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LEFT AND RIGHT CORESIDUATED LATTICES

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Abstract. In this paper, we introduce the pairs of negations and pseudo t-conorms on lattices. As a noncommutative sense, we define left and right coresiduated lattices which are an algebraic structure to deal information systems. We investigate their properties and construct them. Moreover, we give their examples.

1. Introduction

Ward et al. [15] introduced a complete residuated lattice as a generalization of BL-algebras and left continuous t-norms [5, 6, 7]. Many researchers [1-3, 7-8, 15] developed algebraic structures in complete residuated lattices as a formal tool to deal information systems, fuzzy concepts and decision rules in the data analysis. Moreover, Junsheung et al.[9] introduced a complete coresiduated lattice as the generalization of t-conorm. Various fuzzy concept lattices on information systems were studied in complete coresiduated lattices [10,13].

A non-commutative algebraic structure, Turunen [14] introduced a generalized residuated lattice as a generalization of weak-pseudo-BL-algebras and left continuous pseudo-t-norm [4, 5, 6].

In this paper, weak conditions of algebraic structure are needed to analyze large data and divide them into small groups. We introduce left and right coresiduated lattices as a noncommutative sense. We investigate their properties. Our purpose is to create various coresiduated lattices with the pairs of negations and pseudo t-conorms on lattices. As a main result, in Theorem 3.5, we show that if S is a pseudo t-conorm with $S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in I} S(x_i, y)$ and we define $M_2(x, y) =$

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$\bigwedge\{z \in L \mid S(z,y) \geq x\}$, then $(L, \vee, \wedge, S, M_2, \bot, \top)$ is a right coresiduated lattice. Moreover, if S is a pseudo t-conorm with $S(x, \bigwedge_{j\in\Gamma} y_j) = \bigwedge_{j\in J} S(x, y_j)$ and we define $M_1(x,y) = \bigwedge\{z \in L \mid S(y,z) \geq x\}$, then $(L, \vee, \wedge, S, M_1, \bot, \top)$ is a left coresiduated lattice. We give their examples.

In Theorem 3.9, we can obtain generalized (resp. left, right) left and right coresiduated lattices from the pairs of negations and pseudo t-conorms on lattices. We construct them.

2. Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \bot, \top)$ is a lattice with a bottom element $\bot$ and a top element $\top$ instead of $[0,1]$. Moreover, we denote $\bigvee$ and $\bigwedge$ if they exist.

Definition 2.1 ([4, 5])**.** A map $S : L \times L \to L$ is called a *pseudo t-conorm* if it satisfies the following conditions:

(S1) $S(x, S(y,z)) = S(S(x,y), z)$ for all $x, y, z \in L$,

(S2) If $y \leq z$, $S(x,y) \leq S(x,z)$ and $S(y,x) \leq S(z,x)$,

(S3) $S(x, \bot) = S(\bot, x) = x$.

A pseudo t-conorm is called a *t-conorm* if $S(x,y) = S(y,x)$ for $x, y \in L$.

Definition 2.2 ([4, 5])**.** A pair (n_1, n_2) with maps $n_i : L \to L$ is called a *pair of negations* if it satisfies the following conditions:

(N1) $n_i(\top) = \bot, n_i(\bot) = \top$ for all $i \in \{1,2\}$.

(N2) $n_i(x) \geq n_i(y)$ for $x \leq y$ and $i \in \{1,2\}$.

(N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in L$.

3. Left and Right Coresiduated Lattices

Definition 3.1. A structure $(L, \vee, \wedge, S, M_1, \bot, \top)$ is called a *left coresiduated lattice* if it satisfies the following conditions:

(C) S is a pseudo t-conorm,

(LC) $S(x,y) \geq z$ iff $y \geq M_1(z,x)$ for $x, y, z \in L$.

A structure $(L, \vee, \wedge, S, M_2, \bot, \top)$ is called a *right coresiduated lattice* if it satisfies (C) and

(RC) $S(x,y) \geq z$ iff $x \geq M_2(z,y)$, for $x, y, z \in L$.

A structure $(L, \vee, \wedge, S, M_1, M_2, \perp, \top)$ is called a *generalized coresiduated lattice* if it is a left and right coresiduated lattice.

Theorem 3.2. *Let $(L, \vee, \wedge, S, M_1, \perp, \top)$ be a left coresiduated lattice. For each $x, y, z, x_i, y_i \in L$, the following properties are hold.*

(1) $S(y, M_1(x, y)) \geq x$ *and* $S(M_1(x, y), z) \geq M_1(S(x, z), y)$.

(2) *If* $y \leq z$, *then* $M_1(x, z) \leq M_1(x, y)$ *and* $M_1(y, z) \leq M_1(z, x)$.

(3) $S(x, \bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma} S(x, y_i)$

(4) $M_1(x, \bigwedge_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma} M_1(x, y_i)$. *If* $S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in\Gamma} S(x_i, y)$, *the equality holds.*

(5) $M_1(\bigvee_{i\in\Gamma} x_i, y) = \bigvee_{i\in\Gamma} M_1(x_i, y)$.

(6) $M_1(x, \bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma} M_1(x, y_i)$ *and* $M_1(\bigwedge_{i\in\Gamma} x_i, y) \leq \bigwedge_{i\in\Gamma} M_1(x_i, y)$.

(7) $M_1(M_1(x, y), z) = M_1(x, S(y, z))$.

(8) $S(M_1(y, z), M_1(x, y)) \geq M_1(x, z)$.

(9) $M_1(x, z) \geq M_1(S(y, x), S(y, z))$.

(10) $M_1(x, y) \geq M_1(M_1(x, z), M_1(y, z))$.

