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RIEMANN-STIELTJES INTEGRALS AND THEIR REPRESENTING MEASURES

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Abstract. The Riemann-Stieltjes integrals of continuous functions with respect to a function of bounded variation can be represented by a regular, Borel, complex measure. In this paper, we study the link between the Riemann-Stieltjes integral and measure theory using this representation. Specifically, we investigate the Riemann-Stieltjes integrability and its measurability. Furthermore, we derive a criterion for Riemann-Stieltjes integrability through a method different from known proofs. In particular, we calculate the upper and lower Riemann-Stieltjes integrals with respect to a monotone increasing function.

1. Introduction

The Riemann-Stieltjes integral is a natural generalization of the Riemann integral, with its definition detailed in [5]. Its intuitive and accessible nature makes it a valuable tool for undergraduate students to grasp the concept of integration with respect to a weight function before moving on to Lebesgue integrals. Additionally, it proves invaluable in statistics, applicable to both discrete and continuous probabilities without relying on measure theory.

Typically, undergraduate students encounter the Riemann-Stieltjes integral in their real analysis courses and then study measure theory separately, usually in graduate-level courses. However, the natural connection between these topics is often not addressed in measure theory courses.

While some measure theory textbooks discuss Lebesgue's criterion for Riemann integrability and the Lebesgue measurability of Riemann integrable functions, the link between the Riemann-Stieltjes integral and measure theory is frequently overlooked. This paper aims to bridge this gap by presenting the following topics:

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(i) The representing measure for the Riemann-Stieltjes integral as given by the Riesz representation theorem.
(ii) Functions of bounded variation and Borel measures on the real line.
(iii) The criterion for Riemann-Stieltjes integrability within the framework of measure theory.

2. Preliminaries

In this section, we briefly review the basic properties of Riemann-Stieltjes integrals. Most of them are well-known and can be found in the textbook [1].

2.1. Riemann-Stieltjes Integral Let $[a, b]$ be a compact interval in $\mathbb{R}$. In this section, all functions will be assumed to be bounded on $[a, b]$. A *partition* P of $[a, b]$ is a finite set of points

$$P = \{x_0, \dots, x_n\}$$

such that $a, b \in P$ and $a = x_0 < x_1 < \cdots < x_n = b\}$. A partition P' of $[a, b]$ is said to be *finer* than P (or a *refinement* of P) if $P \subset P'$. Let α be a bounded function on $[a, b]$.

The set of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$. The *norm* of a partition P is the largest length of subintervals of P and is denoted by $\|P\|$. We clearly get $\|P'\| \le \|P\|$ whenever $P \subset P'$.

Definition 2.1. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$ and let t_k be a point in the subinterval $[x_{k-1}, x_k]$. A *Riemann-Stieltjes sum* of f with respect to α is a sum of the form

$$S(f, \alpha, P) = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1})).$$

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* with respect to α and we write $f \in R(\alpha)$ if there is a number I having the following property: For every $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that for every partition P finer than P_ϵ and for every choice of the points t_k in $[x_{k-1}, x_k]$ we have

$$|S(f, \alpha, P) - I| < \epsilon.$$

If such a number I exists, then it is uniquely determined and is denoted by

$$\int_a^b f \, d\alpha.$$

The functions f and α are referred to as the *integrand* and the *integrator*. We also say that the Riemann-Stieltjes integral $\int_a^b f d\alpha$ exists. In the case that $\alpha(x) = x$, we write $S(f, P)$ instead of $S(f, \alpha, P)$ and $f \in R$ instead of $f \in R(\alpha)$. Then integral is called a *Riemann integral* and is denoted by

$$\int_a^b f\, dx.$$

Some basic properties of Riemann-Stieltjes integrals are proved in [1] including the linearity. Another useful property related to the linearity of integrators is presented as follows:

Proposition 2.2. *If f, α, and β are bounded on $[a, b]$ and $f \in R(\alpha) \cap R(\beta)$, then $f \in R(\alpha + c\beta)$ for all real numbers c. Moreover, we have*

$$\int_a^b f\, d(\alpha + c\beta) = \int_a^b f\, d\alpha + c \int_a^b f\, d\beta.$$

Integration by parts is one of the most intriguing properties of the Riemann-Stieltjes integral. The proof is standard and can be found in [1].

Theorem 2.3. *Suppose that f and α are bounded functions on $[a, b]$. If $f \in R(\alpha)$, then $\alpha \in R(f)$ and*

$$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a).$$

The Riemann-Stieltjes integrability depends on the continuity of both α and f. Consider the identity function f on a compact interval $[0, 1]$ and the function α defined to have the value 1 for irrational numbers and 0 for rational numbers. It is easy to check that α is not Riemann integrable. Hence $f \notin R(\alpha)$ by Theorem 2.3. This shows that $f \mapsto \int f d\alpha$ is not defined even for continuous functions f if α does not behave well. Therefore, we focus on functions α of bounded variation, which are continuous except at countably many points. Since a real-valued function on $\mathbb{R}$ is of bounded variation if and only if it is the difference of two monotone increasing functions, we begin with monotone increasing functions.

2.2. Monotone Increasing Integrators Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ and f, α be bounded functions on $[a, b]$. Set

$$M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

for each $k = 1, \dots, n$. The *upper sum* $U(f, \alpha, P)$ of f with respect to α is

$$U(f, P) = \sum_{k=1}^{n} M_k \Delta\alpha_k$$

and the *lower sum* $L(f, P)$ of f with respec to α is

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta\alpha_k.$$

Note if α is monotone increasing on $[a, b]$, we have

$$L(f, \alpha, P) \leq S(f, \alpha, P) \leq M(f, \alpha, P).$$

The following observations are clear:

Definition 2.4. Assume that α is monotone increasing and f is bounded on $[a, b]$. The *upper integral* of f with respect to α is defined to be

$$(U) \int_a^b f d\alpha = \inf\{U(f, \alpha, P) : P \in \mathcal{P}[a, b]\}.$$

The *lower integral* of f with respect to α is defined to be

$$(L) \int_a^b f d\alpha = \sup\{L(f, \alpha, P) : P \in \mathcal{P}[a, b]\}$$

We clearly get

$$(L) \int_a^b f d\alpha \leq (U) \int_a^b f d\alpha.$$

A monotone increasing function α can be decomposed into a monotone increasing function γ determined by jumps of α and a continuous monotone increasing function β, as detailed in Theorem 3.8. Using this decomposition, we will provide a complete description of the upper and lower integrals of bounded functions in Theorem 4.1 and Theorem 4.3.

A Riemann-Stieltjes integrability with respect to a monotone increasing function can be characterized by the following Riemann condition with respect to α.

Definition 2.5. Assume that f, α are bounded functions on $[a, b]$. We say that f satisfies *Riemann condition* with respect to α on $[a, b]$ if, for every $\epsilon > 0$, there is a partition P_ϵ such that, whenever P is finer than P_ϵ, we get

$$0 \leq U(f, \alpha, P) - L(f, \alpha, P) < \epsilon.$$

Note that if α is monotone increasing on $[a, b]$, then Riemann's condition is equivalent to that for each $\epsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(f, \alpha, P) - L(f, \alpha, P) < \epsilon.$$

The following theorem is proved in [1].

