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APPLYING THE PUARL MATHEMATICAL MODEL: PSYCHOLOGICAL INSIGHTS FOR RESTRICTING GAMING AND INTERNET ADDICTION

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Abstract. Consider social media and gaming addiction as diseases and consider their harmful effects on the user's health. This research introduces a PUARL epidemic model to prevent their spread among the population through psychological awareness. It calculates the basic reproduction number, analyzes stability, and conducts a sensitivity analysis. The study emphasizes the importance of psychological factors in preventing addiction, highlighting the role of psychological awareness.

1. Introduction

Internet gaming disorder (IGD) was recognized as a non-substance addiction in the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5) published by the American Psychiatric Association in 2013 [11]. This edition stemmed from a comprehensive review of over 250 articles by an international expert group. The DSM-5 outlined nine criteria for diagnosing IGD, including excessive preoccupation with gaming, withdrawal symptoms, increased tolerance, unsuccessful attempts to cut back, loss of interest in other activities, continued gaming despite negative consequences, deception about gaming habits, using gaming to cope with negative emotions, and impaired relationships or opportunities (See Petry et. al [32], Petry et.al [33], Pontes and Griffiths [34]).

Highlighting the significance of including such conditions in diagnostic criteria, efforts have been made to understand their prevalence, as emphasized by the World Health Organization in the context of public health interventions and treatment

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approaches globally (See Saunders et. al [36] and,the World Health Organization (WHO) [47]).

In the contemporary digital landscape, a burgeoning concern revolves around the phenomenon of social media addiction. As postulated by Hou et al. [16], an individual who allocates an excessive amount of time to engagement in popular social media platforms such as Instagram, Twitter, Facebook, YouTube, and others can be classified as exhibiting symptoms of social media addiction. Researchers of diverse origins have embarked on extensive inquiries to gauge the prevalence of this issue, as elucidated by Cheng et al. [9], garnering international recognition in the process.

Darvesh et al. [10] reviewed 160 studies that used 35 different methods to diagnose IGD. King and Paul [21] reviewed 36 studies on IGD cognition. Ko [22] reviewed epidemiologic, neurocognitive, and brain imaging studies related to IGD. Lemmens et al. [25] investigated the reliability and validity of four survey tools for assessing Internet Gaming Disorder (IGD) using the nine criteria outlined in the DSM-5. Mills and Allen [29] examined whether self-control explains the relation between daily need frustration and IGD.

Monacis et al. [30] tested the Italian version of the Bergen Social Media Addiction Scale (BSMAS). Przybylski et al.[35] conclude that the connection between IGD and game involvement was clear, but the relationship with physical, social, and mental health effects varied significantly. Spread of IGD was recognized by Stevens et al. [43]. They found that the worldwide prevalence of IGD was 3.05.

Wong et al. [48] examined the links between the severity of IGD and SMA and their impact on psychological health and sleep quality among university students in Hong Kong. Zaiac and Chang [49] scientific literature reviewed treatment for IGD. Kim et al. [20] identified global variability in gaming disorder prevalence, stressing the need for standardized diagnostic criteria.

Shek et al. [38] have meticulously examined the deleterious repercussions associated with inappropriate utilization of social media. Among these ramifications, the most prominent is recognized to be social media addiction, as underscored in the studies of Murray [31] and Siddiqui [40]. Additionally, Murray emphasizes the pivotal role played by mathematical modeling in comprehending and managing infectious diseases.

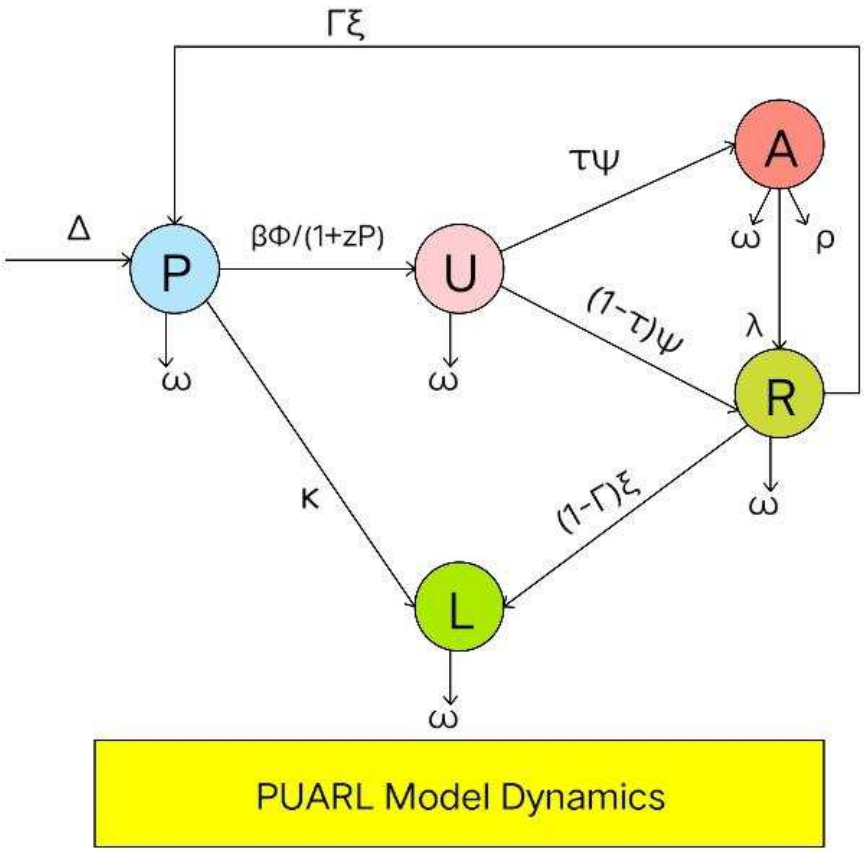

Figure 1. The PUARL model captures the dynamics involving distinct compartments represented by P, U, R, A, and L. The directional arrows signify the flow of population between these compartments, with various parameters symbolically associated with each arrow to represent the transmission processes.

A multitude of scholars, encompassing Alemneh [1], Balakrishnan [2], Chakraborty [8], Cheng [9], Guedes [13], Guo [14], [15], and Li [26], have harnessed the framework of infectious disease dynamics models to investigate a spectrum of addictive behaviors, ranging from gaming, alcohol, and drugs to social media.

Several of these investigations have served as the impetus for the development of analogous models concerning drug and alcohol addiction (See Khajji et.al [19], Ma et.al [27] and, Wang et.al [46]), video game addiction (See Guo and Li [14], [15], [26]), and the detrimental impact of social media on academic achievement (See Ishaku et.al [17], Kolan and Dzandza [23] and, Malak et.al [28]).