(11) $M_1(x, x) = \perp$.

(12) $x \leq y$ *iff* $M_1(x, y) = \perp$.

Proof. (1) Since $M_1(x, y) \geq M_1(x, y)$, by (LC), $S(y, M_1(x, y)) \geq x$. Since

$S(y, S(M_1(x, y), z)) = S(S(y, M_1(x, y)), z) \geq S(x, z)$,

by (LC), $S(M_1(x, y), z) \geq M_1(S(x, z), y)$.

(2) Since $x \leq S(y, M_1(x, y)) \leq S(z, M_1(x, y))$, $M_1(x, z) \leq M_1(x, y)$. Since $S(x, M_1(z, x)) \geq z$ from (1), $y \leq z \leq S(x, M_1(z, x))$. By (LC), $M_1(y, x) \leq M_1(z, x)$.

(3) By (S2), $S(x, \bigwedge_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma} S(x, y_i)$. Since $\bigwedge_{i\in\Gamma} S(x, y_i) \leq S(x, y_i)$, by (LC), $M_1(\bigwedge_{i\in\Gamma} S(x, y_i), x) \leq y_i$ implies $M_1(\bigwedge_{i\in\Gamma} S(x, y_i), x) \leq \bigwedge_{i\in\Gamma} y_i$. Hence $\bigwedge_{i\in\Gamma} S(x, y_i) \leq S(x, \bigwedge_{i\in\Gamma} y_i)$. Thus $\bigwedge_{i\in\Gamma} S(x, y_i) = S(x, \bigwedge_{i\in\Gamma} y_i)$.

(4) By (2), $M_1(x, \bigwedge_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma} M_1(x, y_i)$. If $S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in\Gamma} S(x_i, y)$, then

$$S(\textstyle\bigwedge_{i\in\Gamma} y_i, \bigvee_{i\in\Gamma} M_1(x, y_i)) = \bigwedge_{i\in\Gamma} S(y_i, \bigvee_{i\in\Gamma} M_1(x, y_i)) \geq \bigwedge_{i\in\Gamma} S(y_i, M_1(x, y_i)) \geq x.$$

Hence $\bigvee_{i\in\Gamma} M_1(x, y_i) \geq M_1(x, \bigwedge_{i\in\Gamma} y_i)$.

(5) By (2), $M_1(\bigvee_{i\in\Gamma} x_i, y) \geq \bigvee_{i\in\Gamma} M_1(x_i, y)$. Since

$S(y, \bigvee_{i\in\Gamma} M_1(x_i, y)) \geq \bigvee_{i\in\Gamma} S(y, M_1(x_i, y)) \geq \bigvee_{i\in\Gamma} x_i$,

$\bigvee_{i\in\Gamma} M_1(x_i, y) \geq M_1(\bigvee_{i\in\Gamma} x_i, y)$.

(6) By (2), they are easily proved.

(7) For each $x, y, z \in X$,

$$\begin{aligned} &S(y, S(z, M_1(x, S(y,z)))) = S(S(y,z), M_1(x, S(y,z))) \geq x \\ &\text{iff } S(z, M_1(x, S(y,z))) \geq M_1(x,y) \\ &\text{iff } M_1(x, S(y,z)) \geq M_1(M_1(x,y), z). \end{aligned}$$

Since $S(S(y,z), M_1(M_1(x,y),z)) = S(y, S(z, M_1(M_1(x,y),z))) \geq S(y, M_1(x,y)) \geq x$, $M_1(M_1(x,y),z) \geq M_1(x, S(y,z))$. Hence $M_1(M_1(x,y),z) = M_1(x, S(y,z))$.

(8) Since $S(S(z, M_1(y,z)), M_1(x,y)) \geq S(y, M_1(x,y)) \geq x$, $S(M_1(y,z), M_1(x,y)) \geq M_1(x,z)$.

(9) Since $S(S(y,z), M_1(x,z)) = S(y, S(z, M_1(x,z))) \geq S(y,z)$,

$M_1(x,z) \geq M_1(S(y,x), S(y,z))$.

(10) Since $S(M_1(y,z), M_1(x,y)) \geq M_1(x,z)$, $M_1(x,y) \geq M_1(M_1(x,z), M_1(y,z))$.

(11) Since $S(x, \bot) = x$, by (LC), $M_1(x,x) \leq \bot$. Then $M_1(x,x) = \bot$.

(12) Let $M_1(x,y) = \bot$. Then $y = S(y, \bot) = S(y, M_1(x,y)) \geq x$. Thus $x \leq y$.

If $x \leq y$, then $M_1(x,y) \leq M_1(y,y) = \bot$. Thus $M_1(x,y) = \bot$.

□

Corollary 3.3. *Let $(L, \vee, \wedge, S, M_2, \bot, \top)$ be a right coresiduated lattice. For each $x, y, z, x_i, y_i \in L$, the following properties are hold.*

(1) *If $y \leq z$, then $M_2(x,z) \leq M_2(x,y)$ and $M_2(y,z) \leq M_2(z,x)$.*

(2) *$S(M_2(x,y), y) \geq x$ and $S(x, M_2(y,z)) \geq M_2(S(x,y), z)$.*

(3) *$S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in\Gamma} S(x_i, y)$.*

(4) *$M_2(x, \bigwedge_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma} M_2(x, y_i)$. If $S(x, \bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma} S(x, y_i)$, the equality holds.*