Theorem 2.6. *Assume that α is monotone increasing and f is bounded on $[a, b]$. TFAE.*

(1) $f \in R(\alpha)$ on $[a, b]$
(2) f satisfies Riemann's condition with respect to α on $[a, b]$.
(3) $(U)\int_a^b f d\alpha = (L)\int_a^b f d\alpha$.

In this case, we have

$$(U)\int_a^b f d\alpha = (L)\int_a^b f d\alpha = \int_a^b f d\alpha.$$

2.3. BV functions on $\mathbb{R}$ Let $f : \mathbb{R} \to \mathbb{C}$ be a function and we define the *total variation function* $V(f)$ by:

$$V(f)(x) = \sup\left\{\sum_1^n |f(x_j) - f(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\}.$$

We note that the sums in the definition of $V(f)$ increase when additional subdivision points x_j are added. Therefore, if $a < b$, the value of $V(f)(b)$ remains unaffected if a is included among the subdivision points. Consequently,

$$V(f)(b) - V(f)(a) = \sup\left\{\sum_1^n |f(x_j) - f(x_{j-1})| : n \in \mathbb{N}, a = x_0 < \cdots < x_n = b\right\}.$$

Thus $V(f)$ is a monotone increasing function with values in $[0, \infty]$. If $V(f)(\infty) := \lim_{x\to\infty} V(f)(x)$ is finite, we say that f is of *bounded variation* on $\mathbb{R}$, and denote the space of all such functions f by BV.

Sometimes we use the notation $V(f)([a, b]) = V(f)(b) - V(f)(a)$ to consider the total variation of f on the closed interval $[a, b]$.

Let $BV([a, b])$ be the set of functions on $[a, b]$ whose total variation on $[a, b]$ is finite. If $f \in BV$, the restriction of f to $[a, b]$ is in $BV([a, b])$ for all a, b. Conversely, if $f \in BV([a, b])$ and we extend f to $\mathbb{R}$ by defining $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$, then $f \in BV$.

We present the following three properties of BV functions, which will be useful later. For detailed proofs, refer to [2]. Essentially, BV functions can be expressed as the difference of two monotone increasing functions and they are continuous except at countably many points. Additionally, it is useful to note that if f is BV, the right (and. left) continuity of f is preserved in the bounded variation function $V(f)$.

Lemma 2.7. *If $f \in BV$ and f is real-valued, then $V(f) + f$ and $V(f) - f$ are monotone increasing.*

Theorem 2.8. a. *$f \in BV$ if and only if* $\operatorname{Re} f \in BV$ *and* $\operatorname{Im} f \in BV$.

b. *If $f : \mathbb{R} \to \mathbb{R}$, then $f \in BV$ if and only if f is the difference of two bounded increasing functions.*

c. *If $f \in BV$, then $f(x+) = \lim_{y\to x+} f(y)$ and $f(x-) = \lim_{y\to x-} f(y)$ exist for all $y \in \mathbb{R}$, as do $F(\pm\infty) = \lim_{y\to\pm\infty} F(y)$.*

d. *If $f \in BV$, then the set of points at which f is discontinuous is countable.*

Lemma 2.9. *Suppose that $f \in BV$. Then $V(f)(-\infty) = 0$ and f is right (resp. left) continuous at x if and only if $V(f)$ is right (resp. left) continuous at x.*

If α is BV and f is bounded on $[a, b]$, then the Riemann-Stieltjes integrability of f with respect to α is equivalent to that of f with respect to the total variation function $V(\alpha)$. The complete proof is given below; the neccesity is proved in [1].

Theorem 2.10. *Assume that α is of bounded variation on $[a, b]$. Let $V(x)$ be the total variation of α on $[a, x)$ if $a < x \le b$, and let $V(a) = 0$. Suppose that f is a bounded function on $[a, b]$. $f \in R(\alpha)$ if and only if $f \in R(V)$ on $[a, b]$.*

Proof. If $V(b) = 0$, then α is constant and the result is trivial. So we assume that $V(b) > 0$. Let $M = \sup\{|f(x)| : x \in [a, b]\}$. We need verify that f satisfies Riemann's condition with respect to V.

Given $\epsilon > 0$, choose P_ϵ so that for any partition P finer than P_ϵ and all choices of points t_k and t'_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n}(f(t_k) - f(t'_k))\Delta\alpha_k\right| < \frac{\epsilon}{4} \quad \text{and} \quad V(b) < \sum_{k=1}^{n}|\Delta\alpha_k| + \frac{\epsilon}{4M},$$

where $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ for all $1 \le k \le n$. Let

$$R = \sup\left\{\sum_{k=1}^{n}(f(t_k) - f(t'_k))\Delta\alpha_k : t_k, t'_k \in [x_{k-1}, x_k], 1 \le k \le n\right\}$$

Note that

$$\sum_{k=1}^{n}(M_k(f) - m_k(f))|\Delta\alpha_k| = R < \frac{\epsilon}{2}.$$

Indeed, it is clear that lefthand side is bigger than or equal to R. On the other hand, let $A = \{k : \Delta\alpha_k \geq 0\}$ and $B = \{k : \Delta\alpha_k < 0\}$. Given t_k, t'_k in $[x_{k-1}, x_k]$ we have

$$\sum_{k=1}^{n}(f(t_k) - f(t'_k))|\Delta\alpha_k| \leq \sum_{k\in A}(f(t_k) - f(t'_k))\Delta\alpha_k + \sum_{k\in B}(f(t'_k) - f(t_k))\Delta\alpha_k$$
$$\leq \left|\sum_{k\in A}(f(t_k) - f(t'_k))\Delta\alpha_k + \sum_{k\in B}(f(t'_k) - f(t_k))\Delta\alpha_k\right| \leq R$$

Since t_k, t'_k are arbitrary points in $[x_{k-1}, x_k]$, we have

$$\sum_{k=1}^{n}(M_k(f) - m_k(f))|\Delta\alpha_k| \leq R < \frac{\epsilon}{2}.$$

On the other hand,

$$\sum_{k=1}^{n}(M_k(f) - m_k(f))(\Delta V_k - |\Delta\alpha_k|) \leq 2M\sum_{k=1}^{n}(\Delta V_k - |\Delta\alpha_k|)$$
$$= 2M(V(b) - \sum_{k=1}^{n}|\Delta\alpha_k|) < \frac{\epsilon}{2}.$$

Hence

$$\sum_{k=1}^{n}(M_k(f) - m_k(f))|\Delta V_k| < \epsilon.$$

It follows that f satisfies Riemann's condition and $f \in R(V)$.

