m} + \binom{m+1}{n-2m} + \binom{m}{n-2m-2} \right\}$

(viii) $\sum_{m=0}^{n+3} \left\{ \binom{m+2}{n-2m+3} + \binom{m}{n-2m+1} \right\}$
$= \sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m}{n-2m-2} \right\} + \sum_{m=0}^{n+1} \left\{ \binom{m+2}{n-2m+1} + \binom{m}{n-2m-1} \right\}$

(ix) $\sum_{m=0}^{n} \left\{ \binom{m+2}{n-2m} + \binom{m}{n-2m-2} \right\} = \sum_{m=0}^{n-1} \binom{m+1}{n-2m-1} + \sum_{m=0}^{n+1} \binom{m+1}{n-2m+1}$

Proof. Since $Q(\ell_n) = \ell_n + \ell_{n+1}i + \ell_{n+2}j + \ell_{n+3}k$, $\ell_{n+3} = \ell_{n+1} + \ell_n$ and $\ell_n = P_{n-1} + P_{n+1}$, the proof can be easily completed by applying Theorem 3.5. □

Theorem 3.7. *For the nth Ludovan quaternion $Q(\ell_n)$,*

$$\sum_{m=0}^{n} Q(\ell_m) = Q(\ell_{n+2}) + Q(\ell_{n+3}) - Q(\ell_4).$$

Proof. If $n = 0$ and $n = 1$, then the result is obviously true. We assume that it is true for n. Then, we have

$$\begin{aligned}\sum_{m=0}^{n+1} Q(\ell_m) &= \sum_{m=0}^{n} Q(\ell_m) + Q(\ell_{n+1}) \\ &= Q(\ell_{n+2}) + Q(\ell_{n+3}) - Q(\ell_4) + Q(\ell_{n+1}).\end{aligned}$$

Since $Q(\ell_{n+2}) + Q(\ell_{n+1}) = Q(\ell_{n+4})$, we can get

$$\sum_{m=0}^{n+1} Q(\ell_m) = Q(\ell_{n+3}) + Q(\ell_{n+4}) - Q(\ell_4).$$

Therefore, by induction on n, the proof is completed. □

Theorem 3.8. *For the nth Ludovan quaternion* $Q(\ell_n)$*,*

(i) $\sum_{m=0}^{n} Q(\ell_{2m}) = Q(\ell_{2n+3}) - Q(\ell_1)$
(ii) $\sum_{m=0}^{n} Q(\ell_{2m+1}) = Q(\ell_{2n+4}) - Q(\ell_2)$

Proof. The theorem is proved by induction on n. □

4. Matrix Representations

In this section, we consider a matrix representation of the Padovan quaternions and the Ludovan quaternions.

In [11], the author gave a matrix representation of Padovan quaternions as follows: for $n \geq 1$,

$$\begin{bmatrix} Q(P_n) & Q(P_{n+2}) & Q(P_{n+1}) \\ Q(P_{n+1}) & Q(P_{n+3}) & Q(P_{n+2}) \\ Q(P_{n+2}) & Q(P_{n+4}) & Q(P_{n+3}) \end{bmatrix} = \mathcal{S}^n \begin{bmatrix} Q(P_0) & Q(P_2) & Q(P_1) \\ Q(P_1) & Q(P_3) & Q(P_2) \\ Q(P_2) & Q(P_4) & Q(P_3) \end{bmatrix}, \tag{1}$$

where

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Theorem 4.1. *For the nth Padovan quaternions* $Q(P_n)$*,*

$$N\left(\begin{vmatrix} Q(P_n) & Q(P_{n+2}) & Q(P_{n+1}) \\ Q(P_{n+1}) & Q(P_{n+3}) & Q(P_{n+2}) \\ Q(P_{n+2}) & Q(P_{n+4}) & Q(P_{n+3}) \end{vmatrix}\right) = 13.$$