(5) *$M_2(\bigvee_{i\in\Gamma} x_i, y) = \bigvee_{i\in\Gamma} M_2(x_i, y)$.*

(6) *$M_2(x, \bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma} M_2(x, y_i)$ and $M_2(\bigwedge_{i\in\Gamma} x_i, y) \leq \bigwedge_{i\in\Gamma} M_2(x_i, y)$.*

(7) *$M_2(x, S(y,z)) = M_2(M_2(x,z), y)$.*

(8) *$S(M_2(x,y), M_2(y,z)) \geq M_2(x,z)$.*

(9) *$M_2(x,z) \geq M_2(S(x,y), S(z,y))$.*

(10) *$M_2(x,y) \geq M_2(M_2(x,z), M_2(y,z))$.*

(11) *$M_2(x,x) = \bot$.*

(12) *$x \leq y$ iff $M_2(x,y) = \bot$.*

Theorem 3.4. *Let $(L, \vee, \wedge, S, M_1, M_2, \bot, \top)$ be a generalized coresiduated lattice. For each $x, y, z \in L$, the following properties (1) and (2) are hold.*

(1) *$M_1(M_2(x,y), z) = M_2(M_1(x,z), y)$.*

(2) *$M_1(y,z) \geq M_2(M_1(x,z), M_1(x,y))$ and $M_2(y,z) \geq M_1(M_2(x,z), M_2(x,y))$.*

Let (n_1, n_2) be a pair of negations defined as $n_1(x) = M_1(\top, x)$ and $n_2(x) = M_2(\top, x)$ for each $x \in X$. the following properties (3)-(6) are hold.

(3) *$M_2(x, y) = M_1(n_2(y), n_2(x))$ and $M_1(x, y) = M_2(n_1(y), n_1(x))$ for each $x, y \in X$.*

(4) *$n_1(S(y, z)) = M_1(n_1(y), z)$. Moreover, $n_1(S(y, z)) = M_2(n_2(z), y)$ and $n_2(M_1(x, y)) = S(n_2(x), y)$ for each $x, y, z \in X$.*

(5) *$M_1(x, \bot) = M_2(x, \bot) = x$ for each $x \in X$.*

(6) *For each $k = 1, 2$, $n_k(\bigwedge_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} n_k(x_i)$ and $n_k(\bigvee_{i\in\Gamma} x_i) = n_k(\bigwedge_{i\in\Gamma} x_i)$ for each $x_i \in X$.*

Proof. (1) Since

$$\begin{aligned} &S(z, S(M_1(M_2(x, y), z), y)) = S(S(z, M_1(M_2(x, y), z)), y) \\ &\geq S(M_2(x, y), y) \geq x, \end{aligned}$$

by (LC), $S(M_1(M_2(x, y), z), y) \geq M_1(x, z)$. Thus $M_1(M_2(x, y), z) \geq M_2(M_1(x, z), y)$.

Since $S(S(z, S(M_2(M_1(x, z), y)), y) = S(z, S(M_2(M_1(x, z), y), y)) \geq S(z, M_1(x, z)) \geq x$, $S(z, M_2(M_1(x, z), y)) \geq M_2(x, y)$. Thus $M_2(M_1(x, z), y) \geq M_1(M_2(x, y), z)$.

(2) Since $S(M_1(y, z), M_1(x, y)) \geq M_1(x, z)$, $M_1(y, z) \geq M_2(M_1(x, z), M_1(x, y))$. Since $S(M_2(x, y), M_2(y, z)) \geq M_2(x, z)$, $M_2(y, z) \geq M_1(M_2(x, z), M_2(x, y))$.

(3) By (2), $M_2(x, y) \geq M_1(M_2(\top, y), M_2(\top, x)) = M_1(n_2(y), n_2(x))$. By (2), $M_1(x, y) \geq M_2(M_1(\top, y), M_1(\top, x)) = M_2(n_1(y), n_1(x))$.

Moreover, $M_2(x, y) = M_2(n_1(n_2(x)), n_1(n_2(y))) \leq M_1(n_2(y), n_2(x))$ and $M_1(x, y) = M_1(n_2(n_1(x)), n_2(n_1(y))) \leq M_2(n_1(y), n_1(x))$.

Thus, $M_2(x, y) = M_1(n_2(y), n_2(x))$ and $M_1(x, y) = M_2(n_1(y), n_1(x))$.

(4) By Theorem 3.2(7), $n_1(S(y, z)) = M_1(\top, S(y, z)) = M_1(M_1(\top, y), z) = M_1(n_1(y), z)$. By Corollary 3.3(7), $n_1(S(y, z)) = M_2(\top, S(y, z)) = M_2(M_2(\top, z), y) = M_2(n_2(z), y)$. Since $n_1(S(n_2(y), z)) = M_1(y, z)$, $S(n_2(y), z) = n_2(M_1(y, z))$.

(5) Since $n_1 M_2(x, \bot) = S(\bot, n_1(x)) = n_1(x)$, $M_2(x, \bot) = n_2(n_1(M_2(x, \bot))) = n_2(n_1(x)) = x$. Since $n_2 M_1(x, \bot) = S(n_2(x), \bot) = n_2(x)$, $M_1(x, \bot) = n_1(n_2 M_1(x, \bot))) = n_1(n_2(x)) = x$.

(6) By Theorem 3.2(3,4) and Corollary 3.3(3,4), $n_k(\bigwedge_i x_i) = \bigvee_i n_k(x_i)$ for each $k = 1, 2$. Since $\bigwedge_i x_i = n_2(n_1(\bigwedge_i x_i)) = n_2(\bigvee_i n_1(x_i))$, $\bigwedge_i n_2(x_i) = n_2(\bigvee_i n_1(n_2(x_i))) = n_2(\bigvee_i x_i)$. Other cases are similarly proved.