Conversely, suppose that $f \in R(V)$ and let $\alpha_1 = (V + \alpha)$, $\alpha_2 = (V - \alpha)$. For any $x \leq y$ in $[a, b]$ we get

$$\alpha_1(y) - \alpha_1(x) = \sup_{x=x_0<x_1<\cdots<x_n=y, n\in\mathbb{N}} \left\{\sum_{j=1}^{n}\{|\alpha(x_j) - \alpha(x_{j-1})| + (\alpha(x_j) - \alpha(x_{j-1})\}\right\}$$
$$= 2\sup\left\{\sum_{j=1}^{n}((\alpha(x_j) - \alpha(x_{j-1}))^+ : x = x_0 < x_1 < \cdots < x_n = y, n \in \mathbb{N}\right\}$$
$$\leq 2\sup\left\{\sum_{j=1}^{n}|(\alpha(x_j) - \alpha(x_{j-1})| : x = x_0 < x_1 < \cdots < x_n = y, n \in \mathbb{N}\right\}$$
$$= 2(V(y) - V(x)).$$

Hence $U(f, P, \alpha_1) - L(f, P, \alpha_1) \leq 2(U(f, P, V) - L(f, P, V))$ for any partition of P of $[a, b]$. This shows that $f \in V(\alpha_1)$ and, similarly we have $f \in V(\alpha_2)$. Therefore, $f \in V(\alpha_1 + \alpha_2)$ by Proposition 2.2. □

If $f \in R(\alpha)$ and α is of bounded variation on $[a, b]$, then $f \in R(V)$ on $[a, b]$. So $f \in R(V - \alpha)$ and $f \in R(V + \alpha)$. Conversely, if $f \in R(V - \alpha)$ and $f \in R(V + \alpha)$, then $f \in R(\alpha)$ since $2\alpha = (V + \alpha) - (V - \alpha)$. So the theory of Rieman-Stieltjes interation for integrators of bounded variation can be reduced to the case of increasing integrators.

The following two theorems are useful later. For detailed proof, see [1].

Theorem 2.11. *Assume that bounded functions $f, g \in R(\alpha)$ on $[a, b]$, where α is monotone increasing on $[a, b]$. Let $F(x) = \int_a^x f d\alpha$ and $G(x) = \int_a^x g d\alpha$ for $x \in [a, b]$. Then $f \in R(G)$ and $g \in R(F)$ and we have*

$$\int_a^b fg d\alpha = \int_a^b f dG = \int_a^b g dF.$$

Theorem 2.12. *If f is continuous on $[a, b]$ and if α is of bounded variation on $[a, b]$, then $f \in R(\alpha)$ on $[a, b]$.*

Now we get the basic inequality for Riemann-Stieltjes integrals.

Theorem 2.13. *If α is of bounded variation on $[a, b]$ and $f \in R(\alpha)$, then both f and $|f| \in R(V(\alpha))$. Moreover, we have*

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f(x)| dV(\alpha) \leq \sup_{x \in [a,b]} |f(x)| V(\alpha)([a, b]).$$

Proof. By Theorem 2.10, $f \in R(\alpha)$ if and only if $f \in R(V(\alpha))$. It is shown in [1] that if $f \in R(V(\alpha))$, then $|f| \in R(V(\alpha))$. It is enough to show that

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f(x)| dV(\alpha).$$

Given $\epsilon > 0$, choose P_ϵ so that for any partition P finer than P_ϵ and all choices of points t_k and t'_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^n f(t_k)(\alpha(x_k) - \alpha(x_{k-1})) - \int_a^b f \, d\alpha \right| < \epsilon$$

and

$$\left| \sum_{k=1}^n |f(t'_k)|(V(\alpha)(x_k) - V(\alpha)(x_{k-1})) - \int_a^b |f| \, dV(\alpha) \right| < \epsilon.$$

So

$$\begin{aligned}\left|\int_a^b f\,d\alpha\right| &\le \sum_{k=1}^n |f(t_k)|\,|\alpha(x_k)-\alpha(x_{k-1})| + \epsilon \\ &\le \sum_{k=1}^n |f(t_k)|\,|V(\alpha)(x_k)-V(\alpha)(x_{k-1})| + \epsilon \\ &\le \sum_{k=1}^n |f(t_k)|\,|V(\alpha)(x_k)-V(\alpha)(x_{k-1})| + \epsilon \\ &\le \int_a^b |f| dV(\alpha) + 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get the desired result. □

3. Increasing Functions and its Representing Measures

Recall the following useful theorem, which demonstrates that every Borel measure on $\mathbb{R}$ that is finite on all compact subsets can be obtained from a right-continuous, monotone increasing function on $\mathbb{R}$. For the proof, see [2].

Theorem 3.1. *If $\alpha : \mathbb{R} \to \mathbb{R}$ is any increasing, right-continuous function, there is a unique Borel measure μ_α on $\mathbb{R}$ such that*

$$\mu_\alpha((a,b]) = \alpha(b) - \alpha(a)$$

for any a, b. If β is another such function, we have $\mu_\alpha = \mu_\beta$ if $\alpha - \beta$ is constant. Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all compact subsets and we define

$$\alpha(x) = \begin{cases} \mu((0,x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((x,0] & \text{if } x < 0, \end{cases}$$

then α is increasing and right-continuous and $\mu = \mu_\alpha$.

The completion of μ_α is called the *Lebesgue-Stieltjes measure* associated to α, which will be denoted by $\overline{\mu}_\alpha$. Note that the completion of Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ with respect to μ_α is typicallty larger than $\mathcal{B}_{\mathbb{R}}$.

Now it is natural question to ask whether the Riemann-Stieltjes integral $\int f d\alpha$ is equal to $\int f d\mu_\alpha$ for every Riemann-Stieltjes integrable function f when α is a monotone increasing function. This fact will be established in Theorem 4.5. To prove this, we first observe that the Riemann-Stieltjes integrals induce bounded linear functionals on the space $C([a,b])$ of all complex-valued continuous functions on a compact interval $[a,b]$. Consequently, we expect that the induced Radon measure μ associated to the linear functional is equal to the Lebesgue-Stieltjes measure μ_α and so we get

$$\int_{[a,b]} f d\alpha = \int f \, d\mu_\alpha$$

for every $f \in C([a.b])$. Let's examine these observations carefully.

Let α be a function of bounded variation on $[a,b]$. We assume that α is defined on $\mathbb{R}$ by defining $\alpha(x) = \alpha(a)$ if $x < a$ and $\alpha(x) = \alpha(b)$ if $x > b$. If a right limit of a function f exists at a, we use the notation $f(a+) = \lim_{x \to a+} f(x)$. Similarly, $f(a-)$ is defined as the left limit. Given a function f on a domain A, we define the sup norm $\|f\|_\infty$ as follows:

$$\|f\|_\infty = \sup\{|f(x)| : x \in A\}.$$

Theorem 3.2. *Let α be a function of bounded variation on $[a,b]$. Then there is a unique real-valued Borel measure μ on $\mathbb{R}$ such that for every $f \in C([a,b])$,*

$$\int_a^b f \, d\alpha = \int_{[a,b]} f \mu.$$

In particular, μ is the unique real-valued Borel measure satisfying, for $p < q$,

$$\mu((p,q]) = \alpha(q+) - \alpha(p+).$$

Proof. Every continuous function f on $[a,b]$ is Riemann-Stieltjes integrable and

$$I(f) = \int_a^b f(t) d\alpha(t)$$

is a bounded linear functional on $C[a,b]$. By Proposition 2.13, $|I(f)| \leq \|f\|_\infty V_\alpha(b)$, where $V_\alpha(x)$ is the total variation of α on $[a,x]$, where $a < x \leq b$. By the Riesz representation theorem, there is a real-valued Borel regular measure μ such that $I(f) = \int f d\mu$ for every $f \in C[a,b]$. For each Borel subset E of $\mathbb{R}$, $\nu(E) = \mu([a,b] \cap E)$ is an extension of μ. So we use this extension and assume that μ is defined on $\mathbb{R}$ and $|\mu|(\mathbb{R} \setminus [a,b]) = 0$. By the integration by parts, α is Riemann integrable with respect to $f \in C([a,b])$ and we get

$$I(f) = -\int_a^b \alpha df + f(b)\alpha(b) - f(a)\alpha(a).$$

Suppose that $a < s_n < s < t < t_n < b$ and let $f_n(t)$ be the piecewise linear function such that $f_n(x) = 1$ on $[s,t]$ and 0 on the outside of $[s_n, t_n]$. Note that