Chakraborty et al. [8] posit that social media addiction, in its early stages, may exhibit seemingly benign attributes, akin to addictions associated with gambling, alcohol, and drugs. Nevertheless, it is an evolving predicament that necessitates diligent attention. Moreover, investigations conducted by Shek et al. [38] and Malak et al. [28] explore the intricate nexus between social media addiction and various psychological factors.

Simsek and associates [41], as well as Sujarwoto and their team [44], conducted investigations delving into the association between social media addiction and its influence on the academic performance of students. On a separate note, Alemneh and Alemu [1], along with Guo and Li [[14], [15], [26]], developed an optimal control strategy for a deterministic mathematical model focusing on social media addiction.

Several research studies have shown that video games can be harmful to us. If this habit becomes addictive and starts interfering with our daily tasks, then it can be considered a disease. According to a report published in Bloomberg [4] and cited by Dainik Bhaskar newspaper [5], the World Health Organization recently classified the addiction to video games as a disease.

2. Methodology

In this section, we consider a deterministic mathematical PUARL model for the construction of a social media and gaming addiction framework, based on the following assumptions: the social media addiction phenomenon occurs within a closed environment; factors such as sex, race, and social status do not influence the likelihood of becoming addicted to social media; individuals interact homogeneously (experiencing the same degree of interaction); and social media and gaming addiction is transmitted to non-addicted individuals through peer pressure from those who are addicted.

2.1. Formulation of Model Utilizing mathematical modeling proves to be a valuable methodology for comprehending the intricate dynamics inherent in the progression of epidemics or pandemics. In the context of epidemic modeling, the classical SIR model, initially formulated by Kermack and McKendrick [18], can be subject to refinement through the incorporation of novel compartments, nuanced parametarizations, and diverse incidence rate functions. Building upon the foundational research by Alemneh and Alemu [1], and inspired by the previous work done by Soni

et al. [37] this study introduces a novel non-linear incidence rate, culminating in the development of the PUARL model (as illustrated in Figure 1).

In this model, the human population is divided into five sub-populations based on addiction status. Predisposed individuals (denoted by P) are those who are not yet addicted but are susceptible to social media and gaming addiction, Uncovered individuals (denoted by U) are those who use social media infrequently and do not progress to addiction, Addicted individuals (denoted by A) are those who are addicted to social media/gaming and spend a significant amount of their time on it, Recovered individuals (denoted by R) are those who have overcome social media addiction, Permanently departed individuals (denoted by L) are those who have permanently quit using social media and gaming.

Predisposed individuals enter the population at a rate Δ. These individuals start using social media and gaming due to peer pressure at a contact rate of β from addicted individuals, with a transmission probability ϕ, and move to the uncovered compartment. Some predisposed individuals shift to the sub population of those who do not permanently use social media at a rate κ. Uncovered individuals become addicted and enter the addicted compartment at a rate $\tau\psi$, while the rest of the uncovered individuals recover through treatment at a rate $(1-\tau)\psi$. Addicted individuals either transition to the recovered compartment through education and/or treatment at a rate λ or die from addiction at a rate ρ. Recovered individuals can become predisposed to social media addiction again at a rate $\Gamma\xi$ or completely stop using social media at a rate $(1-\Gamma)\xi$. The entire population has an average death rate of ω. The parameters are detailed in Table 2, and the model's flow diagram is shown in Figure 1.

To examine the psychological impact of predisposed individuals on addiction, we employ the nonlinear incidence rate $\dfrac{\beta\phi}{(1+zP)}$, incorporating the psychological factor rate z specific to predisposed individuals. The parameter z denotes the awareness of social media addiction within the predisposed population, thereby moderating their engagement in a typical and excessive social media and gaming use. When $z = 0$, the incidence rate simplifies to the classical bi-linear form. Similarly, this kind of

incidence rate is taken by Sharma S. and Sharma P.K. [39].

$$
\begin{aligned}
\frac{dP}{dt} &= \Delta + \Gamma\xi R - \frac{\beta\Phi AP}{1+zP} - (\kappa+\omega)P \\
\frac{dU}{dt} &= \frac{\beta\Phi AP}{1+zP} - (\psi+\omega)U \\
\frac{dA}{dt} &= \tau\psi U - (\omega+\lambda+\rho)A \\
\frac{dR}{dt} &= (1-\tau)\psi U + \lambda A - (\omega+\xi)R \\
\frac{dL}{dt} &= \kappa P + (1-\Gamma)\xi R - \omega L
\end{aligned} \tag{1}
$$

where,

$$N = P + U + A + R + L \quad \text{and} \quad P(0) > 0, U(0), A(0), R(0), L(0) \geq 0$$

Involved parameter descriptions and their values are mentioned in Table 2.

2.2. Invariant Region By incorporating the equations from system (1), we arrive at the following expressions:

$$
\begin{aligned}
&\frac{dN}{dt} = \Delta - \omega N - \rho A \\
&\frac{dN}{dt} \leq \Delta - \omega N \quad \text{(Ignoring disease-related deaths)} \\
&\frac{dN}{dt} \to \frac{\Delta}{\omega}, \quad \text{as} \quad t \to \infty
\end{aligned}
$$

Hence the region $\Omega = \{(P, U, A, R, L) \in \Re^5_+, 0 < N(t) \leq \frac{\Delta}{\omega}\}$ remains invariant with the initial conditions.

2.3. Positivity of Solution Commencing with the first equation in the system (1), we obtain,

$$
\begin{aligned}
&\frac{dP}{dt} \leq \Delta - (\kappa+\omega)P \Rightarrow P(t) \leq P(0)e^{-(\kappa+\omega)t} + \frac{\Delta}{\kappa+\omega}(1 - e^{-(\kappa+\omega)t}) \\
&\text{as} \quad t \to \infty, P \to \frac{\Delta}{\kappa+\omega}, \Rightarrow 0 < P(t) \leq \frac{\Delta}{\kappa+\omega}
\end{aligned}
$$

Similarly, we can get, $U(0), A(0), R(0), L(0) \geq 0$
Hence, every viable solution of system (1) exists within the domain denoted by Ω.

2.4. AFEP (Addiction-Free Equilibrium Point) Without addiction $U = A = 0$, so $AFEP = \Lambda = \left(\frac{\Delta}{\kappa+\omega}, 0, 0, 0, \frac{\kappa\Delta}{\omega(\kappa+\omega)}\right)$

2.5. The Basic Reproduction Number The Basic reproduction number (Often denoted as R_0) plays a pivotal role in epidemiology. It serves as a vital metric for quantifying the potential of an infectious disease to spread. Essentially, R_0 represents the average number of new infections initiated by a single infected person within a susceptible population.