□

Theorem 3.5. *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice and $S : L \times L \to L$ be a pseudo t-conorm.*

(1) *If* $S(x, \bigwedge_{j\in J} y_i) = \bigwedge_{j\in J} S(x, y_j)$ *for each* $\{y_j\}_{j\in J}$. *then the following statements* (a), (b) *and* (c) *are equivalent.*

(a) *If* $y \leq z$, *then* $M_1(y,x) \leq M_1(z,x)$. *Moreover, for all* $x, y \in L$, $S(x, M_1(y,x)) \geq y$ *and* $y \geq M_1(S(x,y),x)$.

(b) $M_1(x,y) = \bigwedge\{z \in L \mid S(y,z) \geq x\}$.

(c) $S(y,z) \geq x$ *iff* $z \geq M_1(x,y)$.

(2) *If* $S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in I} S(x_i, y)$ *for each* $\{x_i\}_{i\in I}$, *then* (e), (f) *and* (g) *are equivalent.*

(e) *If* $y \leq z$, *then* $M_2(y,x) \leq M_2(z,x)$. *Moreover, for all* $x, y \in L$, $S(M_2(y,x),x) \geq y$ *and* $x \geq M_2(S(x,y),y)$.

(f) $M_2(x,y) = \bigwedge\{z \in L \mid S(z,y) \geq x\}$.

(g) $S(z,y) \geq x$ *iff* $z \geq M_2(x,y)$.

Proof. (1) (a) $\Rightarrow$ (b). Put $P(x,y) = \bigwedge\{z \in L \mid S(y,z) \geq x\}$. By (a), since $S(y, M_1(x,y)) \geq x$, $P(x,y) \leq M_1(x,y)$.

Suppose there exist $x, y \in L$ such that $P(x,y) \not\geq M_1(x,y)$. Then there exists $z \in L$ such that $z \not\geq M_1(x,y)$ and $S(y,z) \geq x$. By (a),

$$z \geq M_1(S(y,z),y) \geq M_1(x,y).$$

It is a contradiction. Hence $P(x,y) \geq M_1(x,y)$.

(b) $\Rightarrow$ (c). Let $S(y,z) \geq x$. Then $z \geq M_1(x,y)$.

If $M_1(x,y) \leq z$, then $S(y,z) \geq S(y, M_1(x,y)) = S(y, \bigwedge\{z_1 \in L \mid S(y,z_1) \geq x\}) = \bigwedge S(y,z_1) \geq x$.

(c) $\Rightarrow$ (a). Since $S(y,z) \leq S(y,z)$, $M_1(S(y,z),y) \leq z$. Since $M_1(y,x) \leq M_1(y,x)$, $S(x, M_1(y,x)) \geq y$. If $y \geq z$, $S(x, M_1(y,x)) \geq y \geq z$. Hence $M_1(y,x) \geq M_1(z,x)$.

(2) (d) $\Rightarrow$ (e). Put $Q(x,y) = \bigwedge\{z \in L \mid S(z,y) \geq x\}$. By (d), since $S(M_2(x,y),y) \geq x$, $Q(x,y) \leq M_2(x,y)$.

Suppose there exist $x, y \in L$ such that $Q(x,y) \not\geq M_2(x,y)$. Then there exists $z \in L$ such that $z \not\geq M_2(x,y)$ and $S(z,y) \geq x$. By (d),

$$z \geq M_2(S(z,y),y) \geq M_2(x,y).$$

It is a contradiction. Hence $Q(x,y) \geq M_2(x,y)$.

(e) $\Rightarrow$ (f). Let $S(z,y) \geq x$. Then $z \geq M_2(x,y)$.

If $M_2(x,y) \leq z$, then $S(z,y) \geq S(M_2(x,y),y) = S(\bigwedge\{z_2 \in L \mid S(z_2,y) \geq x\}) = \bigwedge S(z_2,y) \geq x$.

(f) $\Rightarrow$ (d). Since $S(z,y) \leq S(z,y)$, $M_2(S(z,y),y) \leq z$. Since $M_2(y,x) \leq M_2(y,x)$, $S(M_2(y,x),x) \geq y$. If $y \geq z$, $S(M_2(y,x),x) \geq y \geq z$. Hence $M_2(y,x) \geq M_2(z,x)$. □

From Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *Let $(L, \vee, \wedge, \top, \bot)$ be a bounded lattice and $S : L \times L \to L$ be a pseudo t-conorm.*

(1) *If $S(\bigwedge_{i\in I} x_i, y) = \bigwedge_{i\in I} S(x_i, y)$ for each $\{x_i\}_{i\in I}$ and we define $M_2(x,y) = \bigwedge\{z \in L \mid S(z,y) \geq x\}$, then $(L, \vee, \wedge, S, M_2, \bot, \top)$ is a right coresiduated lattice.*

(2) *If $S(x, \bigwedge_{j\in J} y_i) = \bigwedge_{j\in J} S(x, y_j)$ for each $\{y_j\}_{j\in J}$ and we define $M_1 : L \times L \to L$ as*

$$M_1(x,y) = \bigwedge\{z \in L \mid S(y,z) \geq x\}.$$

Then $(L, \vee, \wedge, S, M_1, \bot, \top)$ is a left coresiduated lattice.

Example 3.7. (1) Define a map $S : [0,1] \times [0,1] \to [0,1]$ as

$$S(x,y) = \begin{cases} 1, & \text{if } x \geq 0.4,\ y \geq 0.7, \\ x \vee y, & \text{otherwise.} \end{cases}$$

If $S(x,y) = 1$, then $x = 1$ or $y = 1$ and $x \geq 0.4, y \geq 0.7$. Thus, $S(S(x,y),z) = 1 = S(x, S(y,z))$.