$f_n(x) = \int_a^x g_n(x)dx$, where $g_n(x) = 0$ on $[a, s_n] \cup [s, t] \cup [t_n, b]$, $g(x) = \frac{1}{s-s_n}$ on (s_n, s) and $g(x) = \frac{1}{t-t_n}$ on (t, t_n). Then, by Theorem 2.11

$$\int_a^b f_n d\alpha = -\int_a^b \alpha df_n = -\int_a^b \alpha(x)g_n(x)dx.$$

So if $s_n \to s$ and $t_n \to t$, then

$$\begin{aligned}\mu([s,t]) &= \lim_n \int f_n d\mu = \lim_n \int_a^b f_n d\alpha \\ &= \lim_n \left(-\frac{1}{s-s_n}\int_{s_n}^{s} \alpha(x)dx + \frac{1}{t_n - t}\int_t^{t_n} \alpha(x)dx \right) \\ &= \alpha(t+) - \alpha(s-).\end{aligned}$$

Letting $s \to a$, we get

$$\mu((a,t]) = \alpha(t+) - \alpha(a+).$$

If $a < t < t_n < b$, let let $f_n(t)$ be the piecewise linear function such that $f_n(x) = 1$ on $[a, t]$ and 0 on the outside of $[a, t_n]$. Note that $f_n(x) = 1 + \int_a^x g_n(x)dx$, where $g_n(x) = 0$ on $[a, t] \cup [t_n, b]$, and $g(x) = \frac{1}{t-t_n}$ on (t, t_n).

Then by Theorem 2.3 and Theorem 2.11, we have

$$\int_a^b f_n d\alpha = -\int_a^b \alpha df_n - \alpha(a) = -\int_a^b \alpha(x)g_n(x)dx - \alpha(a).$$

So if $t_n \to t$, then

$$\begin{aligned}\mu([a,t]) &= \lim_n \int f_n d\mu = \lim_n \int_a^b f_n d\alpha \\ &= \lim_n \left(\frac{1}{t_n - t}\int_t^{t_n} \alpha(x)dx \right) - \alpha(a) \\ &= \alpha(t+) - \alpha(a).\end{aligned}$$

Since $\mu([a,b]) = \int 1 d\alpha = \alpha(b) - \alpha(a) = \alpha(b+) - \alpha(a-)$, we have $\mu((t,b]) = \alpha(b) - \alpha(t+)$. Therefore, we get $\mu((s,t]) = \alpha(t+) - \alpha(s+)$ for all $s, t \in \mathbb{R}$ with $s < t$ and μ is determined by the right-continuous function $\alpha(t+)$ $(t \in \mathbb{R})$.

Note that ν is another real-valued Borel measure such that $\mu((s,t]) = \nu((s,t])$ for all $s < t$. Since $\{E : \mu(E) = \nu(E), E \text{ is Borel}\}$ is σ-algebra containing the intervals $(s,t]$, it is the Borel σ-algebra. This proves the uniqueness of μ and the proof is done. □

The real-valued Borel regular measure μ induced from the functional $f \mapsto \int_a^b f\, d\alpha$ is called the *representing measure* for α. It will be denoted by μ_α. Recall that

$\alpha = \alpha_1 - \alpha_2$, where α_i are bounded monotone increasing functions. So for each i, there is the Lebesgue-Stieltjes measure μ_i such that

$$\mu_i((s,t]) = \alpha_i(t+) - \alpha_i(s+)$$

for all $s < t$ in $\mathbb{R}$ by Theorem 3.1. Hence $\mu_\alpha = \mu_1 - \mu_2$.

Corollary 3.3. *Let α, β be functions of bounded variation $[a,b]$. Then the following are equivalent:*

(1) $\int_a^b f d\alpha = \int_a^b f d\beta$ for all $f \in C[a,b]$.
(2) There is a constant c such that $\beta(t+) = \alpha(t+) + c$ for all $t \in \mathbb{R}$.

Corollary 3.4. *If μ is a real-valued Borel measure on $[a,b]$ and the function β on $\mathbb{R}$, defined by*

$$\beta(t) = \begin{cases} \mu([a,t]) & t \geq a \\ 0 & t < a, \end{cases}$$

then $\mu = \mu_\beta$.

Note that a function $\alpha : [a,b] \to \mathbb{C}$ is said to be of bounded variation on $[a,b]$ if and only if its real and complex parts are of bounded variation on $[a,b]$. It is easy to see that Theorem 3.2 can be extended to complex-valued functions of bounded variation. The complex-valued Borel regular measure μ_α is called the *representing measure* of BV α. There is a connection between the space of functions of bounded variations (in short, BV) and the space of complex Borel measures on $\mathbb{R}$. Recall the space NBV (N stands for "normalized".) defined by

$$NBV = \{f \in BV : f \text{ is right continuous and } f(-\infty) = 0\}.$$

Given a function $f : [a,b] \to \mathbb{C}$, we extend it to $\mathbb{R}$ by defining $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. let $V(f)(t) = V_f(t)$ be the total variation of f on $(-\infty, t]$. Suppose that $M([a,b])$ be the set of complex-valued Borel regular measures on $\mathbb{R}$ such that $|\mu|(\backslash[a,b]) = 0$. It is a Banach space with the total variation norm. Let $NBV[a,b]$ be the set of complex-valued functions α of bounded variation on a compact interval $[a,b]$. That is, α is BV on $\mathbb{R}$, $V_\alpha(a-) = 0$, $V_\alpha(x) = V_\alpha(b)$ for all $x > b$ and α is right-continuous. The following theorem shows that the total variation is the norm on $NBV([a,b])$ and $NBV([a,b])$ is isometrically isomorphic to $M([a,b])$.

Corollary 3.5. *Let $NBV([a,b])$ be the set of complex-valued functions α of bounded variation on a compact interval $[a,b]$. That is, $V_\alpha(a-) = 0$, $V_\alpha(b) = V_\alpha(x)$ for all*

$x > b$ and α is right-continuous. Suppose that $M([a,b])$ be the set of complex-valued Borel regular measures on $\mathbb{R}$ with $|\mu|(\mathbb{R} \setminus [a,b]) = 0$. Then the mapping $\Psi : M([a,b]) \to NBV([a,b])$, defined by

$$\Psi(\mu)(t) = \begin{cases} \mu([a,t]) & t \geq a \\ 0 & t < a, \end{cases}$$

is a surjective isometric isomorphism. In fact, the restricted measure $\mu|_{[a,t]}$ of μ to Borel σ-algebra on $[a,t]$ corresponds to the restricted function $\Psi(\mu)|_{[a,t]}$, so

$$V(\Psi(\mu))(t) = |\mu|([a,t])$$

for all $t \in [a,b]$.