Proof. Since $|\mathcal{S}| = 1$, we have $|\mathcal{S}^n| = 1$ and hence, from (1),

$$\begin{aligned} \begin{vmatrix} Q(P_n) & Q(P_{n+2}) & Q(P_{n+1}) \\ Q(P_{n+1}) & Q(P_{n+3}) & Q(P_{n+2}) \\ Q(P_{n+2}) & Q(P_{n+4}) & Q(P_{n+3}) \end{vmatrix} &= \begin{vmatrix} Q(P_0) & Q(P_2) & Q(P_1) \\ Q(P_1) & Q(P_3) & Q(P_2) \\ Q(P_2) & Q(P_4) & Q(P_3) \end{vmatrix} \\ &= 2 - 2i + 2j + k. \end{aligned}$$

Hence, we have the norm of $2 - 2i + 2j + k$ as follows:

$$N(2 - 2i + 2j + k) = 2^2 + (-2)^2 + 2^2 + 1^2 = 13.$$

Therefore, the proof is completed □

Now, we consider the matrix representation of the Ludovan quaternions.

Theorem 4.2. *For the nth Ludovan quaternion $Q(\ell_n)$,*

$$\begin{bmatrix} Q(\ell_n) & Q(\ell_{n+2}) & Q(\ell_{n+1}) \\ Q(\ell_{n+1}) & Q(\ell_{n+3}) & Q(\ell_{n+2}) \\ Q(\ell_{n+2}) & Q(\ell_{n+4}) & Q(\ell_{n+3}) \end{bmatrix} = \mathcal{S}^n \begin{bmatrix} Q(\ell_0) & Q(\ell_2) & Q(\ell_1) \\ Q(\ell_1) & Q(\ell_3) & Q(\ell_2) \\ Q(\ell_2) & Q(\ell_4) & Q(\ell_3) \end{bmatrix}$$

Proof. The theorem is proved by induction on n. □

Corollary 4.3. *For the nth Ludovan quaternions $Q(\ell_n)$,*

$$\begin{bmatrix} Q(\ell_n) \\ Q(\ell_{n+1}) \\ Q(\ell_{n+2}) \end{bmatrix} = \mathcal{S}^n \begin{bmatrix} Q(\ell_0) \\ Q(\ell_1) \\ Q(\ell_2) \end{bmatrix}$$

Theorem 4.4. *For the nth Ludovan quaternions $Q(\ell_n)$,*

$$N\left(\begin{vmatrix} Q(\ell_n) & Q(\ell_{n+2}) & Q(\ell_{n+1}) \\ Q(\ell_{n+1}) & Q(\ell_{n+3}) & Q(\ell_{n+2}) \\ Q(\ell_{n+2}) & Q(\ell_{n+4}) & Q(\ell_{n+3}) \end{vmatrix}\right) = 1141.$$

Proof. Since $|\mathcal{S}| = 1$, we have $|\mathcal{S}^n| = 1$ and hence, from Theorem 4.2,

$$\begin{aligned} \begin{vmatrix} Q(\ell_n) & Q(\ell_{n+2}) & Q(\ell_{n+1}) \\ Q(\ell_{n+1}) & Q(\ell_{n+3}) & Q(\ell_{n+2}) \\ Q(\ell_{n+2}) & Q(\ell_{n+4}) & Q(\ell_{n+3}) \end{vmatrix} &= \begin{vmatrix} Q(\ell_0) & Q(\ell_2) & Q(\ell_1) \\ Q(\ell_1) & Q(\ell_3) & Q(\ell_2) \\ Q(\ell_2) & Q(\ell_4) & Q(\ell_3) \end{vmatrix} \\ &= 22 - 22i + 2j + 13k. \end{aligned}$$

Hence, we have the norm of $22 - 22i + 2j + 13k$ as follows:

$$N(22 - 22i + 2j + 13k) = 1141.$$

Therefore, the proof is completed. □

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[a]Professor: Department of Mathematics, Hanseo University, Seosan, 31962, Korea
Email address: gylee@hanseo.ac.kr

[b]Professor: Department of IT Convergence Software, Seoul Theological University, Bucheon, 14754, Korea
Email address: kisoeb@stu.ac.kr