If $S(x,y) < 1$ and $x \geq 0.4, z \geq 0.7$, then $S(S(x,y),z) = 1 = S(x,S(y,z))$.

If $S(x,y) < 1$ and $y < 0.7, z < 0.7$, then $S(S(x,y),z) = (x \vee y) \vee z = x \vee (y \vee z) = S(x,S(y,z))$. Hence $S(S(x,y),z) = S(x,S(y,z))$ for each $x,y,z \in X$. Moreover, (S2) and (S3) are easily proved. Thus S is a pseudo t-conorm.

Since $S(\bigwedge_{i\in\Gamma} x_i, y) = \bigwedge_{i\in I} S(x_i,y)$, by Theorem 3.5, $M_2(x,y) = \bigwedge\{z \in [0,1] \mid S(z,y) \geq x\}$ such that

$$M_2(x,y) = \begin{cases} 0.4, & \text{if } x > y,\ y \geq 0.7, \\ x, & \text{if } x > y,\ y < 0.7, \\ 0, & \text{if } x \leq y. \end{cases}$$

Moreover, $M_1(x,y) = \bigwedge\{z \in [0,1] \mid S(y,z) \geq x\}$ such that

$$M_1(x,y) = \begin{cases} x \wedge 0.7, & \text{if } x > y,\ y \geq 0.4, \\ x, & \text{if } x > y,\ y < 0.4, \\ 0, & \text{if } x \leq y. \end{cases}$$

Define $n_1, n_2 : [0,1] \to [0,1]$ as

$$n_1(x) = M_1(1,x) = \begin{cases} 0.7, & \text{if } 0.4 \leq x < 1, \\ 1, & \text{if } x < 0.4, \\ 0, & \text{if } x = 1. \end{cases}$$

$$n_2(x) = M_2(1,x) = \begin{cases} 0.4, & \text{if } 0.7 \le x < 1, \\ 1, & \text{if } x < 0.7, \\ 0, & \text{if } x = 1. \end{cases}$$

Since $n_2(n_1(0.6)) = n_2(0.7) = 0.4 \ne 0.6$, (n_1, n_2) is not a pair of negations.

(2) Define a map $S : [0,1] \times [0,1] \to [0,1]$ as

$$S(x,y) = \begin{cases} 1, & \text{if } x \ge 0.4,\ y > 0.7, \\ x \vee y, & \text{otherwise.} \end{cases}$$

By a similar way in (1), S is a pseudo t-conorm.

Since $S(\bigwedge x_i, y) = \bigwedge S(x_i, y)$, $M_2(x,y) = \bigwedge\{z \in [0,1] \mid S(z,y) \ge x\}$ such that

$$M_2(x,y) = \begin{cases} 0.4, & \text{if } x > y,\ y > 0.7, \\ x, & \text{if } x > y,\ y \le 0.7, \\ 0, & \text{if } x \le y. \end{cases}$$

By Theorem 3.5, $(L, \vee, \wedge, S, M_2, \perp, \top)$ be a right coresiduated lattice.

It follows $1 = \bigwedge_{n \in N} S(0.5, 0.7 + \frac{1}{n}) \ne S(0.5, \bigwedge_{n \in N} 0.7 + \frac{1}{n}) = S(0.5, 0.7) = 0.5 \vee 0.7 = 0.7$. Since $M_1(0.8, 0.5) = \bigwedge\{z \in [0,1] \mid S(0.5, z) \ge 0.8\} = 0.7$, $M_1(0.8, 0.5) \le 0.7$ but $S(0.5, 0.7) = 0.7 \not\ge 0.8$. Hence $(L, \vee, \wedge, S, M_1, \perp, \top)$ is not a left coresiduated lattice.

(3) Define a map $S : [0,1] \times [0,1] \to [0,1]$ as

$$S(x,y) = \begin{cases} 1, & \text{if } x > 0.4,\ y \ge 0.7, \\ x \vee y, & \text{otherwise.} \end{cases}$$

By a similar way in (1), S is a pseudo t-conorm. Since $S(x, \bigwedge y_i) = \bigwedge S(x, y_i)$, $M_1(x,y) = \bigwedge\{z \in [0,1] \mid S(y,z) \ge x\}$ such that

$$M_1(x,y) = \begin{cases} x \wedge 0.7, & \text{if } x > y,\ y > 0.4, \\ x, & \text{if } x > y,\ y \le 0.4, \\ 0, & \text{if } x \le y. \end{cases}$$

By Theorem 3.5, $(L, \vee, \wedge, S, M_1, \perp, \top)$ is a left coresiduated lattice.

It follows $1 = \bigwedge_{n \in N} S(0.4 + \frac{1}{n}, 0.8) \ne S(\bigwedge_{n \in N} 0.4 + \frac{1}{n}, 0.8) = S(0.4, 0.8) = 0.4 \vee 0.8 = 0.8$. Since $M_2(0.9, 0.8) = \bigwedge\{z \in [0,1] \mid S(z, 0.8) \ge 0.9\} = 0.4$, $M_2(0.9, 0.8) = 0.4$ but $S(0.4, 0.8) = 0.8 \not\ge 0.9$. Hence $(L, \vee, \wedge, S, M_2, \perp, \top)$ is not a right coresiduated lattice.