Proof. It is clear that Ψ is a linear surjective isomorphism by Theorem 3.2. Let α be a function of bounded variation on $[a,b]$ and let $\Psi^{-1}(\alpha) = \mu_\alpha$. For each $f \in C[a,b]$ with $\|f\| \leq 1$, we have

$$\left| \int_{[a,b]} f d\mu_\alpha \right| \leq \left| \int_a^b f d\alpha \right| \leq \|f\|_\infty V(\alpha)(b).$$

Hence $\|\mu_\alpha\| \leq V(\alpha)(b)$. Conversely, if $a = t_0 < t_1 < \cdots < t_n = b$, then

$$\sum_{j=1}^{n} |\alpha(t_j) - \alpha(t_{j-1})| = \sum_{j-1}^{n} |\mu_\alpha((t_{j-1}, t_j])| \leq |\mu_\alpha|([a,b])$$

Hence $V(\alpha)(b) \leq ||\mu_\alpha||$. Hence the proof is done.

Given $t \in (a,b]$, let ν_α be the Borel regular complex measure such that

$$\int_a^t f d\alpha = \int_{[a,t]} f d\nu_\alpha.$$

Then $\nu_\alpha([a,s]) = \alpha(s) - \alpha(a) = \mu_\alpha([a,s])$ for all $a \leq s \leq t$. Hence $\nu_\alpha(E) = \mu_\alpha(E \cap [a,t])$ for all Borel set $E \in [a,t]$ and $V(\alpha)(t) = \|\nu_\alpha\| = |\mu_\alpha|([a,t])$. □

Since $M([a.b]) = C([a,b])^*$, the total variation norm in $M([a,b])$ is complete, hence we get the following result:

Corollary 3.6. *Let $NBV([a,b])$ be the set of complex-valued functions α of bounded variation on a compact interval $[a,b]$ such that $V(\alpha)(a-) = 0$, $V(\alpha)(x) = V(\alpha)(b)$ for all $x > b$ and α is right-continuous. Then the total variation $V(\alpha)(b)$ ($\alpha \in NBV([a,b])$) is a complete norm.*

There is a connection between BV and the space of complex Borel meausres on $\mathbb{R}$. Recall the space NBV (N stands for "normalized".) defined by

$$NBV = \{f \in BV : f \text{ is right continuous and } f(-\infty) = 0\}.$$

Since $[0,1]$ is homeomorphic to the extended real number system $\overline{\mathbb{R}}$, so NBV is isometrically isomorphic to the subspace $NBV_0([0,1])$ of $NBV([0,1])$ consists of $\alpha \in NBV([0,1])$ with $\alpha(0) = 0$ and we get the following from Corollary 3.5.

Corollary 3.7. *Let NBV be the set of complex-valued functions α of bounded variation on $\mathbb{R}$ such that $\lim_{t\to-\infty} \alpha(t) = 0$ and α is right-continuous. Suppose that $M(\mathbb{R})$ be the set of complex-valued Borel regular measures on $\mathbb{R}$. Then the mapping $\Psi : M(\mathbb{R}) \to NBV$, defined by $\Psi(\mu)(t) = \mu((-\infty, t])$ for all $t \in \mathbb{R}$, is a surjective isometric isomorphism. In fact, the restricted measure $\mu|_{(-\infty,t]}$ of μ to Borel σ-algebra on $(-\infty, t]$ corresponds to the restricted function $\Psi(\mu)|_{(-\infty,t]}$, so*

$$V(\Psi(\mu))(t) = |\mu|((-\infty, t])$$

for all $t \in \mathbb{R}$.

A complex Borel measure μ on $\mathbb{R}^n$ is called *discrete* if there is a countable set $\{x_j\} \subset \mathbb{R}^n$ and complex numbers c_j such that $\sum_j |c_j| < \infty$ and $\mu = \sum_j c_j \delta_{x_j}$, where δ_x is the point mass at x. On the other hand, μ is called *continuous* if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^n$. Any complex measure μ can be written uniquely as $\mu = \mu_d + \mu_c$, where μ_d is discrete and μ_c is continuous. Indeed, let $E = \{x : \mu(\{x\}) \neq 0\}$ for all $x \in \mathbb{R}^n$. Then for any countable subset F of E, the series $\sum_{x\in F} \mu(\{x\})$ converges absolutely to $\mu(F)$ and the set $\{x : |\mu(\{x\})| > 1/k\}$ is finite for all k. It follows that E is countable. Hence $\mu_d(A) = \mu(A \cap E)$ is discrete and $\mu_c(A) = \mu(A \setminus E)$ is continuous.

If μ is discrete, then $\mu \perp m$, where m is the Lebesgue measure and if $\mu \ll m$, then μ is continuous. Thus any (regular) complex Borel measure of $\mathbb{R}^n$ can be uniquely

$$\mu = \mu_d + \mu_{ac} + \mu_{sc},$$

where μ_d is discrete, μ_{ac} is absolutely continuous with respect to m, and μ_{sc} is singular continuous measure that is μ_{sc} is continuous but $\mu_{sc} \perp m$.

Now we decompose a monotone increasing function to its jump part and its continuous part.

Let $\gamma : [a,b] \to \mathbb{R}$ be an increasing function such that there is a sequence $\{x_k\} \in [a,b]$ such that $\gamma = \sum_{k=1}^{\infty} J_k(\gamma)$, where

$$J_k(\gamma)(x) = \begin{cases} 0 & \text{if } x < x_k \\ \gamma(x_k) - \gamma(x_k-) & \text{if } x = x_k \\ \gamma(x_k+) - \gamma(x_k-) & \text{if } x > x_k. \end{cases},$$

where right limit (resp. left limit) can be regarded as the right limit (resp. left limit) of extended γ to $\mathbb{R}$ by $\gamma(x) = g(a)$ if $x < a$ and $\gamma(x) = g(b)$ if $x > b$. Note that

$$\sum_{j=1}^{\infty}(\gamma(x_j+) - \gamma(x_j-)) \leq \gamma(b) - \gamma(a) < \infty.$$

Such a γ is called a *monotone increasing function determined by jumps at* $\{x_k\}$. Note that, in this case, the representing measure μ_γ is discrete.

Theorem 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. Then there are two monotone increasing functions f_c, f_d such that f_c is continuous on $\mathbb{R}$, f_d is determined by jumps of f and*

$$f = f_c + f_d.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function and D_f be the set of all discontinuity points of f. Since D_f is countable, we may assume that $D_f = \{x_k : k \in I\}$ for some countble set I and let

$$J_k(f)(x) = J_k(x) = \begin{cases} 0 & \text{if } x < x_k \\ f(x_k) - f(x_k-) & \text{if } x = x_k \\ f(x_k+) - f(x_k-) & \text{if } x > x_k. \end{cases}$$

So $|J_k(x)| \leq f(x_k+) - f(x_k-)$ for all $x \in \mathbb{R}$. Note that for each $a < b$ in $\mathbb{R}$, we get

$$\sum_{D_f \cap [a,b]} (f(x_j+) - f(x_j-)) \leq f(b) - f(a) < \infty.$$

Now define $\gamma(f) : \mathbb{R} \to \mathbb{R}$ to be a monotone increasing function by

$$\gamma(f) = \sum_{k \in I} J_k.$$

The Weierstrass M-test shows that $J(f)$ converges uniformly on each compact interval $[a,b]$ in $\mathbb{R}$. Hence we get $D_f = D_\gamma$. Let $g = f - J_k$. Then clearly, g is monotone increasing and $g(x) = f(x)$ on $(-\infty, x_k)$ and $g(x) = f(x) - (f(x_k+) - f(x_k-)$ on (x_k, ∞). So g is also monotone increasing on (x_k, ∞). Since

$$g(x_k) = f(x_k-) = g(x_k-) = g(x_k+),$$

g is continuous at x_k and g is monotone increasing on $\mathbb{R}$. So the jump at x_k is removed by subtracting f by J_k and the resulting function $f - J_k$ is continuous on $(\mathbb{R} \setminus D_f) \cup \{x_k\}$, monotone increasing on $\mathbb{R}$. The same reasoning shows that $f - (J_k + J_{k+1}) = (g - J_k) - J_{k+1}$ is also monotone increasing and it is continuous on $(\mathbb{R} \setminus D_f) \cup \{x_k, x_{k+1}\}$. Hence $f - \gamma(f) = \lim_n (f - \sum_1^n J_k)$ is monotone increasing and continuous on $\mathbb{R}$. □