This metric provides insights into a disease's contagiousness and aids public health officials in assessing its potential impact. R_0 values can vary significantly among different diseases and are influenced by several factors (Such as the mode of transmission, population density, and individual behaviors). For example, highly contagious diseases like measles can have R_0 values exceeding 10, indicating that each infected individual has the potential to infect more than ten others. In contrast, diseases (Such as influenza, Covid-19 etc.) with lower R_0 values, pose significant risks but tend to spread more slowly.

Understanding the basic reproduction number is essential for implementing effective strategies during disease outbreaks. Researchers and policymakers can make informed decisions about interventions (Such as vaccination campaigns, quarantine measures, or social distancing guidelines) by observing and comparing R_0 values. This number is a key epidemiological metric that determines the proportion of the population that needs to be immune, either through vaccination or previous infection, to achieve herd immunity and control the spread of a disease, as mentioned by Soni M. et al. [42] in their work.

In terms of our model's stability analysis, the calculation of the basic reproduction numberR_0 is of paramount importance. We determine R_0 using the next-generation matrix method given by Van den Driessche [45].

The next-generation matrix (NGM) is constructed by considering two matrices:

1. New Infections Matrix (ν): This matrix represents the rate of new infections in each infected compartment, with each entry (ν_{ij}) indicating the rate of new infections in compartment (i) caused by individuals in compartment (j).

2. Transition Matrix (ζ): This matrix represents the rate at which individuals transition between compartments, excluding new infections, with each entry ζ_{ij} indicating the rate of transfer out of compartment (i) to compartment (j).

The inverse of the transition matrix ζ is then computed. The next-generation matrix is formed by multiplying the new infections matrix ν by the inverse of the transition matrix (ζ): (NGM $= \nu\zeta^{-1}$). The basic reproduction number R_0 is given

by the spectral radius (the largest absolute value of the eigenvalues) of the next-generation matrix.

Let

$$x = (U, A, R, P, U)^T.$$

The equations governing the infected compartments (U, A, R) and uninfected compartments (P and L) within the system (1) are used for this purpose. The system (1) can be rewritten as:

$$\frac{dx}{dt} = \nu - \zeta$$

where,

$$v = \begin{pmatrix} \dfrac{\beta\Phi AP}{1+zP} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} (\psi+\omega)U \\ -\tau\psi U + (\omega+\lambda+\rho)A \\ -(1-\tau)\psi U - \lambda A + (\omega+\xi)R \end{pmatrix}$$

$$\nu_{AFEP} = \begin{pmatrix} \frac{\partial v_{11}}{\partial U} & \frac{\partial v_{11}}{\partial A} & \frac{\partial v_{11}}{\partial R} \\ \frac{\partial v_{22}}{\partial U} & \frac{\partial v_{22}}{\partial A} & \frac{\partial v_{22}}{\partial R} \\ \frac{\partial v_{33}}{\partial U} & \frac{\partial v_{33}}{\partial A} & \frac{\partial v_{33}}{\partial R} \end{pmatrix} \quad \text{and} \quad \zeta_{AFEP} = \begin{pmatrix} \frac{\partial \zeta_{11}}{\partial U} & \frac{\partial \zeta_{11}}{\partial A} & \frac{\partial \zeta_{11}}{\partial R} \\ \frac{\partial \zeta_{22}}{\partial U} & \frac{\partial \zeta_{22}}{\partial A} & \frac{\partial \zeta_{22}}{\partial R} \\ \frac{\partial \zeta_{33}}{\partial U} & \frac{\partial \zeta_{33}}{\partial A} & \frac{\partial \zeta_{33}}{\partial R} \end{pmatrix}$$

$$\nu_{AFEP} = \begin{pmatrix} 0 & \dfrac{\beta\Phi\Delta}{\kappa+\omega+z\Delta} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\zeta_{AFEP} = \begin{pmatrix} (\psi+\omega) & 0 & 0 \\ -\tau\psi & (\omega+\lambda+\rho) & 0 \\ -(1-\tau)\psi & -\lambda & (\omega+\xi) \end{pmatrix}$$

$$\zeta_{AFEP}^{-1} = \frac{Adj\zeta_{AFEP}}{|\zeta_{AFEP}|} = \frac{1}{(\omega+\xi)(\psi+\omega)(\omega+\lambda+\rho)} \times$$
$$\times \begin{bmatrix} (\psi+\omega)(\omega+\lambda+\rho) & 0 & 0 \\ \tau\psi(\omega+\xi) & (\omega+\xi)(\psi+\omega) & 0 \\ \lambda\tau\psi + (1-\tau)\psi(\omega+\lambda+\rho) & \lambda(\psi+\omega) & (\omega+\psi)(\omega+\lambda+\rho) \end{bmatrix}$$

The spectral radius of matrix $v_{AFEP}\zeta_{AFEP}^{-1}$ is the required basic reproduction number R_0.

$$\Rightarrow R_0 = \frac{\beta\Delta\Phi\tau\psi}{(\kappa+\omega+z\Delta)(\psi+\omega)(\omega+\lambda+\rho)} \tag{2}$$

2.6. Local Stability at AFEP The Jacobian matrix at AFEP is given by,

$$J_{AFEP} = \begin{pmatrix} -(\kappa+\omega) & 0 & -\dfrac{\beta\Phi\Delta}{(\kappa+\omega+z\Delta)} & \Gamma\xi & 0 \\ 0 & -(\psi+\omega) & \dfrac{\beta\Phi\Delta}{(\kappa+\omega+z\Delta)} & 0 & 0 \\ 0 & \tau\psi & -(\lambda+\omega+\rho) & 0 & 0 \\ 0 & (1-\tau)\psi & \lambda & -(\xi+\omega) & 0 \\ \kappa & 0 & 0 & (1-\Gamma)\xi & -\omega \end{pmatrix}$$

Three negative eigenvalues are $-\omega$, $-(\omega+\kappa)$ and $-(\omega+\xi)$. The remaining two eigenvalues are given by,

$$x^2 + M_1 x + M_2 = 0 \tag{3}$$

By the Routh-Hurwitz Criteria, negativity of roots exists in equation(3) if and only if $M_1, M_2 > 0$, where $M_1 = \lambda+\rho+\psi+2\omega$ and $M_2 = (\psi+\omega)(\lambda+\rho+\omega)(1-R_0) > 0$ as $R_0 < 1$.

Hence the system is locally asymptotically stable at AFEP whenever $R_0 < 1$.