Theorem 3.8. *Let $(L, \vee, \wedge, \top, \perp)$ be a bounded lattice, $S : L \times L \to L$ be a pseudo t-conorm and (n_1, n_2) a pair of negations. For $i = \{1, ..., 4\}$, we define $M_i : L \times L \to L$*

as follows;

$$\begin{aligned}&M_1(x,y)=n_2S(n_1(x),y),\ \ M_2(x,y)=n_1S(y,n_2(x)),\\&M_3(x,y)=n_2S(y,n_1(x)),\ \ M_4(x,y)=n_1S(n_2(x),y),\\&M_5(x,y)=n_2S(n_1(x),n_1n_1(y)),\\&M_6(x,y)=n_1S(n_2n_2(y),n_2(x)).\end{aligned}$$

The the following properties are hold.

(1) *For each* $y\in Y$, $n_1(y)=M_2(1,y)=M_4(1,y)=M_5(1,y)$ *and* $n_2(y)=M_1(1,y)=M_3(1,y)=M_6(1,y)$.

(2) *For each* $x,y,z\in L$,

$$M_i(M_i(x,z),y)=M_i(x,S(y,z)),i\in\{2,3\}.$$

Moreover, let $x\le y$ *iff* $M_i(x,y)=\bot,i\in\{2,3\}$. *Then* $(L,\vee,\wedge,S,M_i,\bot,\top)$ *is a left coresiduated lattice such that* $S(x,y)\ge z$ *iff* $x\ge M_i(z,y),i\in\{2,3\}$.

(3) *For each* $x,y,z\in L$,

$$M_j(M_j(x,y),z)=M_j(x,S(y,z)),j\in\{1,4\}.$$

Moreover, let $x\le y$ *iff* $M_j(x,y)=\bot,j\in\{1,4\}$. *Then* $(L,\vee,\wedge,S,M_j,\bot,\top)$ *is a right coresiduated lattice such that* $S(x,y)\ge z$ *iff* $y\ge M_j(z,x),j\in\{1,4\}$.

(4) *Let* $x\le y$ *iff* $M_2(x,y)=\bot$ *iff* $M_1(x,y)=\bot$. *Then* $(L,\vee,\wedge,S,M_2,M_1,\bot,\top)$ *is a generalized coresiduated lattice with* $M_2(1,M_1(1,y))=M_1(1,M_2(1,y))=y$ *for each* $y\in L$.

(5) *Let* $x\le y$ *iff* $M_3(x,y)=\bot$ *iff* $M_4(x,y)=\bot$. *Then* $(L,\vee,\wedge,S,M_3,M_4,\bot,\top)$ *is a generalized coresiduated lattice with* $M_3(1,M_4(1,y))=M_4(1,M_3(1,y))=y$ *for each* $y\in L$.

(6) *Let* $x\le y$ *iff* $M_3(x,y)=\bot$ *iff* $M_1(x,y)=\bot$. *Then* $(L,\vee,\wedge,S,M_3,M_1,\bot,\top)$ *is a generalized co-residuated lattice such that* $M_3(1,M_1(1,y))=M_1(1,M_3(1,y))=n_2n_2(y)$ *for each* $y\in L$.

(7) *Let* $x\le y$ *iff* $M_2(x,y)=\bot$ *iff* $M_4(x,y)=\bot$. *Then* $(L,\vee,\wedge,S,M_2,M_4,\bot,\top)$ *is a generalized co-residuated lattice with* $M_2(1,M_4(1,y))=M_4(1,M_2(1,y))=n_1n_1(y)$ *for each* $y\in L$.

(8) *Let* $S(n_1n_1(x),n_1n_1(x))=n_1n_1(S(x,y))$ *for each* $x,y\in X$. *Then* (M_2,M_5) *is a pair with*

$$\begin{aligned}&M_2(M_2(x,z),y)=M_2(x,S(y,z)),\\&M_5(M_5(x,y),z)=M_5(x,S(y,z)),\\&M_2(1,M_4(1,y))=M_4(1,M_2(1,y))=y.\end{aligned}$$

Moreover, let $x \le y$ *iff* $M_2(x,y) = \bot$ *iff* $M_5(x,y) = \bot$. *Then* $(L, \vee, \wedge, S, M_2, M_5, \bot, \top)$ *is a generalized coresiduated lattice such that* $S(x,y) \ge z$ *iff* $x \ge M_2(z,y)$ *iff* $y \ge M_5(z,x)$.

(9) *Let* $S(n_2n_2(x), n_2n_2(y)) = n_2n_2(S(x,y))$ *for each* $x, y \in X$. *Then* (M_6, M_4) *is a pair with*

$$\begin{aligned} &M_6(M_6(x,z),y) = M_6(x, S(y,z)),\\ &M_4(M_4(x,y),z) = M_4(x, S(y,z)),\\ &M_6(1, M_4(1,y)) = M_4(1, M_6(1,y)) = y. \end{aligned}$$

Moreover, let $x \le y$ *iff* $M_4(x,y) = \bot$ *iff* $M_6(x,y) = \bot$. *Then* $(L, \vee, \wedge, S, M_6, M_4, \bot, \top)$ *is a generalized coresiduated lattice such that* $S(x,y) \ge z$ *iff* $x \ge M_6(z,y)$ *iff* $y \ge M_4(z,x)$.

Proof. (1) For each $y \in Y$, $M_2(1,y) = n_1S(y,0) = n_1(y) = M_4(1,y) = M_5(1,y) = n_2S(0, n_1n_1(y))$. Other cases are similarly proved.