Let D_f be the set of all discontinuous points of f. Let μ be the representing measure for a monotone increasing function f. That is, μ is the regular Borel measure satisfying $\mu((a,b]) = f(b+) - f(a+)$ for all real $a < b$. Then the set $\{x : \mu(\{x\}) \neq 0\}$ is the set D_f. This observation leads to the following theorem.

Theorem 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be BV and let μ be its representing measure. Then the following hold:*

a. *The total variation function V of f is represented by $|\mu|$.*
b. *The Jordan decompositions $f_1 = \frac{1}{2}(V+f)$, $f_2 = \frac{1}{2}(V-f)$ of f are represented by μ^+ and μ^-, respectively.*
c. *D_{f_1} and D_{f_2} are disjoint and $D_V = D_f = D_{f_1} \cup D_{f_2}$.*

Proof. Since $|\mu|$ is the representing measure of $V(f)$, (a) and (b) hold. $D_f = D_V$ is proved by Lemma 2.9. Note that $D_{f_1} = \{x \in \mathbb{R} : \mu^+(\{x\}) \neq 0\}$ and $D_{f_2} = \{x \in \mathbb{R} : \mu^-(\{x\}) \neq 0\}$ and $\mu^+ \perp \mu^-$. Hence they are disjoint. Since $|\mu| = \mu^+ + \mu^-$ and

$$\{x \in \mathbb{R} : |\mu|(\{x\}) \neq 0\} = \{x \in \mathbb{R} : \mu^+(\{x\}) \neq 0\} \cup \{x \in \mathbb{R} : \mu^-(\{x\}) \neq 0\}.$$

Therefore, we get $D_V = D_{f_1} \cup D_{f_2}$. □

4. Criterion for Existence of Riemann-Stietjes Integrals

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let $H(x) = \limsup_{y \to x} f(x)$ and $h(x) = \liminf_{y \to x} f(x)$. Suppose that α is continuous, monotone increasing and that μ is its representing measure for α. Then H, h are Borel measurable and we get*

$$(U)\int_a^b f(x)\,d\alpha(x) = \int H\,d\mu, \qquad (L)\int_a^b f(x)\,d\alpha(x) = \int h\,d\mu.$$

Proof. For each partition $P = \{x_0 = a < \cdots < x_n = b\}$, let

$$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\} \text{ and } m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$$

and put

$$G_P = \sum_{j=1}^{n} M_j(f)\chi_{(x_{j-1},x_j]}, \quad g_P = \sum_{j=1}^{n} m_j(f)\chi_{(x_{j-1},x_j]}.$$

Since μ is continuous, we get $U(f,P,\alpha) = \int G_p d\mu$ and $L(f,P,\alpha) = \int g_P d\mu$.

Recall that

$$(U)\int_a^b f(x)\,dx = \sup_P U(f,P,\alpha)$$

and

$$(L)\int_a^b f(x)\,dx = \inf_P L(f,P,\alpha),$$

where P ranges over all partions of $[a,b]$. By taking two sequences and taking their refinements, there is a sequence $\{P_n\}$ such that

$$\lim_n U(f,P_n,\alpha) = (U)\int_a^b f(x)\,d\mu$$

and

$$\lim_n L(f,P_n,\alpha) = (L)\int_a^b f(x)\,d\mu.$$

We may assume that $\|P_n\| < 1/n$ and $P_n \subset P_{n+1}$ for every n. The sequence $G_n = G_{P_n}$ monotonically decreases and the sequence $g_n = g_{P_n}$ monotonically increases. Let $G = \lim_n G_n$ and $g = \lim_n g_{P_n}$. So they are Borel measurable.

We claim that $G = H$ a.e. $[\mu]$. Given $\epsilon > 0$, for each $x \in I = [a,b]$, there is $\delta_x > 0$ such that

$$\sup_{|y-x|<\delta_x,\, y\in I} f(y) < H(x) + \epsilon.$$

Since I is compact and $\{B(x,\delta_x)\}_{x\in I}$ is an open cover of I, there is a finite subcover $\{B(x_j,\delta_j)\}_{j=1}^m$ of I. Let $\delta = \min_j \delta_j$. If $2/n < \delta$, $P_n = \{a = t_0 < \cdots < t_k = b\}$ and if $x \in [t_{j-1},t_j]$,

$$\sup\{f(y) : y \in [x_{j-1},x_j]\} \le \sup_{|y-x|<\delta, y\in I} f(y) < H(x) + \epsilon.$$

Hence $G_n \le H(x) + \epsilon$ and $G \le H(x) + \epsilon$. This holds for arbitrary $\epsilon > 0$, $G \le H$. On the other hand, $H(x) \le G_n(x)$ on each open subinterval (t_{j-1},t_j) of P_n. Let $T = \bigcup_n P_n$, then $H \le G_n$ on $I \setminus T$ for each n and $\mu(T) = 0$. So, $H \le G$ on $I \setminus T$. Therefore $G = H$ on $I \setminus T$. Since T is countable, H is Borel measurable and $G = H$ μ-a.e. The proof that $g = h$ μ-a.e. is similar to the previous case and we omit it. $\square$

The following result is a generalization of Lebesgue's criterion for the Riemann integrability, where, if α is the identity fuunction, its representing measure is the Lebesgue measure.

Theorem 4.2. *Let f be a bounded function and α be a monotone increasing and continuous function on $[a,b]$ and let D be the set of discontinuity points of f. Then $f \in R(\alpha)$ in $[a,b]$ if and only if D has μ-measure zero, where μ is the representing measure for α. In particular, f is $\bar{\mu}$-measurable and we have*

$$\int_a^b f d\alpha = \int f d\overline{\mu},$$

where $\bar{\mu}$ is the completion of μ.

Proof. Note that f is continuous at x if and only if $H(x) = h(x)$. Suppose that $\mu(D) = 0$. Then $h = H = f$ μ-a.e. since $h \leq f \leq H$. Hence

$$(U)\int_a^b f(x)\, d\alpha(x) = (L)\int_a^b f(x)\, d\alpha(x) = \int f d\overline{\mu}.$$

Hence f is Riemann-Stieltjes integrable and

$$\int_a^b f d\alpha = \int f d\overline{\mu}.$$

On the other hand, f is Riemann-Stieltjes integrable, then

$$\int (H - h)\, d\mu = 0.$$

Hnece $\mu(D) = 0$. The proof is done. □

Now we need calculate the upper and lower Riemann-Stieltjes integrals with respect to a monotone increasing function determined by jumps. To achieve this, we need to introduce some basic concepts.