Furthermore, global stability means that the system will return to an equilibrium point regardless of the initial conditions, implying it is the only attractor and all trajectories lead to it. If $R_0 < 1$, the addiction-free equilibrium is globally asymptotically stable meaning the system will eventually converge to the addiction-free state regardless of the initial number of infected individuals. Proving global stability often involves more complex mathematical tools, such as Lyapunov functions or LaSalle's invariance principle [24], which show that all solutions converge to the equilibrium point. Global stability provides a stronger and more comprehensive form of stability. In the next section, we discuss the global asymptotic stability at AFEP.

2.7. Global Asymptotic Stability at AFEP We utilize the Castillo-Chavez theorem [7] to reconfigure the expression of system (1) into the following format:

$$\begin{aligned} X' &= B(X,Y) \\ Y' &= T(X,Y) \quad \text{and} \quad T(X,0) = 0 \end{aligned} \tag{4}$$

where X and Y denoted uninfected and infected populations respectively.

$$X = (P,R,L) \in \Re^3 \quad \text{and} \quad Y = (U,A) \in \Re^2$$

Let $M_0 = (X^*, 0)$ the AFEP of the system (4). It is globally asymptotically stable if the following two conditions satisfied:

H_1: X^* is globally asymptotically stable for $X' = B(X,0)$.

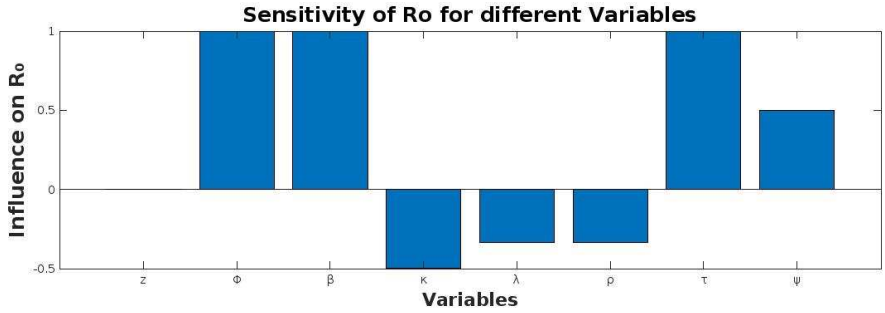

Figure 2. Sensitivity Analysis of various parameter for R_0.

H_2: $S' = D_sT(X^*, 0)S - \widehat{T}(X, S)$, $\widehat{T}(X, S) \geq 0 \quad \forall (X, S) \in \Omega$,
where $D_sT(X^{,}0)$ is the jacobian of $T(X, S)$ and evaluated at $(X^{,}0)$.

$$B(X, Y) = \begin{pmatrix} \Delta + \Gamma\xi R - \dfrac{\beta\Phi AP}{1+zP} - (\kappa+\omega)P \\ (1-\tau)\psi U + \lambda A - (\omega+\xi)R \\ \kappa P + (1-\Gamma)\xi R - \omega L \end{pmatrix}$$

$$T(X, Y) = \begin{pmatrix} \dfrac{\beta\Phi AP}{1+zP} - (\psi+\omega)U \\ \tau\psi U - (\omega+\lambda+\rho)A \end{pmatrix}$$

Consider $X' = (B, 0)$, using H_1 we have,

$$\frac{dP}{dt} = \Delta - (\kappa+\omega)P$$
$$\frac{dR}{dt} = 0$$
$$\frac{dL}{dt} = \kappa P - \omega L$$

$$D_yT(X^*, 0) = \begin{pmatrix} -(\psi+\omega) & \dfrac{\beta\Phi\Delta}{(\kappa+\omega+z\Delta)} \\ \tau\psi & -(\omega+\lambda+\rho) \end{pmatrix}$$

From H_2, we obtain,

$$\widehat{T}(X, Y) = \begin{pmatrix} \beta\Phi A\left(\dfrac{\Delta}{\kappa+\omega+z\Delta} - \dfrac{P}{1+zP}\right) \\ 0 \end{pmatrix}$$

$$\because X^* = \frac{\Delta}{\kappa+\omega} \geq P, \quad \Rightarrow \widehat{T}(X, Y) \geq 0, \quad \forall X, Y \in \Omega.$$

Therefore, according to Lasalle's invariance principle [24], AFEP is globally asymptotically stable.

Parameter	sign
β	+ve
Φ	+ve
ρ	-ve
κ	-ve
λ	-ve
τ	+ve
ψ	+ve
z	-ve

Table 1. Sensitivity indices

2.8. Addiction Endemic Equilibrium Point (AEEP) If social media addiction continues to exist among the population, the model will exhibit a state known as the endemic equilibrium point AEEP $= (P^*, U^*, A^*, R^*, L^*)$. One can derive it by equating each equation in system (1) to zero, i.e., $\frac{dP}{dt} = \frac{dU}{dt} = \frac{dA}{dt} = \frac{dR}{dt} = \frac{dL}{dt} = 0$, we obtain,

$$
\begin{aligned}
U^* &= \frac{(\kappa+\omega)P^* - \Delta}{\frac{\Gamma\xi(\psi(1-\tau)(\omega+\lambda+\rho)+\lambda\tau\psi)}{(\omega+\xi)(\omega+\lambda+\rho)} - (\psi+\omega)} \\
A^* &= \left(\frac{\tau\psi}{\omega+\lambda+\rho}\right)U^* \\
P^* &= \frac{(\omega+\psi)}{\beta\Phi}\left(\frac{\omega+\lambda+\rho}{\tau\psi} + zU^*\right) \\
R^* &= \left(\frac{\psi(1-\tau)(\omega+\lambda+\rho)+\lambda\tau\psi}{(\omega+\xi)(\omega+\lambda+\rho)}\right)U^* \\
L^* &= \frac{\kappa P^+(1-\Gamma)\xi R^*}{\omega}
\end{aligned}
$$

To investigate the local stability of addiction equilibrium point and the nature of bifurcation, we used the method introduced by Castillo-Chavez, C., & Song, B. [6] in their research work.