(2) For each $x, y, z \in X$,

$$\begin{aligned} &M_2(M_2(x,z),y) = M_2(n_1(S(z,n_2(x)),y)\\ &= n_1S(y, S(z, n_2(x)) = n_1S(S(y,z), n_2(x)) = M_2(x, S(y,z)), \end{aligned}$$

$$\begin{aligned} &M_3(M_3(x,z),y) = M_3(n_2(S(z,n_1(x)),y)\\ &= n_2S(y, S(z, n_1(x)) = n_2S(S(y,z), n_1(x)) = M_3(x, S(y,z)). \end{aligned}$$

Since $M_i(z, S(x,y)) = \bot$ iff $M_i(M_i(z,y),x) = \bot$ for each $i \in \{2,3\}$, by Theorem 3.2(12), $z \le S(x,y))$ iff $M_i(z,y) \le x$. Hence $(L, \vee, \wedge, S, M_i, \bot, \top)$ is a left coresiduated lattice

(3) For each $x, y, z \in X$,

$$\begin{aligned} &M_1(M_1(x,y),z) = M_1(n_2(S(n_1(x),y),z)\\ &= n_2S(S(n_1(x),y),z) = n_2S(n_1(x), S(y,z)) = M_1(x, S(y,z)), \end{aligned}$$

$$\begin{aligned} &M_4(M_4(x,y),z) = M_4(n_1(S(n_2(x),y),z)\\ &= n_1S(S(n_2(x),y),z) = n_1S(n_2(x), S(y,z)) = M_4(x, S(y,z)). \end{aligned}$$

Since $M_j(z, S(x,y)) = \bot$ iff $M_i(M_j(z,x),y) = \bot$ for each $j \in \{1,4\}$, by Theorem 3.2(12), $z \le S(x,y))$ iff $M_j(z,x) \le y$. Hence $(L, \vee, \wedge, S, M_j, \bot, \top)$ is a right coresiduated lattice.

(4),(5),(6) and (7) are easily proved from (1)-(3).

(8) It follows from

$$\begin{aligned}&M_5(M_5(x,y),z) = M_5(n_2(S(n_1(x),n_1n_1(y)),z)\\&= n_2S(S(n_1(x),n_1n_1(y)),n_1n_1(z))\\&= n_2S(n_1(x),S(n_1n_1(y),n_1n_1(z)))\\&(S(n_1n_1(y),n_1n_1(z)) = n_1n_1(S(y,z)))\\&= n_2S(n_1(x),n_1n_1(S(y,z))))\\&= M_5(x,S(y,z)).\end{aligned}$$

(9) It follows from

$$\begin{aligned}&M_6(M_6(x,z),y) = M_6(n_1(S(n_2n_2(z),n_2(x)),y)\\&= n_1S(n_2n_2(y),S(n_2n_2(z),n_2(x)))\\&= n_1S(S(n_2n_2(y),n_2n_2(z)),n_2(x)))\\&(S(n_2n_2(y),n_2n_2(z)) = n_2n_2(S(y,z)))\\&= n_1S(n_2n_2(S(y,z)),n_2(x)))\\&= M_6(x,S(y,z)).\end{aligned}$$

□

Example 3.9. Put $L = \{(x,y) \in R^2 \mid (0,1) \leq (x,y) \leq (2,3)\}$ where $(0,1)$ is the bottom element and $(2,3)$ is the top element where

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow y_1 < y_2 \ \text{ or } \ y_1 = y_2, x_1 \leq x_2.$$

A map $S : L \times L \to L$ is defined as

$$S((x_1,y_1),(x_2,y_2)) = (x_2 + x_1y_2, y_1y_2) \wedge (2,3).$$

(1) (S1) $S(S((x_1,y_1),(x_2,y_2)),(x_3,y_3)) = S((x_1,y_1),S((x_2,y_2),(x_3,y_3)))$ from:

$$\begin{aligned}&S(S((x_1,y_1),(x_2,y_2)),(x_3,y_3))\\&= S((x_2+x_1y_2,y_1y_2)\wedge(2,3),(x_3,y_3))\\&= (x_3+x_2y_3+x_1y_2y_3,y_1y_2y_3)\wedge(2,3).\\&S((x_1,y_1),S((x_2,y_2),(x_3,y_3)))\\&= S((x_1,y_1),(x_3+x_2y_3,y_2y_3)\wedge(2,3))\\&= (x_3+x_2y_3+x_1y_2y_3,y_1y_2y_3)\wedge(2,3).\end{aligned}$$

(S2) If $(x_1,y_1) \leq (x_2,y_2)$, then $y_1 < y_2$ or $y_1 = y_2, x_1 \leq x_2$. Thus

$$\begin{aligned}&S((x_1,y_1),(x_3,y_3)) = (x_3+x_1y_3,y_1y_3)\wedge(2,3)\\&\leq (x_3+x_2y_3,y_2y_3)\wedge(2,3) = S((x_2,y_2),(x_3,y_3)).\end{aligned}$$

(S3) For each $(x_1,y_1) \in L$,

$$S((x_1,y_1),(0,1)) = (x_1,y_1) = S((0,1),(x_1,y_1)).$$

Then S is a pseudo t-conorm but not t-conorm because

$$(2,2) = S((-1,2),(3,1)) \neq S((3,1),(-1,2)) = (5,2).$$

(2) We define a pair (n_1, n_2) as follows

$$n_1(x,y) = (2 - \frac{3x}{y}, \frac{3}{y}), \ \ n_2(x,y) = (\frac{2-x}{y}, \frac{3}{y}).$$