Let f be a function defined on $[x, x+\delta)$ for some $\delta > 0$. We define its right-limsup and right-liminf as follows:

$$H_r(x) = \lim_{\delta \to 0} [\sup\{f(y) : x \leq y \leq x + \delta\}]$$
$$h_r(x) = \lim_{\delta \to 0} [\inf\{f(y) : x \leq y \leq x + \delta\}].$$

Likewise, we define its left-limsup and left-liminf. Let f be a function defined on $(x - \delta, x]$ for some $\delta > 0$ and let

$$H_l(x) = \lim_{\delta \to 0} [\sup\{f(y) : x - \delta \leq y \leq x\}]$$
$$h_l(x) = \lim_{\delta \to 0} [\inf\{f(y) : x - \delta \leq y \leq x\}].$$

Note that $H_l(x) = h_l(x)$ if and only if $f(x-)$ exists and $f(x-) = f(x)$. Similarly, $H_r(x) = h_r(x)$ if and only if $f(x+)$ exists and $f(x+) = f(x)$.

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and γ be a monotone increasing function determined by jumps at $\{x_k\}$. Then*

$$(U)\int_a^b f d\gamma = \sum_1^\infty H_r(x_k)(\gamma(x_k+) - \gamma(x_k)) + \sum_1^\infty H_l(x_k)(\gamma(x_k) - \gamma(x_k-))$$

$$(L)\int_a^b f d\gamma = \sum_1^\infty h_r(x_k)(\gamma(x_k+) - \gamma(x_k)) + \sum_1^\infty h_l(x_k)(\gamma(x_k) - \gamma(x_k-)).$$

Proof. Let $\gamma_n = \sum_{k=1}^n J_k(\gamma)$. We will show that

$$(U)\int_a^b f d\gamma_n = \sum_1^n H_r(x_k)(\gamma(x_k+) - \gamma(x_k)) + \sum_1^n H_l(x_k)(\gamma(x_k) - \gamma(x_k-)).$$

Let $P_m = \{a = s_0 < s_1 < x_1 < t_1 < s_2 < x_2 < t_2 < \cdots < s_n < x_n < t_n < t_{n+1} = b\}$ be a partition of $[a,b]$, where $\max\{t_j - s_j : 1 \le j \le n\} \le 1/m$. Then,

$$\begin{aligned} U(f, P_m, \gamma_n) = &\sum_{i=1}^n \sup\{f(x) : x \in [x_i, t_i]\}(\gamma(t_i) - \gamma(x_i)) \\ &+ \sum_{i=1}^n \sup\{f(x) : x \in [s_i, x_i]\}(\gamma(x_i) - \gamma(s_i)). \end{aligned}$$

For any partition Q of $[a,b]$ we get

$$\lim_m U(f, Q \cup P_m, \gamma_n) = \sum_1^n H_r(x_k)(\gamma(x_k+) - \gamma(x_k)) + \sum_1^n H_l(x_k)(\gamma(x_k) - \gamma(x_k-)).$$

So

$$(U)\int_a^b f d\gamma_n \le \lim_m U(f, P_m, \gamma_n).$$

Given $\epsilon > 0$, there is a partition P of $[a,b]$ such that

$$U(f, P, \gamma_n) < (U)\int_a^b f d\gamma + \epsilon.$$

Then

$$U(f, P \cup P_m, \gamma_n) \le U(f, P, \gamma_n) < (U)\int_a^b f d\gamma + \epsilon.$$

for all m. Hence $\lim_m U(f, P \cup P_m, \gamma_n) \le (U)\int_a^b f d\gamma + \epsilon$. Since $\epsilon > 0$ is arbitrary, $\lim_m U(f, P \cup P_m, \gamma_n) \le (U)\int_a^b f d\gamma$. The first proof is done.

Let $M = \sup\{|f(x) : x \in [a,b]\}$. Then we get

$$|U(f,P,\gamma) - U(f,P,\gamma_n)| = |U(f,P,\gamma - \gamma_n)| \le \sum_{k=n}^{\infty} M(\gamma(x_k+) - \gamma(x_k-))$$

for every partition of P. Given $\epsilon > 0$, there is N such that $\sum_{k=N}^{\infty} M(\gamma(x_k+) - \gamma(x_k-)) < \epsilon$. Then $U(f,P,\gamma - \gamma_n) < \epsilon$ for every partition P of $[a,b]$ and for every $n \ge N$. For each $n \ge N$, choose a partition P_n such that

$$U(f,P_n,\gamma_n) < (U)\int_a^b f d\gamma_n + \epsilon \text{ and } U(f,P_n,\gamma) < (U)\int_a^b f d\gamma + \epsilon.$$

If $n \ge N$, then

$$\begin{aligned}(U)\int_a^b f(x) d\gamma &\le U(f,P_n,\gamma) \le U(f,P_n,\gamma_n) + \epsilon \\ &\le (U)\int_a^b f d\gamma_n + 2\epsilon\end{aligned}$$

$$\begin{aligned}(U)\int_a^b f d\gamma &\ge U(f,P_n,\gamma) - \epsilon > U(f,P_n,\gamma_n) - 2\epsilon \\ &\ge (U)\int_a^b f d\gamma_n - 3\epsilon\end{aligned}$$

Hence

$$(U)\int_a^b f d\gamma = \lim_n (U)\int_a^b f d\gamma_n$$

and we get the desired result. The proof on the lower Riemann-Stieltjes integral is similar and we omit the proof. □

Theorem 4.4. *Given a monotone increasing function $\alpha : [a,b] \to \mathbb{R}$, let $\{x_k\}$ be the set of discontinuity points of α. Then $\beta = \alpha - \sum_{k=1}^{\infty} J_k(\alpha)$ is monotone increasing and continuous. Let $\gamma = \sum_{k=1}^{\infty} J_k(\alpha)$. Then*

$$(U)\int_a^b f d\alpha = (U)\int_a^b f d\beta + (U)\int_a^b f d\gamma$$

$$(L)\int_a^b f d\alpha = (L)\int_a^b f d\beta + (L)\int_a^b f d\gamma.$$

In particular, $f \in R(\alpha)$ in $[a,b]$ if and only if $f \in R(\beta)$ and $f \in R(\gamma)$

Proof. $U(f,P,\alpha) = U(f,P,\beta) + U(f,P,\gamma) \ge (U)\int_a^b f d\beta + (U)\int_a^b f d\gamma$ and

$$(U)\int_a^b f d\alpha \ge (U)\int_a^b f d\beta + (U)\int_a^b f d\gamma.$$

On the other hand, if P and Q are partitions of $[a, b]$,

$$(U)\int_a^b f d\alpha \leq U(f, P \cup Q, \alpha) = U(f, P \cup Q, \beta) + U(f, P \cup Q, \gamma)$$
$$\leq U(f, P, \beta) + U(f, Q, \gamma).$$

So we get

$$(U)\int_a^b f d\alpha \leq (U)\int_a^b f d\beta + (U)\int_a^b f d\gamma.$$

□

In summary, we get the following result:

Theorem 4.5 (Criterion for Riemann-Stieltjes integrability)**.** *Let f be a bounded function and α be a nondecreasing function on $[a, b]$ with discontinuity points $\{x_k\}$. Let μ be the representing measure of α and $\bar{\mu}$ be the completion of μ. Suppose that μ_c be the representing measure of the continuous part of α and D be the set of discontinuity points of f. Then $f \in R(\alpha)$ in $[a, b]$ if and only if (i) $\mu_c(D) = 0$ (ii) $f(x+)$ exists and $f(x+) = f(x)$ whenever $\alpha(x+) - \alpha(x) > 0$ and $f(x-)$ exists and $f(x-) = f(x)$ whenever $\alpha(x) - \alpha(x-) > 0$. In particular, if $f \in R(\alpha)$, f is $\bar{\mu}$-measurable and*

$$\int_a^b f d\alpha = \int f d\bar{\mu}_c + \sum_{k=1}^{\infty} f(x_k)(\alpha(x_k+) - \alpha(x_k-)) = \int f d\bar{\mu}.$$

Theorem 4.5 is presented in [3], and a more direct proof is provided in [4]. The proof given here differs somewhat from these two proofs, as it is derived from the comprehensive description of the upper and lower Riemann-Stieltjes integrals.