Theorem 2.1. *If $R_0 > 1$, then the endemic equilibrium AEEP of system (1) is locally asymptotically stable in Ω and the system (1) exhibits forward bifurcation at $R_0 = 1$.*

Proof. Let $P = x_1, U = x_2, A = x_3, R = x_4, L = x_5$ the system (1) can be written as:

$$\begin{aligned}
\frac{dx_1}{dt} &= \Delta + \Gamma\xi R - \frac{\beta\Phi x_1 x_3}{1+zx_1} - (\kappa+\omega)x_1 \equiv g_1 \\
\frac{dx_2}{dt} &= \frac{\beta\Phi x_1 x_3}{1+zx_1} - (\psi+\omega)x_2 \equiv g_2 \\
\frac{dx_3}{dt} &= \tau\psi x_2 - (\omega+\lambda+\rho)x_3 \equiv g_3 \\
\frac{dx_4}{dt} &= (1-\tau)\psi x_2 + \lambda x_3 - (\omega+\xi)x_4 \equiv g_4 \\
\frac{dx_5}{dt} &= \kappa x_1 + (1-\Gamma)\xi x_4 - \omega x_5 \equiv g_5
\end{aligned} \tag{5}$$

We consider the transmission rate β as bifurcation parameter so that $R_0 = 1$ iff

$$\beta = \beta^* = \frac{(\kappa+\omega+z\Delta)(\psi+\omega)(\omega+\lambda+\rho)}{\Delta\Phi\tau\psi}.$$

Then the linearizing matrix equation at AFEP is given by:

$$J_{AFEP} = \begin{pmatrix}
-(\kappa+\omega) & 0 & -\dfrac{\beta\Phi\Delta}{(\kappa+\omega+z\Delta)} & \Gamma\xi & 0 \\
0 & -(\psi+\omega) & \dfrac{\beta\Phi\Delta}{(\kappa+\omega+z\Delta)} & 0 & 0 \\
0 & \tau\psi & -(\lambda+\omega+\rho) & 0 & 0 \\
0 & (1-\tau)\psi & \lambda & -(\xi+\omega) & 0 \\
\kappa & 0 & 0 & (1-\Gamma)\xi & -\omega
\end{pmatrix}$$

To compute the right eigenvector, $u = (u_1, u_2, u_3, u_4, u_5)^T$, we consider $J_{AFEP}.u = 0$. Then the system (5) becomes

$$\begin{aligned}
-(\kappa+\omega)u_1 - \frac{\beta\Phi\Delta}{\kappa+\omega+z\Delta}u_3 + \Gamma\xi u_4 &= 0 \\
-(\psi+\omega)u_2 + \frac{\beta\Phi\Delta}{\kappa+\omega+z\Delta}u_3 &= 0 \\
\tau\psi u_2 - (\lambda+\omega+\rho)u_3 &= 0 \\
(1-\tau)\psi u_2 + \lambda u_3 - (\xi+\omega)u_4 &= 0 \\
\kappa u_1 + (1-\Gamma)\xi u_4 - \lambda\omega u_5 &= 0
\end{aligned} \tag{6}$$

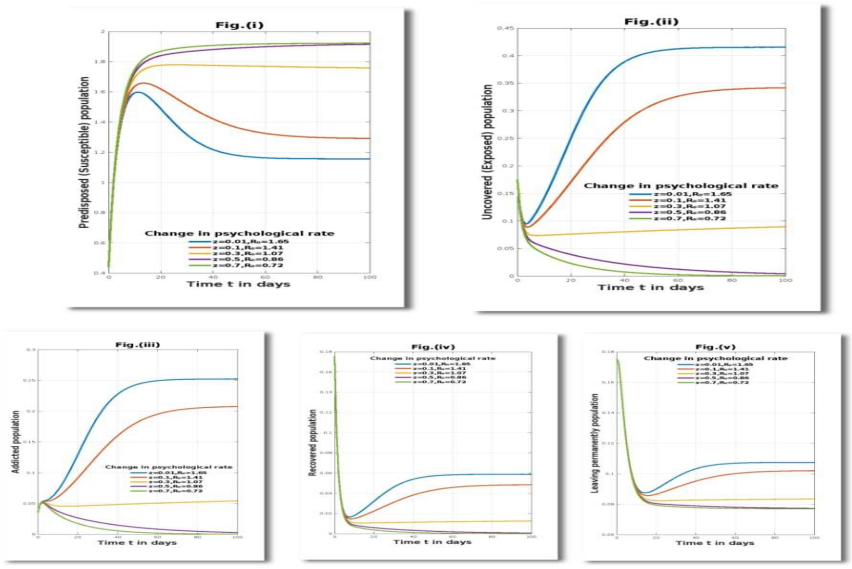

Figure 3. Effect of changing in psychological awareness rate on (i) Predisposed population, (ii) Uncovered population, (iii) Addicted Population, (iv) Recovered population, (v) Leave permanently population.

Solving above set of equations (6) we obtain,

$$u_1 = \frac{\psi\Gamma\xi(1-\tau)(\omega+\rho+\lambda) + \Gamma\psi\lambda\tau\xi - (\psi+\omega)(\xi+\omega)(\omega+\rho+\lambda)}{(\kappa+\omega)(\xi+\omega)(\omega+\rho+\lambda)}u_2$$

$$u_3 = \frac{\tau\psi}{\omega+\lambda+\rho}u_2$$

$$u_4 = \frac{(1-\tau)(\omega+\rho+\lambda)\psi + \tau\psi\lambda}{(\xi+\omega)(\omega+\lambda+\rho)}u_2$$

$$u_5 = \frac{\zeta - \kappa\tau\lambda\Gamma\psi\xi + \omega(\kappa+\omega)\lambda\tau\psi}{\omega(\kappa+\omega)(\xi+\omega)(\omega+\lambda+\rho)}u_2$$

where $\zeta = (\omega+\lambda+\rho)(-\kappa\psi\Gamma\xi(1-\tau) + \kappa(\xi+\omega)(\psi+\omega) + \omega\psi(\kappa+\omega)(1-\tau)(1-\Gamma))$. The left eigenvector $v = (v_1, v_2, v_3, v_4, v_5)$, can be obtain by using $v.J_{AFEP} = 0$, where

$$v_1 = 0, v_4 = 0, v_5 = 0, v_3 = \frac{\omega+\psi}{\tau\psi}v_2.$$

where v_2 is calculated to ensure that the eigenvectors satisfy the condition $u.v = 1$. From the derivatives of g_2 and g_3, the only ones that are nonzero are:

$$\frac{\partial^2 g_2}{\partial x_1 \partial x_3} = \frac{\partial^2 g_2}{\partial x_3 \partial x_1} = \frac{\beta\phi}{(1+zx_1^*)^2} \quad \text{and} \quad \frac{\partial^2 g_2}{\partial x_3 \partial \beta} = \frac{\Phi x_1^*}{(1+zx_1^*)}$$

The direction of the bifurcation at $R_0 = 1$ is determined by the signs of the bifurcation coefficients 'a' and 'b', obtained from the above partial derivatives, given respectively by:

$$a = \frac{2v_2u_1u_3\beta\Phi}{(1+zx_1^*)^2} = \frac{2v_2u_1u_3\Phi\beta(\kappa+\omega)^2(\Psi+\omega)(\omega+\lambda+\rho)}{\Delta\tau\Psi(\kappa+\omega+z\Delta)} < 0 \tag{7}$$

$$b = \frac{v_2u_3\Phi x_1^*}{(1+zx_1^*)} = \frac{v_2u_3\Phi\Delta}{\kappa+\omega+z\Delta} > 0 \tag{8}$$

Since denominator of a is a square term, it must be positive. The remaining term is the same as in the previous work done by Alemneh[1], hence a must be negative and b must be positive. In forward bifurcation the disease-free equilibrium loses its stability and an endemic equilibrium that is stable emerges as the basic reproduction number rises by one.