Then (n_1, n_2) is a pair of negations from:

$$n_1(n_2(x,y)) = (x,y), \ \ n_2(n_1(x,y)) = (x,y).$$

(3)

$$\begin{aligned}
&M_1((x_1,y_1),(x_2,y_2)) = n_2S(n_1(x_1,y_1),(x_2,y_2))\\
&= n_2S((2-\tfrac{3x_1}{y_1}, \tfrac{3}{y_1}),(x_2,y_2))\\
&= (\tfrac{(2-x_2)y_1}{3y_2} + \tfrac{3x_1-2y_1}{3}, \tfrac{y_1}{y_2}) \vee (0,1),\\
&M_1((-1,2),(-5,2)) = (0,1), \quad (-1,2) \not\leq (-5,2).
\end{aligned}$$

$$\begin{aligned}
&M_2((x_1,y_1),(x_2,y_2)) = n_1S((x_2,y_2),n_2(x_1,y_1))\\
&= n_1S((x_2,y_2),(\tfrac{2-x_1}{y_1}, \tfrac{3}{y_1}))\\
&= (2 - \tfrac{2-x_1+3x_2}{y_2}, \tfrac{y_1}{y_2}) \vee (0,1),\\
&M_2((4,2),(3,2)) = (0,1), \quad (4,2) \not\leq (3,2),
\end{aligned}$$

$$\begin{aligned}
&M_3((x_1,y_1),(x_2,y_2)) = n_2S((x_2,y_2),n_1(x_1,y_1))\\
&= n_2S((x_2,y_2),(2-\tfrac{3x_1}{y_1}, \tfrac{3}{y_1})) = n_2(2-\tfrac{3x_1}{y_1}+\tfrac{3x_2}{y_1}, \tfrac{3y_2}{y_1})\\
&= (\tfrac{x_1}{y_2} - \tfrac{x_2}{y_2}, \tfrac{y_1}{y_2}) \vee (0,1),\\
&M_3((x_1,y_1),(x_2,y_2)) = (0,1) \text{ iff } (x_1,y_1) \leq (x_2,y_2),
\end{aligned}$$

By Theorem 3.8(2), $(L, \vee, \wedge, S, M_3, \bot, \top)$ is a left coresiduated lattice such that $S((x_1,y_1),(x_2,y_2)) \geq (x_3,y_3)$ iff $(x_1,y_1) \geq M_3((x_3,y_3),(x_2,y_2))$.

$$\begin{aligned}
&M_4((x_1,y_1),(x_2,y_2)) = n_1S(n_2(x_1,y_1),(x_2,y_2))\\
&= n_1S((\tfrac{2-x_1}{y_1}, \tfrac{3}{y_1}),(x_2,y_2)) = n_1(x_2 + (\tfrac{2-x_1}{y_1})y_2, \tfrac{3y_2}{y_1})\\
&= (\tfrac{-x_2y_1}{y_2} + x_1, \tfrac{y_1}{y_2}) \vee (0,1),\\
&M_4((x_1,y_1),(x_2,y_2)) = (0,1) \text{ iff } (x_1,y_1) \leq (x_2,y_2).
\end{aligned}$$

By Theorem 3.8(3), $(L, \vee, \wedge, S, M_4, \bot, \top)$ is a right coresiduated lattice such that $S((x_1,y_1),(x_2,y_2)) \geq (x_3,y_3)$ iff $(x_2,y_2) \geq M_4((x_3,y_3),(x_1,y_1))$. Moreover, $(L, \vee, \wedge, S, M_3, M_4, \bot, \top)$ is a generalized coresiduated lattice

(4) Since $n_1(n_1(x,y)) = (3x - 2y + 2, y)$,

$$\begin{aligned}
&n_1n_1S((x_1,y_1),(x_2,y_2)) = n_1n_1(x_2 + x_1y_2, y_1y_2)\\
&= (3x_2 + 3x_1y_2 - 2y_1y_2 + 2, y_1y_2) = S(n_1n_1(x_1,y_1), n_1n_1(x_2,y_2)).
\end{aligned}$$

$$\begin{aligned}
&M_5((x_1,y_1),(x_2,y_2)) = n_2S(n_1(x_2,y_2), n_1n_1(x_2,y_2))\\
&= n_2S((2-\tfrac{3x_1}{y_1}, \tfrac{3}{y_1}),(3x_2 - 2y_2 + 2, y_2))\\
&= n_2(3x_2 - 2y_2 + 2 + (2-\tfrac{3x_1}{y_1})y_2, \tfrac{3y_2}{y_1})\\
&= (\tfrac{-x_2y_1}{y_2} + x_1, \tfrac{y_1}{y_2}) \vee (0,1)\\
&= M_4((x_1,y_1),(x_2,y_2)).
\end{aligned}$$

(5) Since $n_2(n_2(x,y)) = (\frac{1}{3}(x+2y-2), y)$,

$$\begin{aligned} & n_2n_2(S(x_1,y_1),(x_2,y_2)) = n_2n_2(x_2+x_1y_2, y_1y_2) \\ & = (3x_2+3x_1y_2-2y_1y_2+2, y_1y_2)) \\ & = S(n_2n_2(x_1,y_1), n_2n_2(x_2,y_2)). \end{aligned}$$

$$\begin{aligned} & M_6((x_1,y_1),(x_2,y_2)) = n_1S(n_2n_2(x_2,y_2), n_2(x_1,y_1)) \\ & = n_1S((\tfrac{x_2+2y_2-2}{3}, y_2), (\tfrac{2-x_1}{y_1}, \tfrac{3}{y_1})) \\ & = n_1(\tfrac{-x_1+x_2+2y_2}{y_1}, \tfrac{3y_2}{y_1}) \\ & = (\tfrac{x_1}{y_2} - \tfrac{x_2}{y_2}, \tfrac{y_1}{y_2}) \vee (0,1) = M_3((x_1,y_1),(x_2,y_2)). \end{aligned}$$

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