The following result is proved in [4]. Here we provide an alternative proof based on measure theory.

Proposition 4.6. *Let f be a bounded function. Suppose that D is the set of discontinuity points of f and D_α is the set of discontinuities of α. Then $\mu_c(D) = 0$ if and only if, given $\epsilon > 0$, there is a (finite or countable) sequence of open intervals $\{(a_k, b_k)\}$ in $\mathbb{R}$ such that $D \setminus D_\alpha \subset \bigcup_k (a_k, b_k)$ and*

$$\sum_k (\alpha(b_k) - \alpha(a_k)) < \epsilon.$$

Proof. (Sufficiency) Let β be the continuous part of α. Then The condition is satisfied with β and it means that $\mu_c(D \setminus D_\alpha) = \mu_c(D) = 0$ by the outer regularity of μ_c.

(Necessity) Suppose that $\mu_c(D) = \mu(D \setminus D_\alpha) = 0$. Then there is an open set V in $\mathbb{R}$ such that $D \setminus D_\alpha \subset V$ such that $\mu(V) < \epsilon$. Since V is an open set in $\mathbb{R}$, let $V = \bigcup_{j=1}^{\infty}(a_j, b_j)$ as the union of disjoint open intervals. So

$$\mu(V) = \sum_{j=1}^{\infty}(\alpha(b_j-) - \alpha(a_j+)) < \epsilon.$$

Note that given $(s,t) \subset \mathbb{R}$, chosose a sequence $s_n < t_n$ such that $\{s_n\}$ decreasingly converges to s, $\{t_n\}$ increasingly converges to t and $s_n, t_n \in \mathbb{R} \setminus (D \cup D_\alpha)$ for all n, where α is regarded as a function in $\mathbb{R}$ by extension.

Then $\sum_{n=1}^{\infty}(\alpha(t_n + 1) - \alpha(t_n)) = \alpha(b-) - \alpha(t_1)$ and $\sum_{n=1}^{\infty}(\alpha(s_n) - \alpha(s_{n+1}) = \alpha(s_1) - \alpha(a+)$. Hence

$$\alpha(b-) - \alpha(a+) = \sum_{n=1}^{\infty}(\alpha(t_{n+1}) - \alpha(t_n)) + \alpha(t_1) - \alpha(s_1) + \sum_{n=1}^{\infty}(\alpha(s_n) - \alpha(s_{n+1})).$$

So there is a set $A = \{s_n : n \geq 1\} \cup \{t_n : n \geq 1\} \subset (s,t) \setminus (D \cup D_\alpha)$ such that $\mu(A) = 0$ and

$$(s,t) \setminus A = \bigcup_{j=1}^{\infty}(x_{n+1}, x_n) \cup (x_1, y_1) \cup \bigcup_{n=1}^{\infty}(y_n, y_{n+1}).$$

For each interval j, there is a countable set $A_j \subset (a_j, b_j) \setminus (D \cup D_\alpha)$ such that

$$(a_j, b_j) \setminus A_j = \bigcup_{k=1}^{\infty}(a_{j,k}, b_{j,k})$$

as the union of disjoint open intervals,

$$A_j = \{a_{j,k} : k \geq 1\} \cup \{b_{j,k} : k \geq 1\}$$

and $\mu(A_j) = 0$. So

$$\alpha(b_j-) - \alpha(a_j+) = \mu((a_j, b_j)) = \mu((a_j, b_j) \setminus A_j) = \sum_{k=1}^{\infty}(\alpha(a_{j,k}) - \alpha(b_{j,k})).$$

So we get

$$D \setminus D_\alpha \subset (V \setminus D_\alpha) \setminus \bigcup_j A_j = \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}(a_{j,k}, b_{j,k}),$$

where $(a_{j,k}, b_{j,k})$'s are disjoint open intervals.

$$\mu(V \setminus D_\alpha) = \mu((V \setminus D_\alpha) \setminus \bigcup_j A_j) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(\alpha(b_{j,k}) - \alpha(b_{j,k})) < \epsilon.$$

So we get the desired result. □

Now we conclude with an alternative proof of Theorem 2.10, based on a criterion of Riemann-Stieltjes integrability (Theorem 4.5) and measure theory.

Theorem 4.7. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and α be in $BV[a, b]$. Then $f \in R(\alpha)$ if and only if $f \in R(V)$, where $V = V(\alpha)$ is the total variation function of α.*

Proof. Now we give another proof based on the measure theory. For the direct proof, see the proof of Theorem 2.10. Let μ be the representing measure of α. Then $|\mu|$ is the representing measure of V. Then the positive part μ^+, and the negative part μ^- of μ are representing measures of $\alpha_1 = \frac{1}{2}(V + f)$ and $\alpha_2 = \frac{1}{2}(V - f)$. Given a function g, let D_g be the set of all discontinuities of g. Then $D_V = D_{\alpha_1} \cup D_{\alpha_2}$, this follows easily from the fact that $D_g = \{x : \mu_g(\{x\}) > 0\}$ if g is monotone increasing and μ_g is the representing measure of g. Note also that $D_V = D_\alpha$ by Theorem 2.9. Let P and N be Borel sets such that $P \cup N = [a, b]$ and $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = \mu(E \cap N)$ for every Borel set E. Hence $D_{\alpha_1} \subset P$ and $D_{\alpha_2} \subset N$. Since $f \in R(V)$, $|\mu|(D_f \setminus D_V) = 0$.

$$\begin{aligned}\mu^+(D_f \setminus D_{\alpha_1}) &= \mu^+((D_f \setminus (D_{\alpha_1} \cup D_{\alpha_2}) \cup (D_f \cap D_{\alpha_2}) \setminus D_{\alpha_1}) \\ &= \mu^+(D_f \setminus D_V) \le |\mu|(D_f \setminus D_V) = 0.\end{aligned}$$

Similarly, $\mu^-(D \setminus D_{\alpha_2}) = 0$. Suppose that $\alpha_1(x+) > \alpha_1(x)$, then $V(x+) > V(x)$ and so $f(x+)$ exists and $f(x+) = f(x)$. Similarly, if $\alpha_1(x-) < \alpha(x-)$, then $f(x-)$ exists and $f(x-) = f(x)$. Therefore, $f \in R(\alpha_1)$. The same proof shows that $f \in R(\alpha_2)$. Since $\alpha = \frac{1}{2}(\alpha_1 - \alpha_2)$, we get $f \in R(\alpha)$ by Proposition 2.2. □

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