The following are the primary traits of forward bifurcation:

1. There is no endemic equilibrium close to disease- free equilibrium when $R_0 < 1$, which is one of the forward bifurcation's primary characteristics. In other words, when $R_0 < 1$, the disease- free equilibrium frequently exists alone.
2. Low endemicity when R_0 is marginally more than unity.

From equation (7) 'a' is consistently negative and from eq.(8) 'b' is consistently positive, the bifurcation is forward. When $R_0 < 1$, occurs in the forward bifurcation, a modest influx of infected people won't cause massive outbreaks, the disease dies out over time, and the related disease-free state is asymptotically stable. On the other hand, if $R_0 > 1$, the disease will continue to exist in a stable endemic equilibrium. Since the unique endemic equilibrium point is locally asymptotically stable for $R_0 > 1$ and system (1) displays forward bifurcation at $R_0 = 1$, the AFEP and the endemic equilibrium point cannot coexist. □

3. Sensitivity Analysis

To illustrate the influence of individual parameters on the transmission of social media addiction in our model, we conducted a sensitivity analysis. This analysis

was carried out following the methodology described in reference [3]. We employed the concept of the normalized forward sensitivity index to quantify how a variable R_0 is affected by variations in a specific parameter,χ, is defined by:

$$\Lambda_{\chi}^{R_0} = \frac{\partial R_0}{\partial \chi} \times \frac{\chi}{R_0},$$

where χ represents all the basic parameters and

$$R_0 = \frac{\beta \Delta \Phi \tau \psi}{(\kappa + \omega + z\Delta)(\psi + \omega)(\omega + \lambda + \rho)}.$$

Then

$$\begin{aligned}
\Lambda_{\beta}^{R_0} &= \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = 1 > 0 \\
\Lambda_{\Phi}^{R_0} &= \frac{\partial R_0}{\partial \Phi} \times \frac{\Phi}{R_0} = 1 > 0 \\
\Lambda_{\tau}^{R_0} &= \frac{\partial R_0}{\partial \tau} \times \frac{\tau}{R_0} = 1 > 0 \\
\Lambda_{\psi}^{R_0} &= \frac{\partial R_0}{\partial \psi} \times \frac{\psi}{R_0} = \frac{\omega}{(\psi + \omega)} > 0 \\
\Lambda_{\rho}^{R_0} &= \frac{\partial R_0}{\partial \rho} \times \frac{\rho}{R_0} = -\frac{\rho}{\rho + \omega + \lambda} < 0 \\
\Lambda_{\lambda}^{R_0} &= \frac{\partial R_0}{\partial \lambda} \times \frac{\lambda}{R_0} = -\frac{\lambda}{\lambda + \rho + \omega} < 0 \\
\Lambda_{\kappa}^{R_0} &= \frac{\partial R_0}{\partial \kappa} \times \frac{\kappa}{R_0} = -\frac{\kappa}{(\kappa + \omega)} < 0 \\
\Lambda_{z}^{R_0} &= -\frac{\partial R_0}{\partial z} \times \frac{z}{R_0} = -\frac{z\Delta}{\kappa + \omega + z\Delta} < 0
\end{aligned}$$

When taking all these parameter values as 0.001, we obtain the Figure 2.

Obviously, the parameters Φ, β, τ and ψ display positive influence, whereas the parameters κ, λ and ρ display negative influence. It means decreasing the value of positive influence parameter and increasing the value of negative influence parameter can lead to reduce the value of R_0. Here we should note that R_0 is independent from the psychological constant parameter z, but as a part of incidence rate it affects the transmission directly.

4. Numerical Simulation

We utilize the following parameter values to fulfil the purpose of simulation.

Using the above values we obtain the value of R_0 is $1.6827 > 1$. To illustrate the effectiveness of psychological rate (z) on various compartments of the model, we fix all the parameter values in Table 2 and then make a change in the psychological rate (see Figure 3).

Except for the predisposed population, it's evident that all other population groups are inversely related to the psychological rate. In other words, as the rate increases, these populations decrease comparatively. The number of recovered and permanently displaced individuals also declines as exposure and addiction decrease. In conclusion, the basic reproduction number is directly tied to the psychological awareness rate z, significantly impacting disease propagation dynamics and playing a crucial role in preventing its spread.

5. Results and Discussion

We utilized the PUARL model to examine the patterns of social media addiction within various segments of the population. In our model, we consider a nonlinear incidence rate, which can be expressed as: $\frac{\beta\Phi AP}{1+zP}$, with the psychological rate z. When we set z equal to zero, our model aligns with the earlier model proposed by Alemneh and Alemu [1]. To the best of our knowledge, our model stands out as the initial one to incorporate the psychological awareness aspect of social media addiction dynamics within the susceptible population. We've also identified the local and global stability of the AFEP, revealing that the system is asymptotically stable when the value of $R_0 < 1$, but it becomes unstable when $R_0 > 1$. Regarding the local stability of AEEP, it signifies that the system exhibits asymptotic stability when the value of $R_0 > 1$. Our sensitivity analysis highlights that the factors with the most positive impact are the transmission rate (β), contact rate (ϕ), and proportion rate (τ), while the most detrimental factor is (κ), which represents the portion of the population that remains addicted to social media without giving up. The psychological rate plays a direct role in determining the basic reproduction number, and simulations emphasize the crucial need for managing addiction's spread effectively.

6. Conclusion

The primary concern for social media users is the problem of addiction.

Symbol	Description	Value	Source
β	Transmission rate of addiction to predisposed population	0.9	Assumed
ρ	Death rate induced due to disease	0.01	Assumed
Φ	Contact rate of predisposed population with addicted population	0.8	Assumed
ψ	Individual that leave uncovered class	0.25	[10]
Δ	Recruitment rate of predisposed population	0.5	Assumed
ξ	Individual that leave recovered population	0.4	[12]
κ	That predisposed population which don't give up from adopting social media	0.01	Assumed
λ	Transmission rate from addicted to recovered population	0.028	Assumed
τ	Proportion of uncovered population that join addicted population	0.7	[10]
ω	Natural death rate	0.25	[30]
z	Psychological rate	0.01	Assumed
Γ	Proportion of recovered population predisposed to Social media addiction	0.35	[18]
P	Predisposed population (Those who are not addicted but may be addicted to social media)	25000	Assumed
U	Uncovered population (Those who adopt social media less frequently but do not process to the addicted stage)	10000	Assumed
A	Addicted population (Those who are addicted to social media and spent most of their time on it)	2000	Assumed
R	Recovered population (Those who are recovered to social media addiction)	10000	Assumed
L	Leave the population (Those who do not adopt social media and quit its use forever)	10000	Assumed

Table 2. Parameter and symbol with their description and value

To prevent addiction, individuals can gain psychological awareness about the risks associated with excessive and improper social media use. Research underscores the importance of the psychological factor in shaping social media addiction dynamics. Undeniably, simulations indicate that increasing psychological awareness can reduce the prevalence of addiction in society.

7. Conflict of Interest

There is not any kind of conflict.

References

1. Alemneh, H.T. & Alemu, N.Y.: Mathematical modeling with optimal control analysis of social media addiction. Infectious Disease Modelling **6** (2021), 405-419. https://doi.org/10.1016/j.idm.2021.01.011
2. Balakrishnan, J. & Griffiths, M.D.: Social media addiction: What is the role of content in YouTube?. Journal of behavioral addictions **6** (2017), no. 3, 364-377. https://doi.org/10.1556/2006.6.2017.058
3. Blower, S.M. & Dowlatabadi, H.: Sensitivity and uncertainty analysis of complex models of disease transmission: an HIV model, as an example. International Statistical Review/Revue Internationale de Statistique, 229-243. https://doi.org/10.2307/1403510
4. https://www.bloomberg.com/news/articles/2022-02-23/video-game-addiction-recognized-by-world-health-organization accessed on 2 January 2024.
5. https://www.bhaskar.com/who-told-mobile-game-addiction-in-the-category-of-disorder-kept-brain-disease-5902462.html/ accessed on 2 January 2024.
6. Castillo-Chavez, C. & Song, B.: Dynamical models of tuberculosis and their applications. Math. Biosci. Eng. **1** (2004), no. 2, 361-404. https://doi.org/10.3934/mbe.2004.1.361
7. Castillo-Chavez, C., Blower, S., van den Driessche, P., Kirschner, D. & Yakubu, A.A.(Eds.): Mathematical approaches for emerging and reemerging infectious diseases: models, methods, and theory (Vol. 126). Springer Science & Business Media, 2002. https://doi.org/10.1007/978-1-4613-0065-6
8. Chakraborty, K., Basu, D., & Kumar, K.V.: Internet addiction: Consensus, controversies, and the way ahead. East Asian Archives of Psychiatry **20** (2010), no. 3, 123-132. https://search.informit.org/doi/10.3316/informit.935942388038735
9. Cheng, C., Lau, Y.C., Chan, L. & Luk, J.W.: Prevalence of social media addiction across 32 nations: Meta-analysis with subgroup analysis of classification schemes

and cultural values. Addictive behaviors **117** (2021), 106845. https://doi.org/10.1016/j.addbeh.2021.106845

10. Darvesh, N., Radhakrishnan, A., Lachance, C.C., Nincic, V., Sharpe, J.P., Ghassemi, M., ... & Tricco, A.C.: Exploring the prevalence of gaming disorder and Internet gaming disorder: A rapid scoping review. Systematic reviews **9** (2020), 1-10. https://doi.org/10.1186/s13643-020-01329-2
11. Edition, F.: Diagnostic and statistical manual of mental disorders. Am Psychiatric Assoc. **21** (2013), no. 21, 591-643. https://doi.org/10.1176/appi.books.9780890425596
12. Gentile, D.A., Bailey, K., Bavelier, D., Brockmyer, J.F., Cash, H., Coyne, S.M., ... & Young, K.: Internet gaming disorder in children and adolescents. Pediatrics **140**(Supplement_2) (2017), S81-S85. https://doi.org/10.1542/peds.2016-1758H
13. Guedes, E., Nardi, A.E., Guimaraes, F. M.C.L., Machado, S. & King, A.L.S.: Social networking, a new online addiction: a review of Facebook and other addiction disorders. Medical Express **3** (2016), M160101. https://doi.org/10.5935/MedicalExpress.2016.01.01
14. Guo, Y. & Li, T.: Optimal control and stability analysis of an online game addiction model with two stages. Mathematical Methods in the Applied Sciences **43** (2020), no. 7, 4391-4408. https://doi.org/10.1002/mma.6200
15. Guo, Y. & Li, T.: Fractional-order modeling and optimal control of a new online game addiction model based on real data. Communications in Nonlinear Science and Numerical Simulation **121** (2023), 107221. https://doi.org/10.1016/j.cnsns.2023.107221
16. Hou, Y., Xiong, D., Jiang, T., Song, L. & Wang, Q.: Social media addiction: Its impact, mediation, and intervention. Cyberpsychology **13** (2019), no. 1. https://doi.org/10.5817/CP2019-1-4
17. Ishaku, A., Musa, B.S., Sanda, A. & Bakoji, A.M.: Mathematical assessment of social media impact on academic performance of students in higher institution. IOSR J. Math. **2018** (2018), 72-79. https://doi.org/10.9790/5728-1401017279
18. Kermack, W.O. & McKendrick, A.G.: A contribution to the mathematical theory of epidemics. Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character **115** (1927), no. 772, 700-721. https://doi.org/10.1098/rspa.1927.0118

19. Khajji, B., Labzai, A., Balatif, O. & Rachik, M.: Mathematical modeling and analysis of an alcohol drinking model with the influence of alcohol treatment centers. International Journal of Mathematics and Mathematical Sciences **2020** (2020), 1-12. https://doi.org/10.1155/2020/4903168
20. Kim, H.S., Son, G., Roh, E.B., Ahn, W.Y., Kim, J., Shin, S.H., ... & Choi, K.H.: Prevalence of gaming disorder: A meta-analysis. Addictive behaviors **126** (2022), 107183. https://doi.org/10.1016/j.addbeh.2021.107183
21. King, D.L. & Delfabbro, P.H.: The cognitive psychology of Internet gaming disorder. Clinical psychology review **34** (2014), no. 4, 298-308. https://doi.org/10.1016/j.cpr.2014.03.006.
22. Ko, C.H.: Internet gaming disorder. Current Addiction Reports **1** (2014), 177-185. https://doi.org/10.1007/s40429-014-0030-y
23. Kolan, B.J. & Dzandza, P.E.: Effect of social media on academic performance of students in Ghanaian Universities: A case study of University of Ghana, Legon. Library Philosophy and Practice (2018), 0_1-24. https://core.ac.uk/download/pdf/189476898.pdf
24. LaSalle, J.P.: The stability of dynamical systems, society for industrial and applied mathematics, Philadelphia, pa., 1976. In With an appendix:"Limiting equations and stability of nonautonomous ordinary differential equations" by Z. Artstein, Regional Conference Series in Applied Mathematics (1976). https://doi.org/10.1137/1.9781611970432
25. Lemmens, J.S., Valkenburg, P M. & Gentile, D.A.: The internet gaming disorder scale. Psychological assessment **27** (2015), no. 2, 567. https://doi.org/10.1037/pas0000062
26. Li, T., & Guo, Y.: Stability and optimal control in a mathematical model of online game addiction. Filomat **33** (2019), no. 17, 5691-5711. https://doi.org/10.2298/FIL1917691L
27. Ma, S.H., Huo, H.F., & Meng, X.Y.: Modelling alcoholism as a contagious disease: a mathematical model with awareness programs and time delay. Discrete Dynamics in Nature and Society **2015** (2015). https://doi.org/10.1155/2015/260195
28. Malak, M.Z., Shuhaiber, A.H., Al-amer, R.M., Abuadas, M.H. & Aburoomi, R.J.: Correlation between psychological factors, academic performance and social media addiction: model-based testing. Behaviour & Information Technology **41** (2022), no. 8, 1583-1595. https://doi.org/10.1080/0144929X.2021.1891460
29. Mills, D.J. & Allen, J.J.: Self-determination theory, internet gaming disorder, and the mediating role of self-control. Computers in Human Behavior **105** (2020), 106209. https://doi.org/10.1016/j.chb.2019.106209

30. Monacis, L., De Palo, V., Griffiths, M.D. & Sinatra, M.: Social networking addiction, attachment style, and validation of the Italian version of the Bergen Social Media Addiction Scale. Journal of behavioral addictions**6** (2017), no. 2, 178-186. https://doi.org/10.1556/2006.6.2017.023
31. Murray, J.D.: Mathematical biology: I. An introduction. Interdisciplinary applied mathematics. Mathematical Biology, Springer, 17, 2002. https://doi.org/10.1007/b98868
32. Petry, N.M., Rehbein, F., Gentile, D.A., Lemmens, J.S., Rumpf, H.J., Moßle, T., ... & O'Brien, C.P.: An international consensus for assessing internet gaming disorder using the new DSM-5 approach. Addiction **109** (2014), no. 9, 1399-1406. https://doi.org/10.1111/add.12457
33. Petry, N.M., Rehbein, F., Ko, C.-H. & O'Brien, C.P.: (2015). Internet gaming disorder in the DSM-5. Current Psychiatry Reports **17** (2015), no. 9, 72. https://doi.org/10.1007/s11920-015-0610-0
34. Pontes, H.M. & Griffiths, M.D.: Measuring DSM-5 internet gaming disorder: Development and validation of a short psychometric scale. Computers in Human Behavior **45** (2015), 137-143. https://doi.org/10.1016/j.chb.2014.12.006
35. Przybylski, A.K., Weinstein, N. & Murayama, K.: Internet gaming disorder: Investigating the clinical relevance of a new phenomenon. American Journal of Psychiatry **174** (2017), no. 3, 230-236. https://doi.org/10.1176/appi.ajp.2016.16020224
36. Saunders, J.B., Hao, W., Long, J., King, D.L., Mann, K., Fauth-Buhler, M. ... & Poznyak, V.: Gaming disorder: Its delineation as an important condition for diagnosis, management, and prevention. Journal of behavioral addictions **6** (2017), no. 3, 271-279. https://doi.org/10.1556/2006.6.2017.039
37. Sharma, S., Soni, M. & Sharma, D.R.K.: 'Unveiling the Psychological Consequences of Social Media Addiction: A PUARL Model Approach'. Available at SSRN 4625112. https://dx.doi.org/10.2139/ssrn.4625112
38. Shek, D.T., Tang, V.M. & Lo, C.Y.: Internet addiction in Chinese adolescents in Hong Kong: assessment, profiles, and psychosocial correlates. The Scientific World Journal **8** (2008), 776-787. https://doi.org/10.1100/tsw.2008.104
39. Sharma, S. & Sharma, P.K.: A study of SIQR model with Holling type-II incidence rate. Malaya Journal of Matematik **9** (2021), no. 01, 305-311. https://doi.org/10.26637/MJM0901/0052
40. Siddiqui, S. & Singh, T.: Social media its impact with positive and negative aspects. International journal of computer applications technology and research **5** (2016), no. 2, 71-75. http://dx.doi.org/10.7753/IJCATR0502.1006
41. Simsek, A., Elciyar, K. & Kizilhan, T.: A comparative study on social media addiction of high school and university students. Contemporary educational technology **10** (2019), no. 2, 106-119. https://doi.org/10.30935/cet.554452

42. Soni, M., Sharma, R.K. & Sharma, S.: The basic reproduction number and herd immunity for COVID-19 in India. Indian Journal of Science and Technology **14** (2021), no. 35, 2773-2777. https://doi.org/10.17485/IJST/v14i35.797
43. Stevens, M.W., Dorstyn, D., Delfabbro, P.H. & King, D.L.: Global prevalence of gaming disorder: A systematic review and meta-analysis. Australian & New Zealand Journal of Psychiatry. **55** (2021), no. 6, 553-568. https://doi.org/ 10.1177/0004867420962851
44. Sujarwoto, Saputri, R.A.M. & Yumarni, T.: Social media addiction and mental health among university students during the COVID-19 pandemic in Indonesia. International journal of mental health and addiction **21** (2023), no. 1, 96-110. https://doi.org/10.1177/0004867420962851
45. Van den Driessche, P. & Watmough, J.: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Mathematical biosciences **180** (2002), no. 1-2, 29-48. https://doi.org/10.1016/S0025-5564(02)00108-6
46. Wang, X.Y., Huo, H.F., Kong, Q.K. & Shi, W.X.: Optimal control strategies in an alcoholism model. In Abstract and Applied Analysis **2014** (2014, January), Hindawi. https://doi.org/10.1155/2014/954069
47. https://www.who.int/standards/classifications/frequently-asked-questions/gaming-disorder accessed 2 january 2024.
48. Wong, H.Y., Mo, H.Y., Potenza, M.N., Chan, M.N.M., Lau, W.M., Chui, T.K. ... & Lin, C.Y.: Relationships between severity of internet gaming disorder, severity of problematic social media use, sleep quality and psychological distress. International journal of environmental research and public health **17**, no. 6, 1879. https://doi.org/10.3390/ijerph17061879
49. Zajac, K., Ginley, M.K. & Chang, R.: Treatments of internet gaming disorder: a systematic review of the evidence. Expert review of neurotherapeutics **20**, no. 1, 85-93. https://doi.org/10.1080/14737175.2020.1671824

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